Forcer results on deep-inelastic scattering and related quantities

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- DIS and related quantities in perturbative QCD
- Low moments of four-loop splitting and coefficient functions
- All-N results for large-n_f parts of splitting functions, $\gamma_{\rm cusp}^{(3)}\Big|_{n_f^2}$

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Deep-inelastic scattering in perturbative QCD

Inclusive lepton-proton scattering: structure functions F_a . Up to $\mathcal{O}(1/Q^2)$:

$$x^{\eta_a} F_a^{\,p}(x,Q^2) \;=\; \sum_i \int_x^1 \frac{d\xi}{\xi}\, c_{a,i}\!\left(\frac{x}{\xi}, \alpha_{\mathsf{S}}(\mu^2), \frac{\mu^2}{Q^2}\right) f_i^{\,p}(\xi,\mu^2)$$

Coefficient functions: renormalization / factorization scale $\mu = \mathcal{O}(Q)$

Parton densities f_i : renormalization-group evolution (convolution $\otimes$, above)

$$\frac{\partial}{\partial \ln \mu^2}\, f_i(\xi,\mu^2) \;=\; \sum_k \left[P_{ik}(\alpha_{\mathsf{S}}(\mu^2)) \otimes f_k(\mu^2)\right](\xi)$$

Splitting functions ($\Leftrightarrow$ twist-2 anomalous dimensions) & coefficient funct's:

$$P \;=\; \alpha_{\mathsf{S}}\, P^{(0)} + \alpha_{\mathsf{S}}^2\, P^{(1)} + \alpha_{\mathsf{S}}^3\, P^{(2)} + \alpha_{\mathsf{S}}^4\, P^{(3)} + \ldots$$

$$c_a \;=\; \underbrace{\alpha_{\mathsf{S}}^{\,n_a}\Big[\, c_a^{(0)} + \alpha_{\mathsf{S}}\, c_a^{(1)} + \alpha_{\mathsf{S}}^2\, c_a^{(2)}}_{\text{NNLO: first quantitative error estimate}} + \alpha_{\mathsf{S}}^3\, c_a^{(3)} + \alpha_{\mathsf{S}}^4\, c_a^{(4)} + \ldots \Big]$$

NNLO: first quantitative error estimate

N$^{n>2}$LO: for high precision (α_{S} from DIS); slow convergence (Higgs in pp)

. . . , Alekhin, Blümlein, Moch ('16); . . . , Anzai et al. ('15), Anastasiou et al. ('16)

Flavour decomposition of the evolution (I)

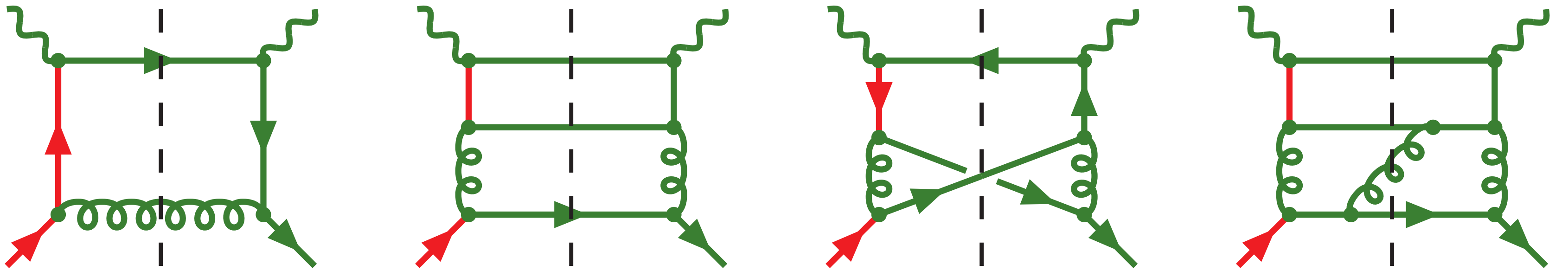
Quark-gluon and gluon-quark splitting functions: (anti-)flavour independent

$$P_{\rm gq} \;\equiv\; P_{{\rm gq}_i} \;=\; P_{{\rm g\bar q}_i} \;, \qquad P_{\rm qg} \;\equiv\; 2n_f P_{{\rm q}_i{\rm g}} \;=\; 2n_f P_{{\rm \bar q}_i{\rm g}}$$

$\Rightarrow$ quark-(anti-)quark differences $q_i - q_k$ and $q_i - \bar{q}_k$ decouple from g

General structure of the (anti-)quark (anti-)quark splitting functions

$$P_{{\rm q}_i{\rm q}_k} \;=\; P_{{\rm \bar q}_i{\rm \bar q}_k} \;=\; \delta_{ik} P^{\rm v}_{\rm qq} + P^{\rm s}_{\rm qq}$$
$$P_{{\rm q}_i{\rm \bar q}_k} \;=\; P_{{\rm \bar q}_i{\rm q}_k} \;=\; \delta_{ik} P^{\rm v}_{\rm q\bar q} + P^{\rm s}_{\rm q\bar q}$$



$P^{\rm v}_{\rm qq} = \mathcal{O}(\alpha_{\rm s})$ $\qquad$ $P^{\rm s}_{\rm qq},\, P^{\rm s}_{\rm q\bar q} \;:\; \alpha_{\rm s}^2$ $\qquad$ $P^{\rm v}_{\rm q\bar q} \;:\; \alpha_{\rm s}^2$ $\qquad$ $P^{\rm s}_{\rm q\bar q} \neq P^{\rm s}_{\rm qq} \;:\; \alpha_{\rm s}^3$

$\Rightarrow$ three types of independent difference (non-singlet, ns) combinations

Flavour decomposition cont'd, even vs odd N

$2(n_f-1)$ flavour asymmetries of $q_i \pm \bar{q}_i$ + one total valence distribution

$$q^{\pm}_{\mathrm{ns},ik} \;=\; q_i \pm \bar{q}_i - (q_k \pm \bar{q}_k) \;, \qquad q^{\mathrm{v}}_{\mathrm{ns}} \;=\; \textstyle\sum_{r=1}^{n_f} (q_r - \bar{q}_r)$$

with

$$P^{\pm}_{\mathrm{ns}} \;=\; P^{\mathrm{v}}_{\mathrm{qq}} \pm P^{\mathrm{v}}_{\mathrm{q\bar{q}}}$$

$$P^{\mathrm{v}}_{\mathrm{ns}} \;=\; P^{\mathrm{v}}_{\mathrm{qq}} - P^{\mathrm{v}}_{\mathrm{q\bar{q}}} + n_f (P^{\mathrm{s}}_{\mathrm{qq}} - P^{\mathrm{s}}_{\mathrm{q\bar{q}}}) \;\equiv\; P^{-}_{\mathrm{ns}} + P^{\mathrm{s}}_{\mathrm{ns}}$$

Flavour-singlet quark distribution q_{s} : maximal coupling to g

$$q_{\mathrm{s}} \;=\; \sum_{r=1}^{n_f} (q_r + \bar{q}_r) \;, \qquad \frac{d}{d\ln\mu^2} \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix} \;=\; \begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix}$$

with (ps = 'pure singlet')

