Forcer: a FORM program for 4-loop massless propagators

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Introduction

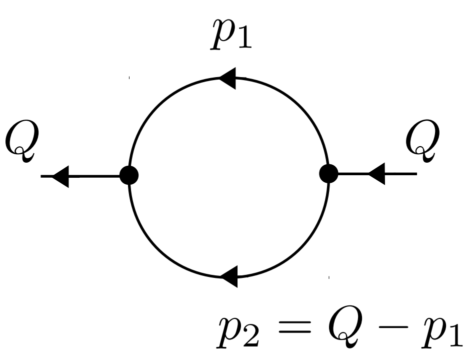
Introduction

- Massless propagator-type Feynman integrals: many physics applications ⇨ Andreas's talk

- At the 1-loop:

$$F(\underbrace{n_1, n_2}) = \int \frac{d^D p_1}{(2\pi)^D} \frac{1}{(p_1^2)^{n_1} (p_2^2)^{n_2}}$$

indices: powers of the denominators



  - Dimensional regularization: $D = 4 - 2\epsilon$
  - The external momenta Q is off-shell: $Q^2 \neq 0$

Feynman integral calculus

- The "standard" way of Feynman integral calculus

  (1) Reduction to master integrals (MIs) via integration-by-parts identities (IBPs) [Chetyrkin, Tkachov '81]

  (2) Evaluation of MIs

- Generic IBP reduction algorithms for any processes
  - Laporta's algorithm, Baikov's method, Lee's "LiteRed", ...
    [Laporta '01] [Baikov '96; '05] [Lee '12; '13]

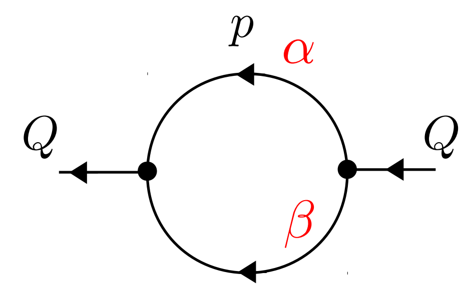
    AIR, FIRE, Reduze,...

- "Mincer" algorithm [Chetyrkin, Tkachov '81]
  - Specialized reduction for massless propagator-type integrals (up to 3-loops), very efficient

“Mincer” approach

General 1-loop formula

- General formula for arbitrary indices α and β



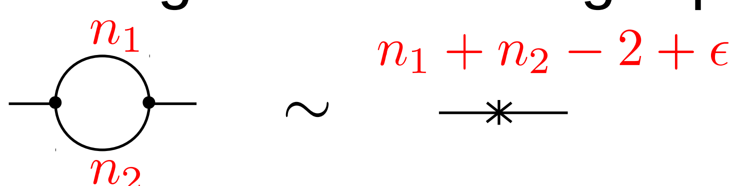
$$\int \frac{d^D p}{(2\pi)^D} \frac{p^{\mu_1} \dots p^{\mu_n}}{(p^2)^\alpha [(Q-p)^2]^\beta}$$ allowing numerators ($n \geq 0$)

$$= \frac{1}{(4\pi)^2} \frac{1}{(Q^2)^{\alpha+\beta-2+\epsilon}} \sum_{\sigma=0}^{\lfloor n/2 \rfloor} G(\alpha,\beta,n,\sigma)(Q^2)^\sigma \left[\frac{1}{\sigma!} \left(\frac{\Box_p}{4} \right)^\sigma p^{\mu_1} \dots p^{\mu_n} \right]_{p=Q}$$

$$G(\alpha,\beta,n,\sigma) = (4\pi)^\epsilon \frac{\Gamma(\alpha+\beta-\sigma-2+\epsilon)}{\Gamma(\alpha)\Gamma(\beta)} B(2-\epsilon-\alpha+n-\sigma, 2-\epsilon-\beta+\sigma)$$