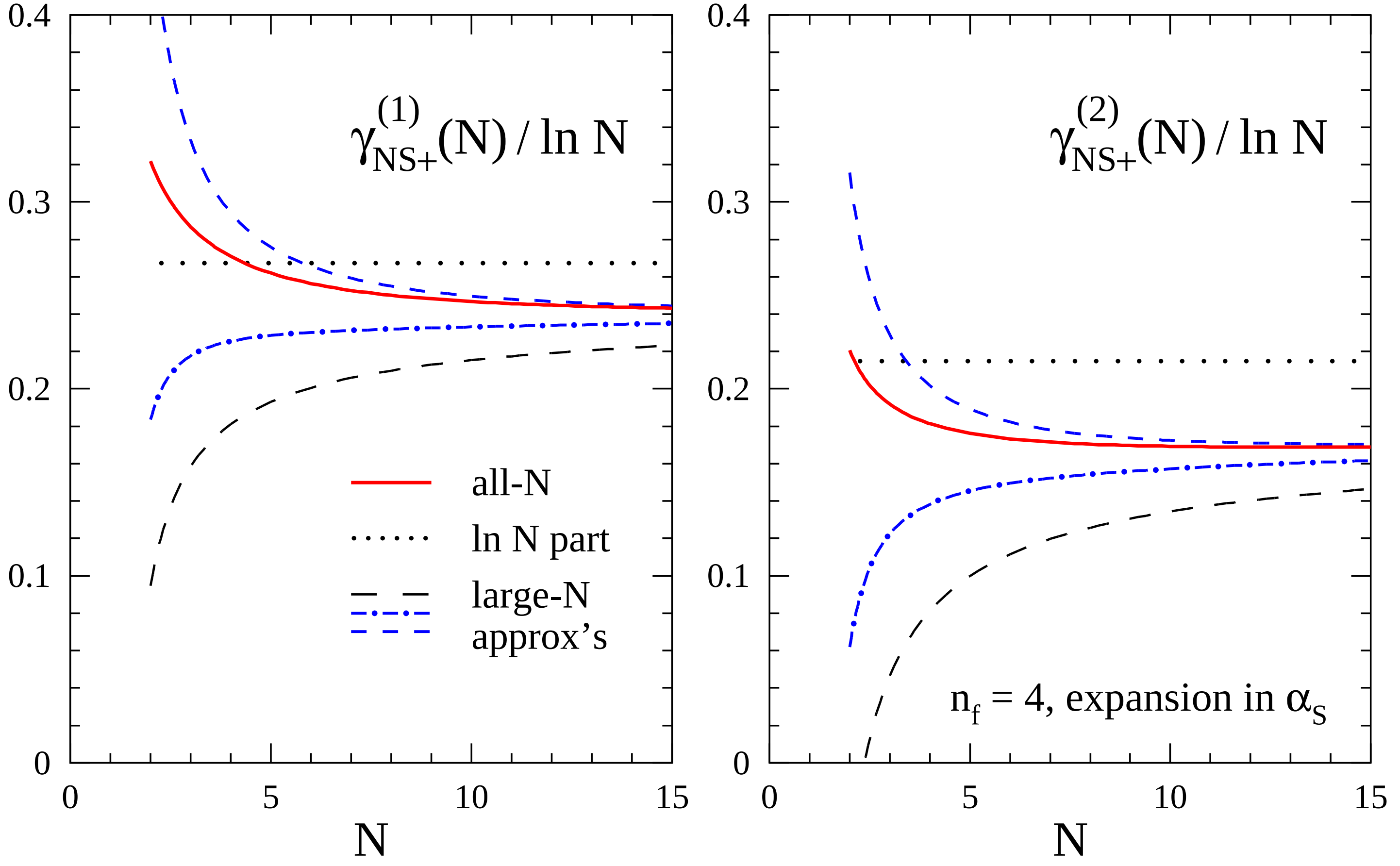
$$P_{\mathrm{qq}} \;=\; P^{+}_{\mathrm{ns}} + n_f (P^{\mathrm{s}}_{\mathrm{qq}} + P^{\mathrm{s}}_{\mathrm{\bar{q}q}}) \;\equiv\; P^{+}_{\mathrm{ns}} + P_{\mathrm{ps}}$$

Coefficient functions: analogous decompositions

Mellin-space calculations such as OPE, 'NIKHEF method': even or odd N

$F_{2,L}$, e.m. and $\nu+\bar{\nu}$ CC, F_{Higgs} (for P_{gi}): even N. F_3 , $\nu+\bar{\nu}$: odd N

Large-N expansion of $\gamma_{\rm ns}$ at NLO and NNLO



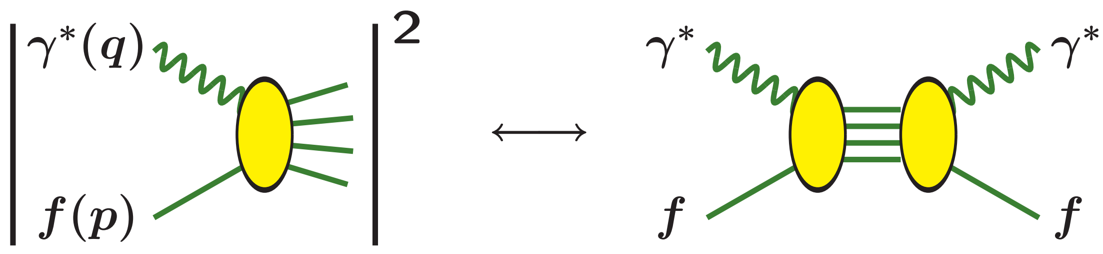
$$\gamma_{\rm ns}^{(n)\pm,\rm v}(N) \;=\; A_n \ln N - B_n + C_n N^{-1} \ln N - D_n + \mathcal{O}_\pm(N^{-2}) \;\; \text{in } \overline{\rm MS}$$

A_n: n-loop cusp anomalous dimension. C_n: known functions of $A_{k<n}$

Korchemsky (89); Dokshitzer, Marchesini, Salam (05)

Calculation of 4-loop DIS with Forcer

Optical theorem: probe-parton total cross sections $\leftrightarrow$ forward amplitudes



Dispersion relation in x: coefficient of $(2p\cdot q)^N$ $\leftrightarrow$ N-th Mellin moment

$$A(N) \;=\; \int_0^1 dx\, x^{N-1} A(x)$$

Fixed N: harmonic projection $\rightarrow$ self-energy integrals $\rightarrow$ Mincer / Forcer

3-loop: Larin, van Ritbergen, Vermaseren [, Nogueira] ('93, '96); Retey, Vermaseren ('00)
for same quantities as here; Moch, Rogal ('07) for $\nu - \bar{\nu}$ charged-current cases; ...