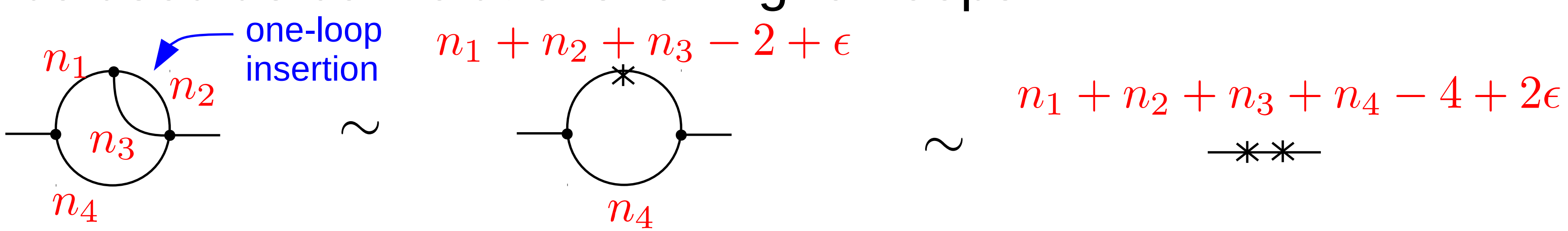
[Chetyrkin, Kataev, Tkachov '80; Chetyrkin, Tkachov '81]

- The result gets a non-integer power $1/(Q^2)^\epsilon$



$\ast$: non-integer part ϵ

- Can be used as convolutions for higher loops

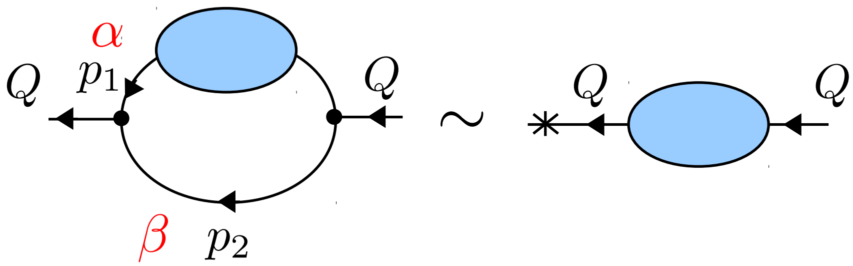


"Carpet" 1-loop integral

- We can perform another type of 1-loop integral (outer loop of p_1) for arbitrary indices α and β possibly with numerators

$p_i \cdot p_j$, $p_i \cdot Q$

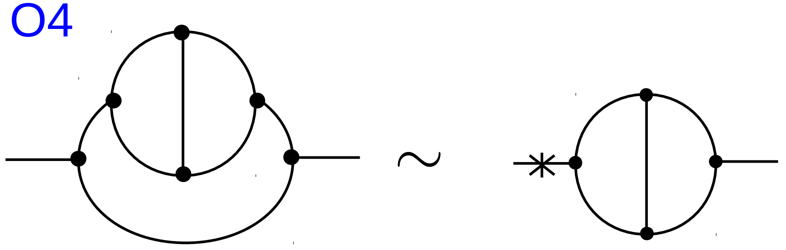
(loop momenta in the blob)



- Used for the following topology at the 3-loop level

[Chetyrkin, Tkachov '81]

O4



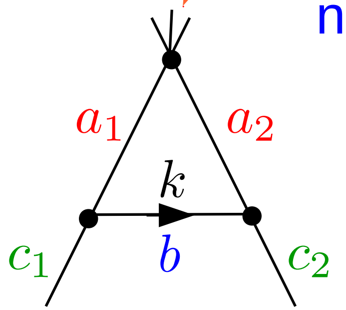
Triangle rule

[Chetyrkin, Tkachov '81]

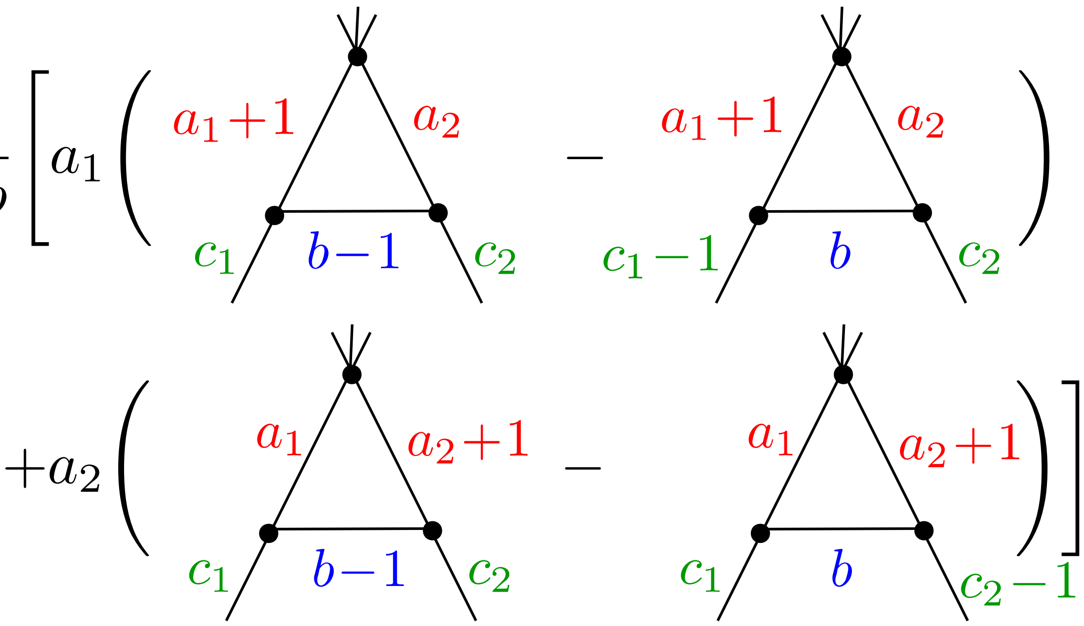
- IBP ($\frac{\partial}{\partial k} \cdot k$) in one-loop triangle-shaped (sub-)diagrams

any # of lines

numerator $k^{\mu_1} \dots k^{\mu_N}$ OK



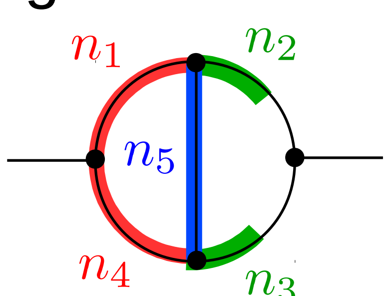
$$= \frac{1}{D + N - a_1 - a_2 - 2b}$$



Decreases b or c_1 or c_2 by 1
at the cost of increasing a_1 or a_2
in the right-hand side

- From positive integer indices, recursive use of the triangle rule makes $b = 0$ or $c_1 = 0$ or $c_2 = 0$ (removal of a line)

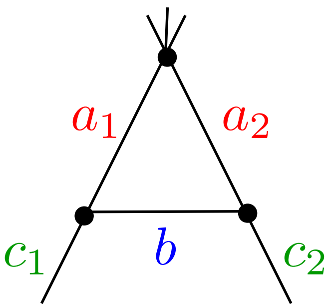
⇒ sums of integrals in simpler topologies



Diamond rule

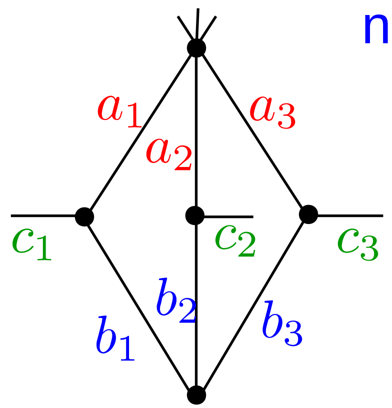
[Ruijl, TU, Vermaseren '15]