Projection on structure functions, $D = 4 - 2\varepsilon$ dimensions, mass factorization

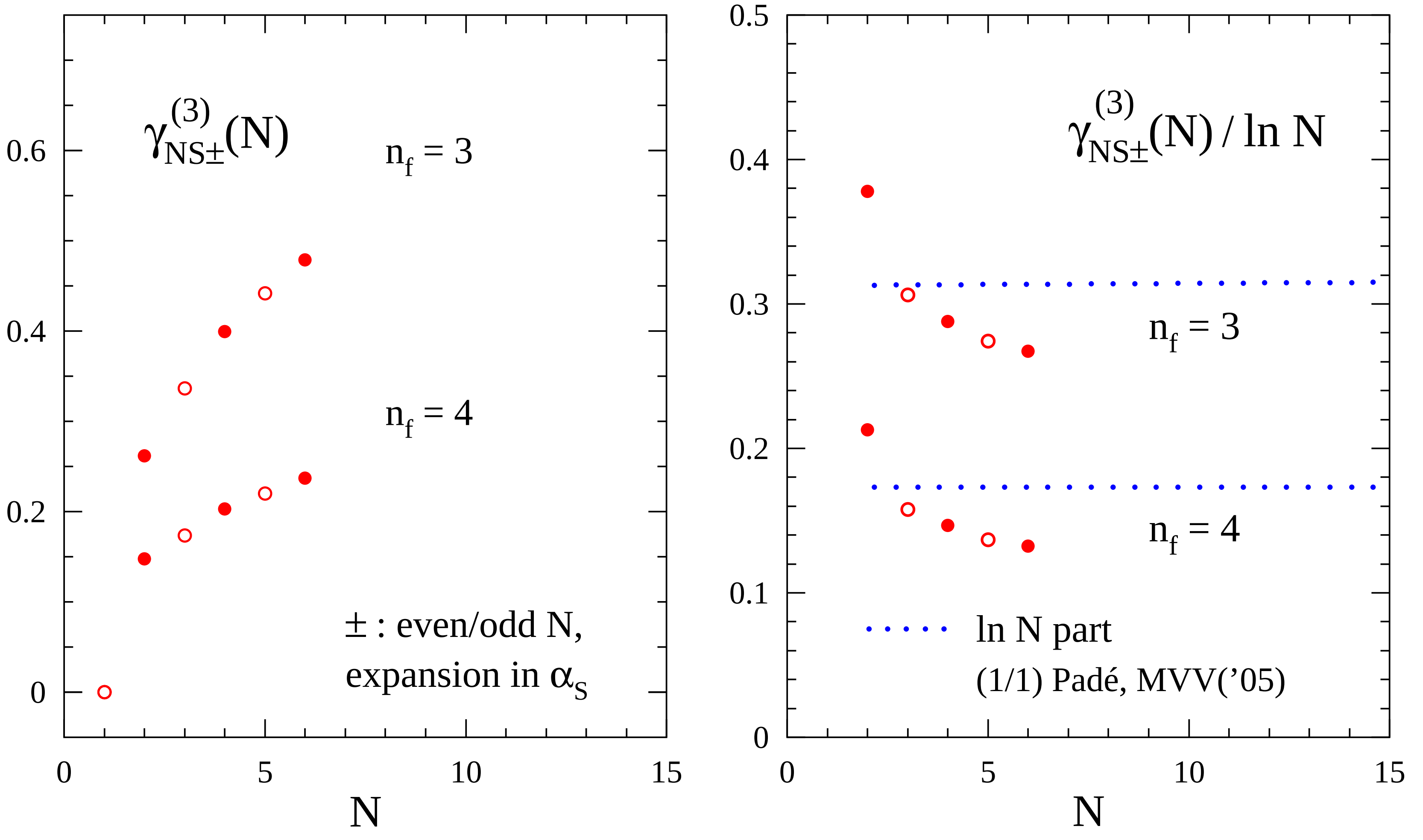
ε^{-1}: splitting functions $P_{\rm ik}^{(n)}(N) = -\,\gamma_{\rm ik}^{(n)}(N)$, ϵ^0 : n-loop coefficient fct's

Example: analytic result for $\gamma_{gg}^{(3)}$ at $N=4$

$$
\begin{aligned}
\gamma_{gg}^{(3)}(N=4) \;=\; & C_A^4 \left(\frac{1502628149}{3375000} + \frac{1146397}{11250}\,\zeta_3 - \frac{504}{5}\,\zeta_5 \right) + \frac{d_A^{abcd}\,d_A^{abcd}}{n_a} \left(\frac{21623}{150} \right. \\
& \left. + \frac{15596}{15}\,\zeta_3 - \frac{6048}{5}\,\zeta_5 \right) - n_f C_A^3 \left(\frac{20580892841}{72900000} + \frac{12550223}{22500}\,\zeta_3 - \frac{8613}{25}\,\zeta_4 - \frac{4316}{27}\,\zeta_5 \right) \\
& + n_f \frac{d_R^{abcd}\,d_A^{abcd}}{n_a} \left(\frac{160091}{675} + \frac{80072}{225}\,\zeta_3 - \frac{48016}{45}\,\zeta_5 \right) - n_f C_A^2 C_F \left(\frac{4212122951}{41006250} \right. \\
& \left. - \frac{1170784}{5625}\,\zeta_3 + \frac{418198}{1125}\,\zeta_4 - \frac{17636}{45}\,\zeta_5 \right) + n_f C_A C_F^2 \left(\frac{1913110089023}{26244000000} + \frac{39313783}{101250}\,\zeta_3 \right. \\
& \left. + \frac{26741}{750}\,\zeta_4 - \frac{3082}{5}\,\zeta_5 \right) + n_f C_F^3 \left(\frac{34764568601}{2099520000} - \frac{958343}{40500}\,\zeta_3 - \frac{18997}{2250}\,\zeta_4 + \frac{908}{45}\,\zeta_5 \right) \\
& - n_f^2 C_A^2 \left(\frac{3250393649}{218700000} - \frac{2969291}{20250}\,\zeta_3 + \frac{1566}{25}\,\zeta_4 + \frac{1276}{135}\,\zeta_5 \right) - n_f^2 C_F^2 \left(\frac{275622924731}{26244000000} \right. \\
& \left. - \frac{253369}{10125}\,\zeta_3 + \frac{1078}{225}\,\zeta_4 \right) + n_f^2 C_A C_F \left(\frac{136020246173}{3280500000} - \frac{1672751}{10125}\,\zeta_3 + \frac{15172}{225}\,\zeta_4 \right) \\
& + n_f^2 \frac{d_R^{abcd}\,d_R^{abcd}}{n_a} \left(\frac{75788}{675} + \frac{3008}{15}\,\zeta_3 - \frac{20416}{45}\,\zeta_5 \right) + n_f^3 C_F \left(\frac{1780699}{24300000} - \frac{484}{675}\,\zeta_3 \right) \\
& - n_f^3 C_A \left(\frac{20440457}{21870000} - \frac{1888}{405}\,\zeta_3 \right) .
\end{aligned}
$$

This line: Bennett, Gracey ('97)

N^{3}LO corrections to $\gamma_{\mathrm{ns}}^{\pm}(N)$ at low moments



$N = 2, 3, 4$: agreement with Baikov, Chetyrkin ('06, ...); Velizhanin ('12, '14)

Consistent with (but not sufficient to improve on) the Padé estimate of $\gamma_{\mathrm{cusp}}^{(3)}$

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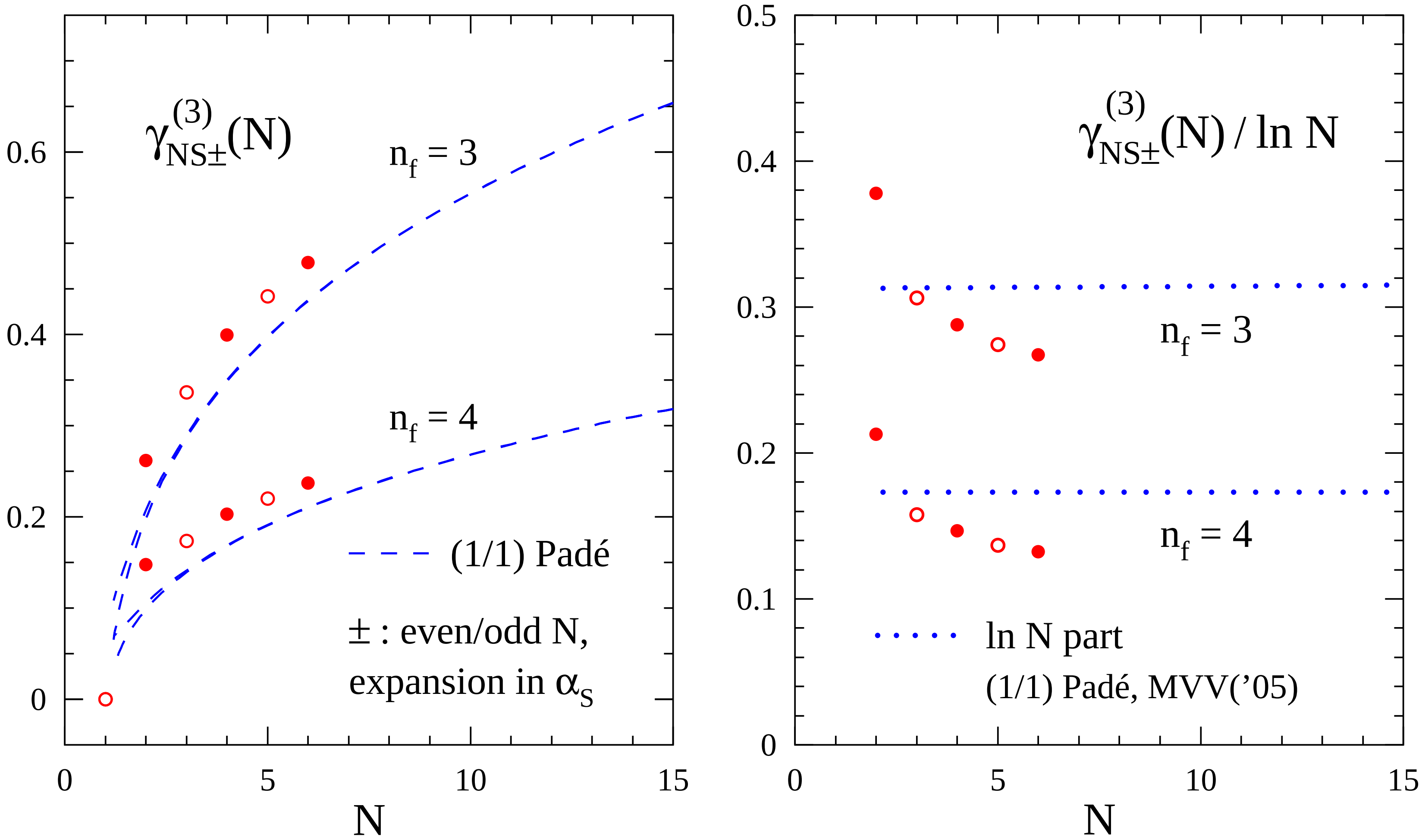

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Consistent with (but not sufficient to improve on) the Padé estimate of $\gamma^{(3)}_{\rm cusp}$

α_s expansions of $\gamma_{qq}(N, n_f = 4)$ at low N

Even-N non-singlet combination

$$\gamma^+_{ns}(2,4) = 0.28294\,\alpha_s\,(1 + 0.7987\,\alpha_s + 0.5451\,\alpha_s^2 + 0.5215\,\alpha_s^3 + \ldots)$$

$$\gamma^+_{ns}(4,4) = 0.55527\,\alpha_s\,(1 + 0.6851\,\alpha_s + 0.4564\,\alpha_s^2 + 0.3659\,\alpha_s^3 + \ldots)$$

$$\gamma^+_{ns}(6,4) = 0.71645\,\alpha_s\,(1 + 0.6497\,\alpha_s + 0.4368\,\alpha_s^2 + 0.3307\,\alpha_s^3 + \ldots)$$

Odd-N cases, γ^{a}_{ns} for $a = -, v$: $\gamma^{a}_{ns}(1, n_f) = 0$ (fermion number)

$$\gamma^{a}_{ns}(3,4) = 0.44210\,\alpha_s\,(1 + 0.7218\,\alpha_s + 0.4767\,\alpha_s^2 + 0.3921\,\alpha_s^3 + \ldots + \delta_{av}\left[0.0144\,\alpha_s^2 + 0.0045\,\alpha_s^3 + \ldots\right])$$

$$\gamma^{a}_{ns}(5,4) = 0.64369\,\alpha_s\,(1 + 0.6636\,\alpha_s + 0.4434\,\alpha_s^2 + 0.3421\,\alpha_s^3 + \ldots + \delta_{av}\left[0.0032\,\alpha_s^2 + 0.0024\,\alpha_s^3 + \ldots\right])$$

Flavour-singlet quark-quark anomalous dimension

$$\gamma_{qq}(2,4) = 0.28294\,\alpha_s\,(1 + 0.6219\,\alpha_s + 0.1461\,\alpha_s^2 + 0.3662\,\alpha_s^3 + \ldots)$$

$$\gamma_{qq}(4,4) = 0.55527\,\alpha_s\,(1 + 0.6803\,\alpha_s + 0.4278\,\alpha_s^2 + 0.3459\,\alpha_s^3 + \ldots)$$

α_s expansion cont'd, supersymmetric relation

$$\gamma_{\rm qg}(2,4) \;=\; -0.21221\,\alpha_s\,(1 + 0.9004\,\alpha_s - 0.1028\,\alpha_s^2 - 0.2367\,\alpha_s^3 + \ldots)$$

$$\gamma_{\rm qg}(4,4) \;=\; -0.11671\,\alpha_s\,(1 - 0.2801\,\alpha_s - 0.9986\,\alpha_s^2 + 0.1297\,\alpha_s^3 + \ldots)$$

Lower row of the matrix: $\gamma_{\rm gi}(2,n_f) = -\gamma_{\rm qi}(2,n_f)$, **momentum sum rule**

$$\gamma_{\rm gq}(4,4) \;=\; -0.07781\,\alpha_s\,(1 + 1.1152\,\alpha_s + 0.8234\,\alpha_s^2 + 0.8833\,\alpha_s^3 + \ldots)$$

$$\gamma^{+}_{\rm gg}(4,4) \;=\; 1.21489\,\alpha_s\,(1 + 0.3835\,\alpha_s + 0.1220\,\alpha_s^2 + 0.2406\,\alpha_s^3 + \ldots)$$

Relative N^{3}LO corrections small in all cases, | coeff's$(N, n_f = 3,\ldots,6)| \lesssim 1$

α_{S} expansion cont'd, supersymmetric relation

$$
\begin{aligned}
\gamma_{\rm qg}(2,4) &= -0.21221\,\alpha_{\mathsf{S}}\,(1 + 0.9004\,\alpha_{\mathsf{S}} - 0.1028\,\alpha_{\mathsf{S}}^2 - 0.2367\,\alpha_{\mathsf{S}}^3 + \ldots) \\
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\end{aligned}
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\gamma_{\rm gg}^{+}(4,4) &= 1.21489\,\alpha_{\mathsf{S}}\,(1 + 0.3835\,\alpha_{\mathsf{S}} + 0.1220\,\alpha_{\mathsf{S}}^2 + 0.2406\,\alpha_{\mathsf{S}}^3 + \ldots)
\end{aligned}
$$

Relative N^{3}LO corrections small in all cases, $|\,$coeff's$(N, n_f = 3,\ldots,6)| \lesssim 1$