- Extension of the triangle rule to multi-loop diamond-shaped (sub-)diagrams



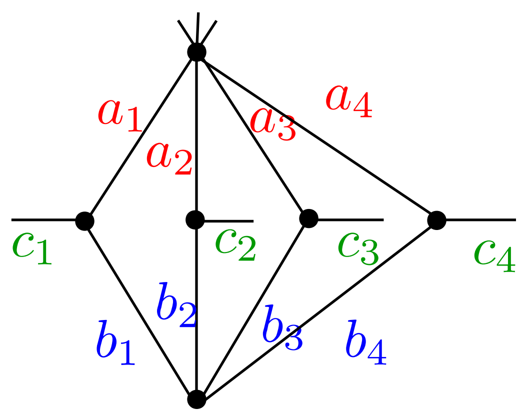
any # of lines

numerator OK



Increases a_1, a_2, a_3

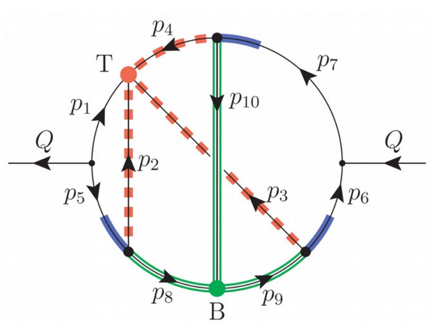
Decreases b_1, b_2, b_3
c_1, c_2, c_3



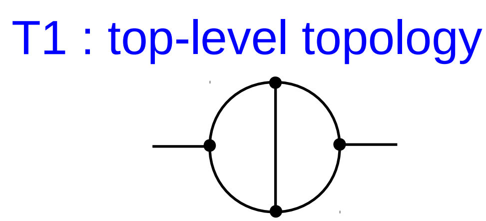
Increases a_1, a_2, a_3, a_4

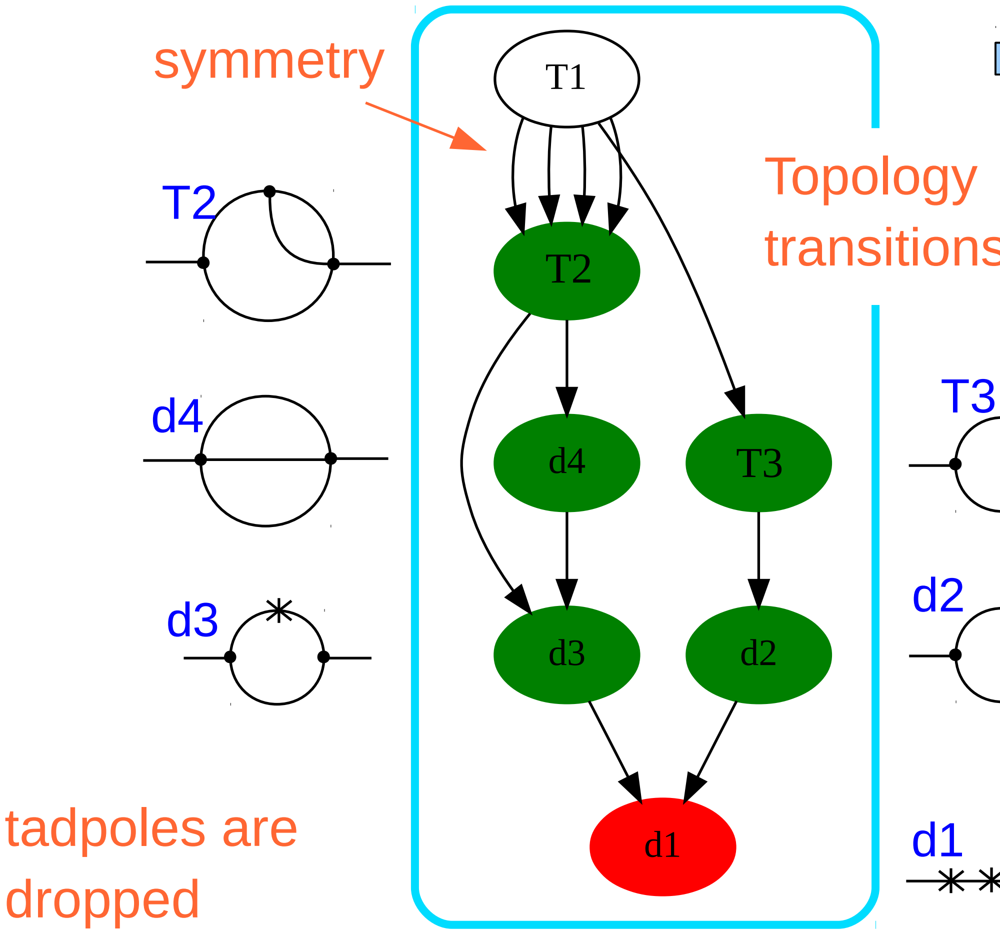
Decreases b_1, b_2, b_3, b_4
c_1, c_2, c_3, c_4