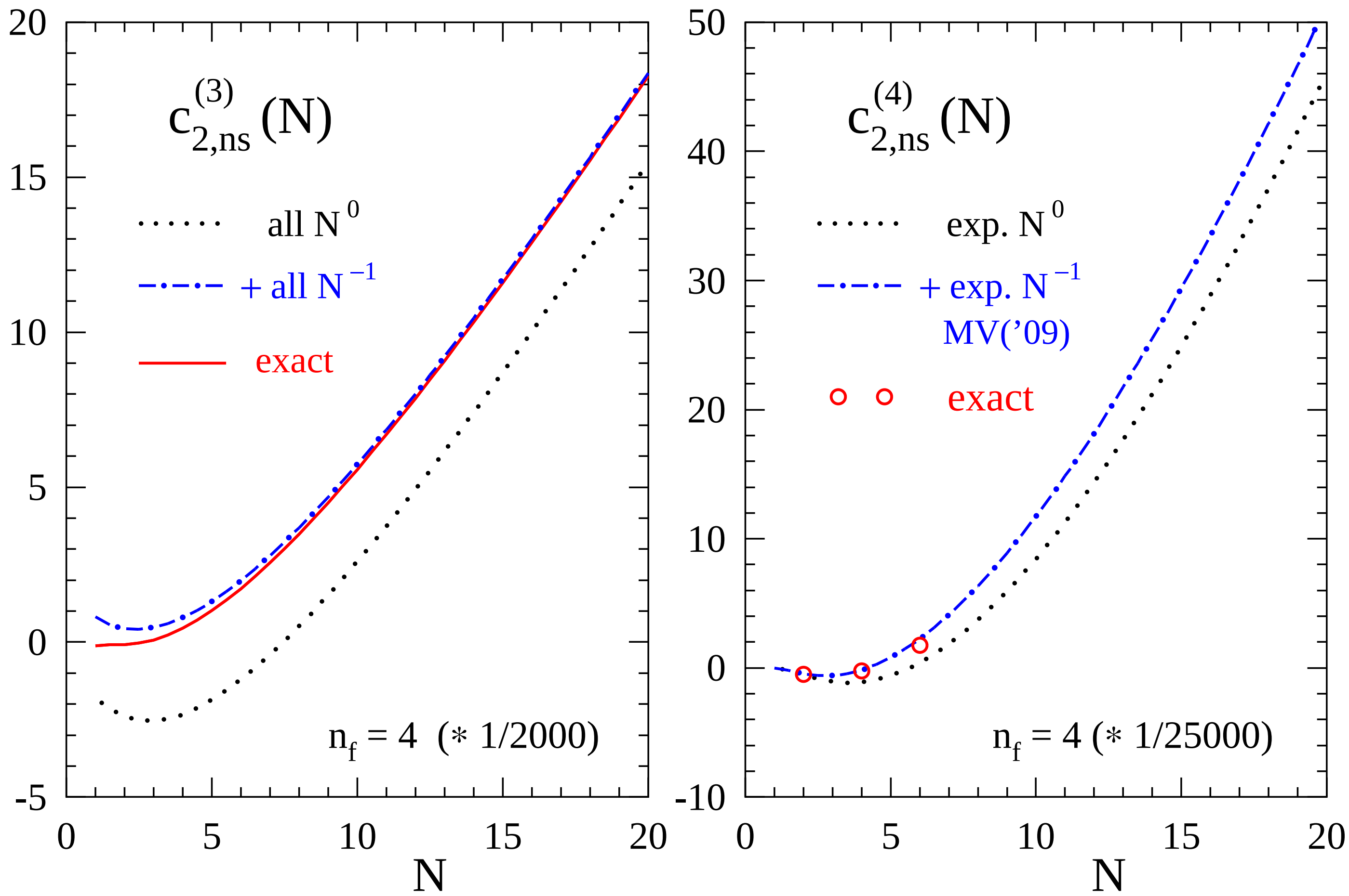
Supersymmetric relation: put $n_f = C_F = C_A \equiv n_c$, consider combination

$$
\Delta_{\rm S}^{(n)}(N) \;=\; -\,\gamma_{\rm qq}^{(n)}(N) - \gamma_{\rm gq}^{(n)}(N) + \gamma_{\rm qg}^{(n)}(N) + \gamma_{\rm gg}^{(n)}(N)
$$

In $\overline{\rm MS}$: $\Delta_{\rm S}^{(n)}(4) =$ integer$_1(n)/$integer$_2(n)\; n_c^{n+1}$ also holds for $n = 3$ if

$$
\frac{1}{2n_c}\frac{d_{\rm RA}^{(4)}}{n_c} = \frac{1}{2n_c}\frac{d_{\rm AA}^{(4)}}{n_a} = 2n_c\frac{d_{\rm RR}^{(4)}}{n_a} = \frac{d_{\rm RA}^{(4)}}{n_a} = \frac{d_{\rm RR}^{(4)}}{n_c}\;, \qquad d_{\rm xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd}
$$