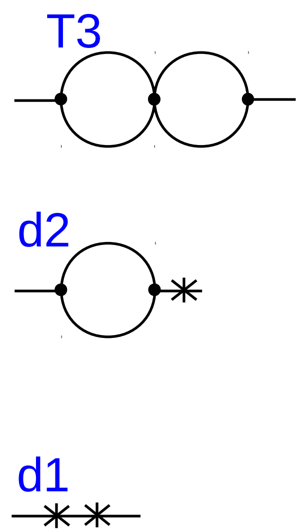
Appear from 4-loops



2-loop topologies







- All integrals can be evaluated by using the triangle rule and performing one-loop integrals

  ⇨ Expressed in terms of Γ -functions

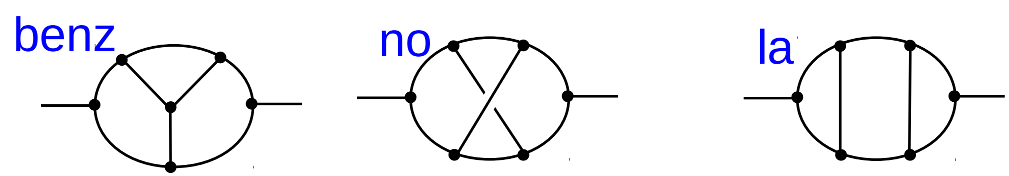
Topologies in which
- One loop-insertion
- Triangle rule