Third- and fourth-order contributions to $C_{2,\mathrm{ns}}$

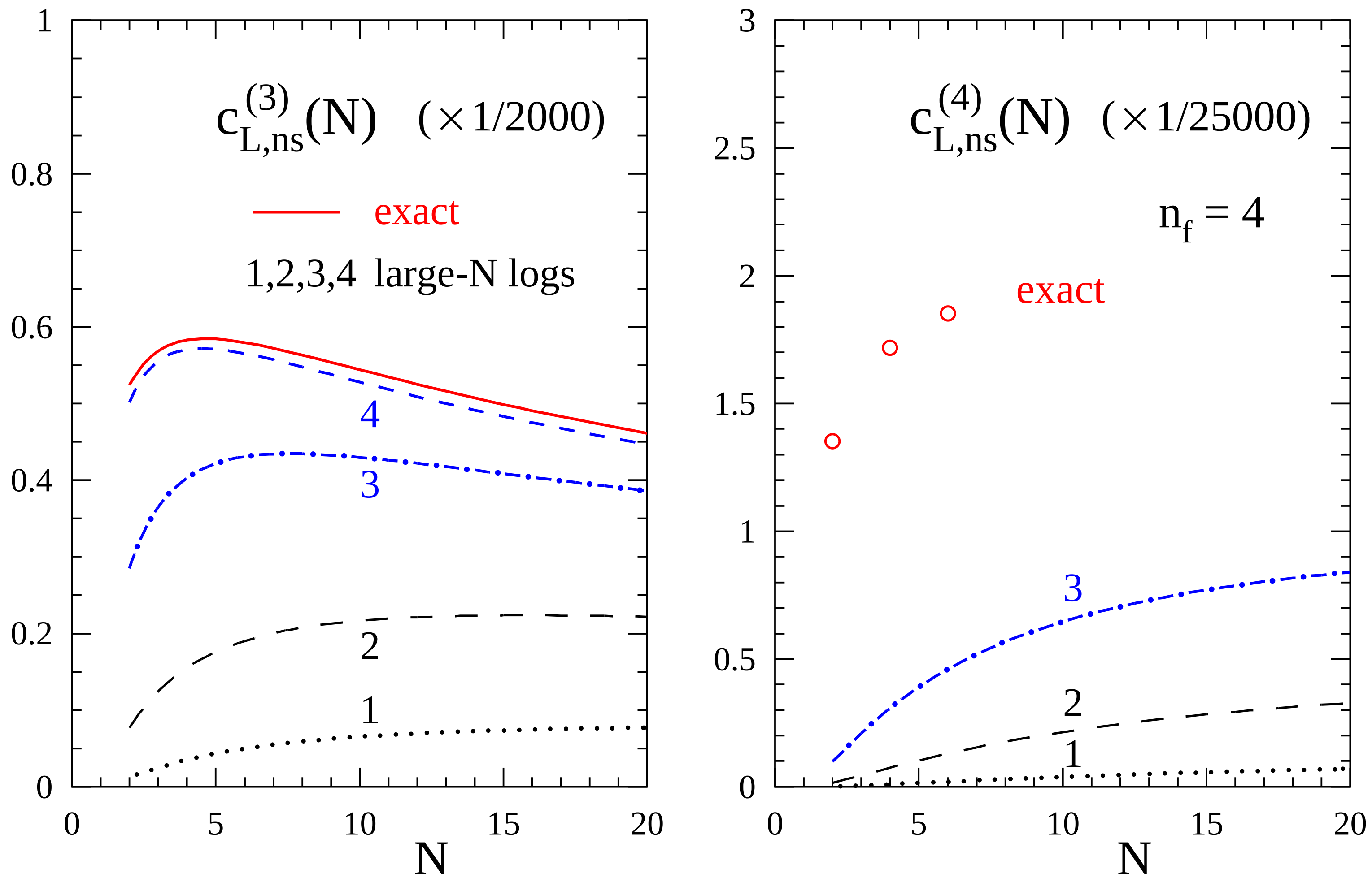


Exp. N^0 : $\ln^8 N \ldots \ln^2 N$ from N^{3}LL soft-gluon exponentiation MVV ('05)

Exp. N^{-1}: $\ln^7 N \ldots \ln^4 N$, resummation via phys. kernel / D-dim. structure

Moch, A.V. ('09) / Almasy, Soar, A.V. ('10)

Third- and fourth-order contributions to $C_{L,\mathrm{ns}}$



Large-N terms: $N^{-1}(\ln^{2n-2} N \ldots 1)$ at n^{th} order in α_s

Double-logarithmic resummation (three terms at order α_s^4) Moch, A.V. ('09)

α_s series of charged-current $C_{a,\mathrm{ns}}(N, n_f = 4)$

$$C_{2,\mathrm{ns}}(2,4) = 1 + 0.0354\,\alpha_s - 0.0231\,\alpha_s^2 - 0.0613\,\alpha_s^3 - 0.4746\,\alpha_s^4 + \ldots$$
$$C_{2,\mathrm{ns}}(4,4) = 1 + 0.4828\,\alpha_s + 0.4711\,\alpha_s^2 + 0.4727\,\alpha_s^3 - 0.2458\,\alpha_s^4 + \ldots$$
$$C_{2,\mathrm{ns}}(6,4) = 1 + 0.8894\,\alpha_s + 1.2053\,\alpha_s^2 + 1.7571\,\alpha_s^3 + 1.7748\,\alpha_s^4 + \ldots$$

$$C_{L,\mathrm{ns}}(2,4) = 0.14147\,\alpha_s\,(1 + 1.7270\,\alpha_s + 3.7336\,\alpha_s^2 + 9.5619\,\alpha_s^3 + \ldots)$$
$$C_{L,\mathrm{ns}}(4,4) = 0.08488\,\alpha_s\,(1 + 2.5619\,\alpha_s + 6.9208\,\alpha_s^2 + 20.251\,\alpha_s^3 + \ldots)$$
$$C_{L,\mathrm{ns}}(6,4) = 0.06063\,\alpha_s\,(1 + 3.1557\,\alpha_s + 9.6370\,\alpha_s^2 + 30.572\,\alpha_s^3 + \ldots)$$

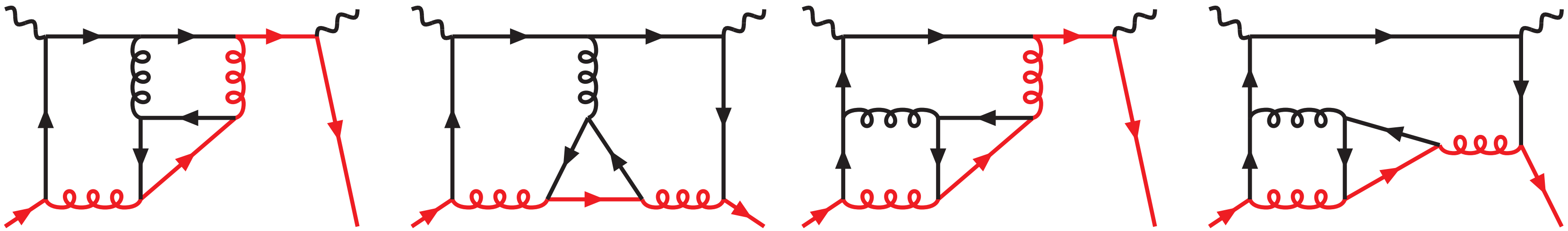
$$C_{3,\mathrm{ns}}(1,4) = 1 - 0.3183\,\alpha_s - 0.3293\,\alpha_s^2 - 0.4467\,\alpha_s^3 - 1.0512\,\alpha_s^4 + \ldots + \delta_{\mathrm{av}}\,[0.0533\,\alpha_s^3 + 0.1999\,\alpha_s^4 + \ldots]$$
$$C_{3,\mathrm{ns}}(3,4) = 1 + 0.1326\,\alpha_s - 0.0852\,\alpha_s^2 - 0.5202\,\alpha_s^3 - 2.2510\,\alpha_s^4 + \ldots + \delta_{\mathrm{av}}\,[0.0202\,\alpha_s^3 + 0.0805\,\alpha_s^4 + \ldots]$$
$$C_{3,\mathrm{ns}}(5,4) = 1 + 0.6166\,\alpha_s + 0.6042\,\alpha_s^2 + 0.4214\,\alpha_s^3 - 1.3217\,\alpha_s^4 + \ldots + \delta_{\mathrm{av}}\,[.00788\,\alpha_s^3 + 0.0422\,\alpha_s^4 + \ldots]$$

$N = 1$ (GLS sum rule): agreement with Baikov, Chetyrkin, Kühn [, Rittinger] ('10, '12)

All-N results for n_f^3 and n_f^2 parts of $\gamma_{\mathrm{ik}}^{(3)}$ (I)

Non-singlet: n_f^3 part by Gracey ('94), confirmed. New: n_f^2 terms determined

Top-level diag's: 3-loop n_f case of MVV ('02) with a 1-loop gluon propagator



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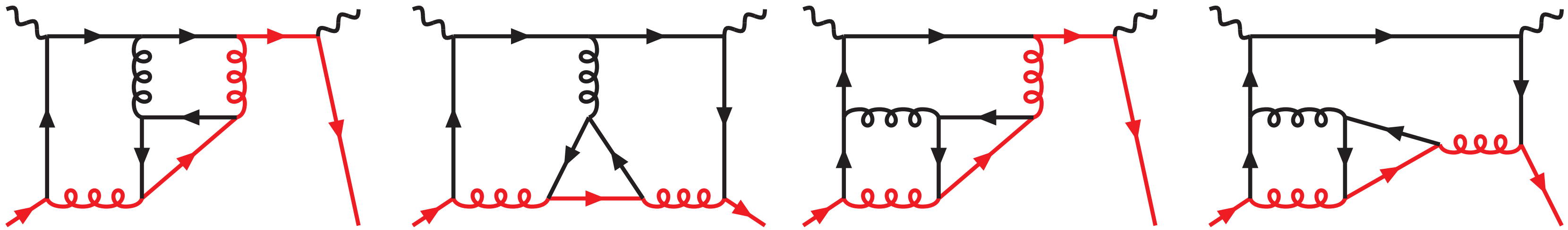

Colour-factor decomposition of the non-singlet$_\pm$ cases

$$\begin{aligned}\gamma_{\rm ns}^{(3)\pm}|_{n_f^2} &= C_F n_f^2 \,\{\, C_F\, 2A \qquad\qquad + (C_A - 2C_F) B_\pm \} \\ &= C_F n_f^2 \,\{\, C_F (2A - 2B_\pm) + \; C_A\, B_\pm \}\end{aligned}$$

$A(N)$: large n_c part, same for +/-, even & odd N, $S_{m_1,m_2,\ldots}(N)$ with $m_i > 0$

If $A(N)$ is known: get $B_\pm(N)$ from C_F parts in 2nd line, only 2-loop diag's

Forcer: $N \leq 20$, even & odd, for $A(N)$; $N \leq 42$, even or odd, for $B_\pm(N)$

Analytic forms in N obtained by LLL-based algorithm, cf. $\Delta\gamma_{\rm gj}^{(2)}$ of MVV ('14)

All-N results for n_f^3 and n_f^2 parts of $\gamma_{\rm ik}^{(3)}$ (II)

Example: large-n_c contribution (sums at N, $D_i = (N+i)^{-1}$, $\eta = D_0 - D_1$)

$$\gamma_{\rm ns}^{(3)}(N)|_{C_F n_c n_f^2} = \frac{127}{18} + \frac{1}{81}\Big(\frac{20681}{2}\,\eta + 2119\,S_1 - 2275\,\eta^2 - 20460\,D_1^2 + 3392\,S_1\eta - 5036\,S_2\Big) + \frac{4}{81}\Big(118\,\eta^3 - 886\,D_1^3 - 914\,S_1\eta^2 - 848\,S_1 D_1^2 - 152\,S_{1,2} - 416\,S_2\eta - 152\,S_{2,1} + 1148\,S_3\Big) + \frac{8}{27}\Big(- 57\,D_1^4 + 18\,S_1\eta^3 - 24\,S_1 D_1^3 + 2\,S_2\eta^2 + 128\,S_2 D_1^2 - 8\,S_3\eta + 40\,S_{1,3} + 80\,S_{2,2} + 120\,S_{3,1} - 159\,S_4\Big) + \frac{8}{9}\Big(- 6\,\eta^5 - 12\,D_1^5 + 10\,S_1\eta^4 - 24\,S_1 D_1^4 + 8\,S_2\eta^3 + 4\,S_3\eta^2 - 8\,S_3 D_1^2 + 4\,S_{3,1}\eta - 8\,S_{1,3,1} + 4\,S_{1,4} - 8\,S_{2,3} - 16\,S_{3,2} - 2\,S_4\eta - 20\,S_{4,1} + 24\,S_5\Big) + \text{ much simpler } \zeta_3, \zeta_4 \text{ terms}$$

All-N results for n_f^3 and n_f^2 parts of $\gamma_{\rm ik}^{(3)}$ (II)

Example: large-n_c contribution (sums at N, $D_i = (N+i)^{-1}$, $\eta = D_0 - D_1$)

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& - 5036\,S_2\Big) + \frac{4}{81}\Big(118\,\eta^3 - 886\,D_1^3 - 914\,S_1\eta^2 - 848\,S_1 D_1^2 - 152\,S_{1,2} - 416\,S_2\eta \\
& - 152\,S_{2,1} + 1148\,S_3\Big) + \frac{8}{27}\Big(-57\,D_1^4 + 18\,S_1\eta^3 - 24\,S_1 D_1^3 + 2\,S_2\eta^2 + 128\,S_2 D_1^2 \\
& - 8\,S_3\eta + 40\,S_{1,3} + 80\,S_{2,2} + 120\,S_{3,1} - 159\,S_4\Big) + \frac{8}{9}\Big(-6\,\eta^5 - 12\,D_1^5 + 10\,S_1\eta^4 \\
& - 24\,S_1 D_1^4 + 8\,S_2\eta^3 + 4\,S_3\eta^2 - 8\,S_3 D_1^2 + 4\,S_{3,1}\eta - 8\,S_{1,3,1} + 4\,S_{1,4} - 8\,S_{2,3} \\
& - 16\,S_{3,2} - 2\,S_4\eta - 20\,S_{4,1} + 24\,S_5\Big) \;+\; \text{much simpler } \zeta_3, \zeta_4 \text{ terms}
\end{aligned}$$

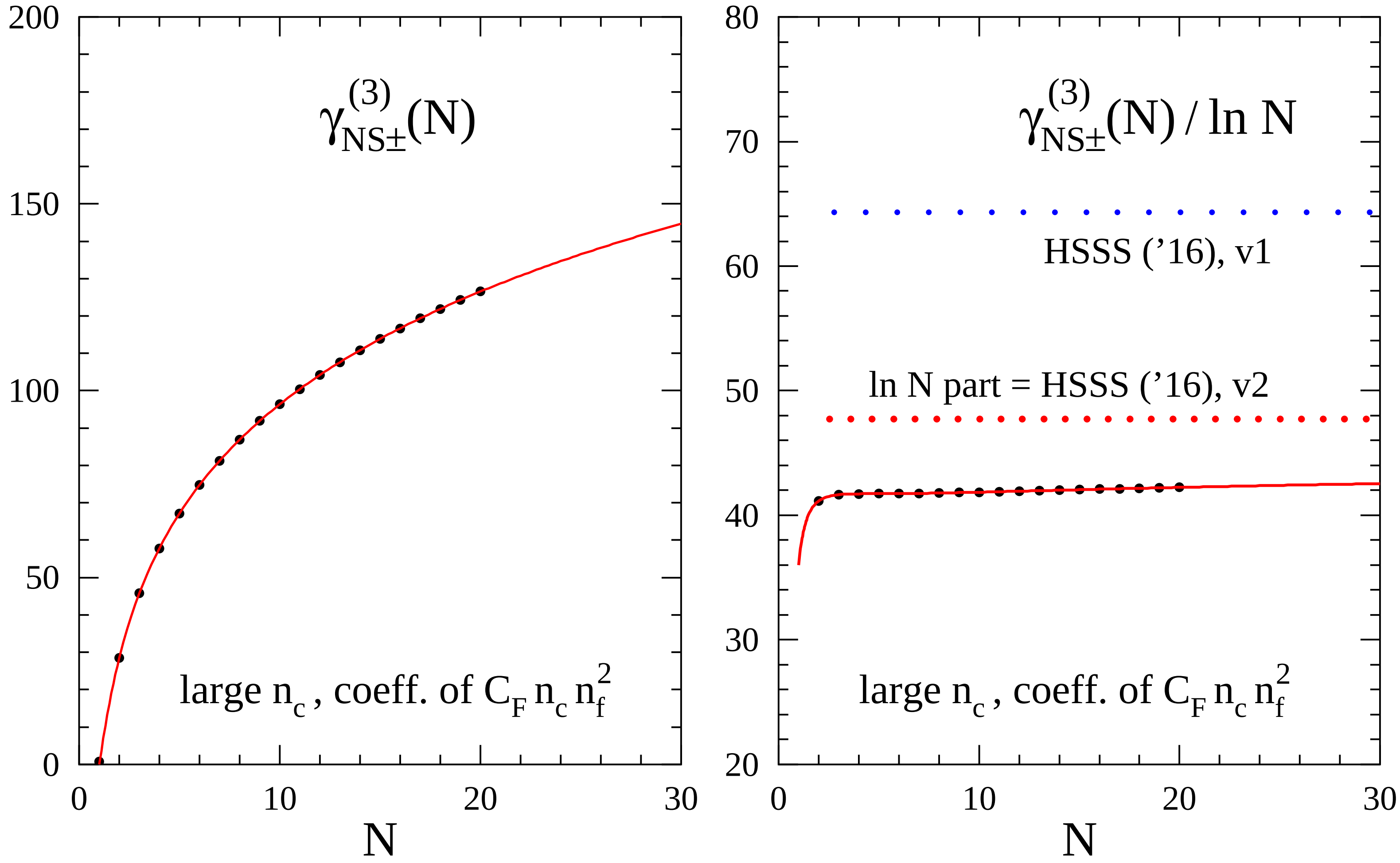
Quark cusp anomalous dimension, from $\ln N$ coefficient in large-N limit:

$$\begin{aligned}
\gamma_{\rm cusp}^{(3)} \;=\; & \ldots + C_F C_A n_f^2\Big(\frac{923}{81} - \frac{608}{81}\zeta_2 + \frac{2240}{27}\zeta_3 - \frac{112}{3}\zeta_4\Big) \\
& + C_F^2 n_f^2\Big(\frac{2392}{81} - \frac{640}{9}\zeta_3 + 32\,\zeta_4\Big) - C_F n_f^3\Big(\frac{32}{81} - \frac{64}{27}\zeta_3\Big)
\end{aligned}$$

Singlet: n_f^3 done for ps, gq and gg – two linear combinations long known

Gracey ('96); Bennett, Gracey ('97). n_f^3 for qg much harder, in progress

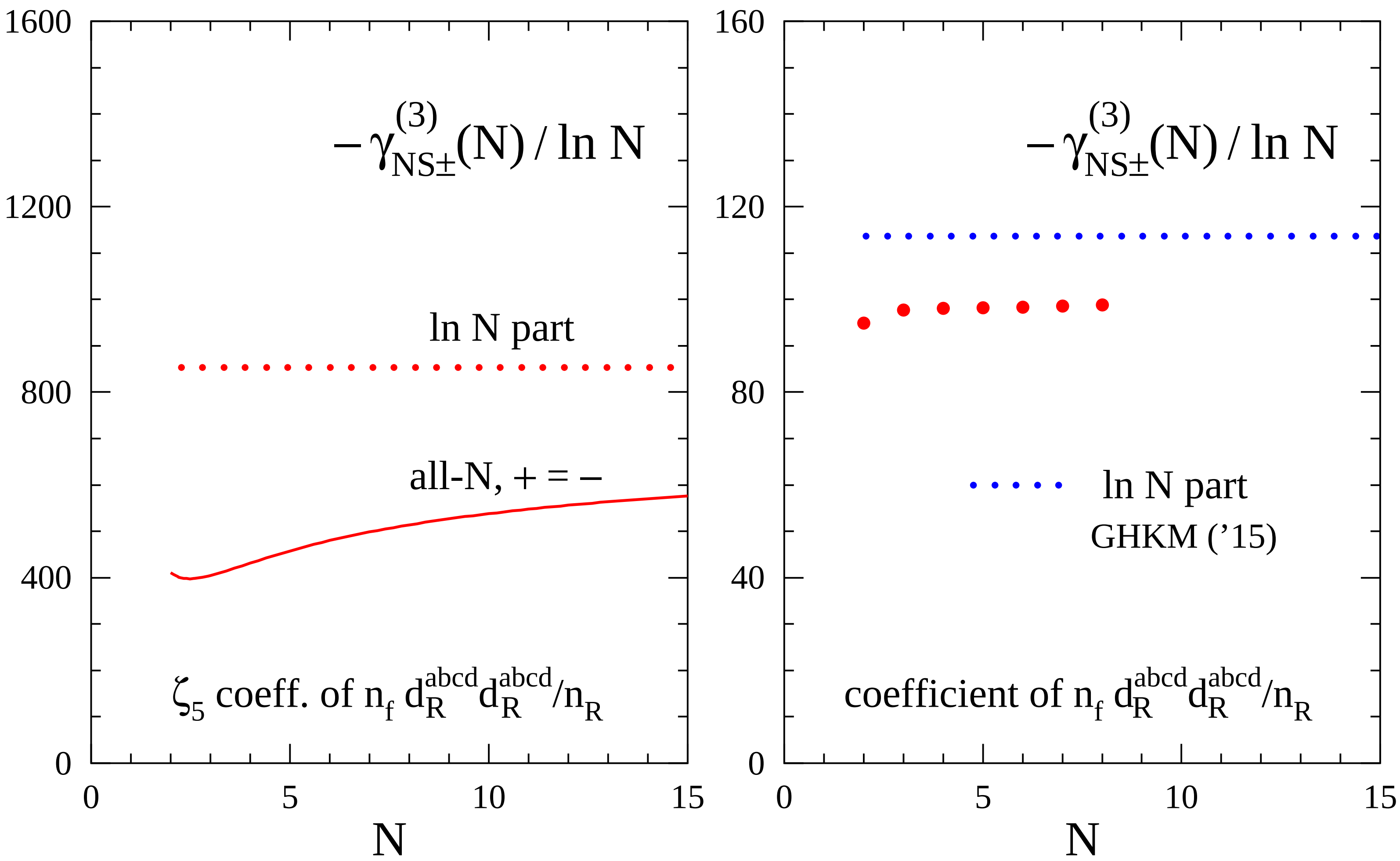
n_f^2 part of $\gamma_{\rm ns}^{(3)}(N)$ in the large-n$_c$ limit



Dokshitzer, Marchesini, Salam ('05) **prediction of $N^{-1}\ln N$ correct**

$\ln N$ coefficient: error spotted/fixed in **Henn, Smirnov^2, Steinhauser ('16)**

n_f quartic-Casimir contribution to $\gamma_{\rm ns}^{(3)}(N)$



So far all-N form only for ζ_5 part – wrong in Velizhanin ('14). $N \leq 8$ values consistent with $\gamma_{\rm cusp}|_{(d_R^{abcd})^2}$ of Grozin, Henn, Korchemski, Marquardt ('15)

Summary and Outlook

Forcer: new FORM program for parametric reduction of 4-loop massless propagator integrals — Ruijl, Ueda, Vermaseren

$\Rightarrow$ Calculation of fixed moments of structure functions in DIS at order α_S^4, moments of N^{3}LO splitting fct's & coefficient fct's for F_L, N^{4}LO for $F_{2,3}$

Hardest (11-line non-planar) topologies: very time-consuming (unlike β-fct.)

$N \geq 4$: 4-loop time $\gg 10^3 \cdot$ 3-loop time, $N \rightarrow N+2$: weight $\rightarrow$ weight $+4$

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Non-singlet $N \leq 6$, singlet $N = 2,\ 4$ done: corr's to splitting fct's small

High-n_f contributions to splitting functions (non-singlet n_f^2, singlet n_f^3):
Sufficiently high N possible for (LLL) determination of functional forms in N

$\Rightarrow$ $C_F n_f^2 (C_F, C_A)$ parts of the 4-loop cusp anomalous dimension $\gamma_{\rm cusp}^{(3)}$
Evidence for non-vanishing contribution to $\gamma_{\rm cusp}^{(3)}$ of $n_f d_R^{abcd} d_R^{abcd} / n_c$

Next aim: more moments $\to$ approx. $P_{\rm ik}^{(3)}(x)$ for use with N^{3}LO coeff. fct's