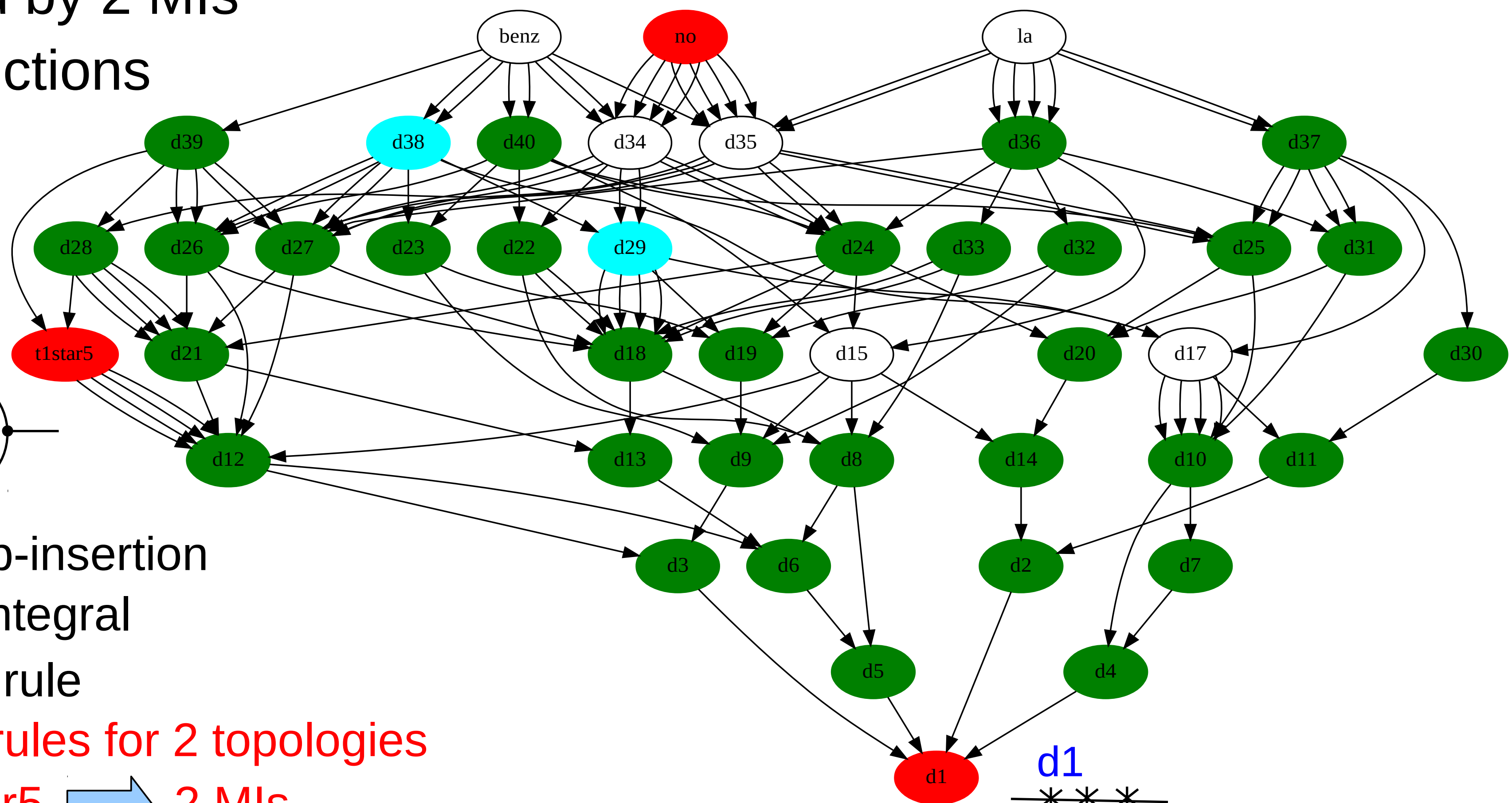
can be applied

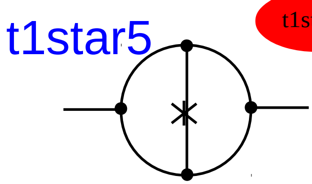
3-loop topologies

- All integrals are expressed by 2 MIs and Γ-functions

3 top-level topologies







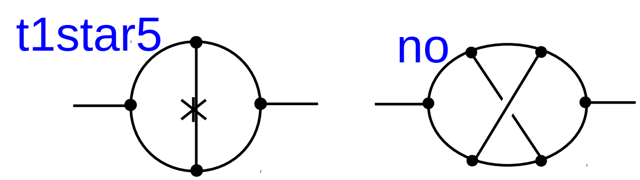
- One loop-insertion
- Carpet integral
- Triangle rule
- Special rules for 2 topologies

no, t1star5 ⇒ 2 MIs

Mincer approach

Algorithm: [Chetyrkin, Tkachov ’81]
Schoonschip implementation: [Gorishny, Larin, Surguladze, Tkachov ’89]
Form implementation: [Larin, Tkachov, Vermaseren ’91]

- Many topologies for massless propagator-type integrals can be reduced to simpler ones by
  - performing one-loop integrals ⇐ makes a big difference from general algorithms
  - use of triangle rules to remove one of lines
- Special cases (where we need to solve IBPs) are not too many
  - Only 2 topologies up to 3-loops



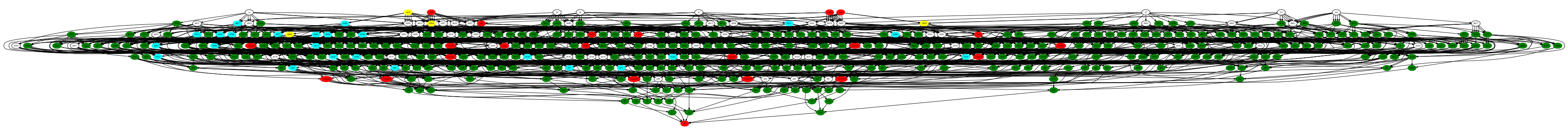
- In principle, this approach can be extended to the 4-loop level
  - Laurent series of the MIs in ϵ at the 4-loop are known

Baikov, Chetyrkin ’10 (via glue-and-cut symmetry)
Lee, Smirnov^2 ’11 (up to weight 12, via DRA, Mellin-Barnes, PSLQ)

Extension to the 4-loop level: “Forcer”

4-loop topologies

11 top-level topologies

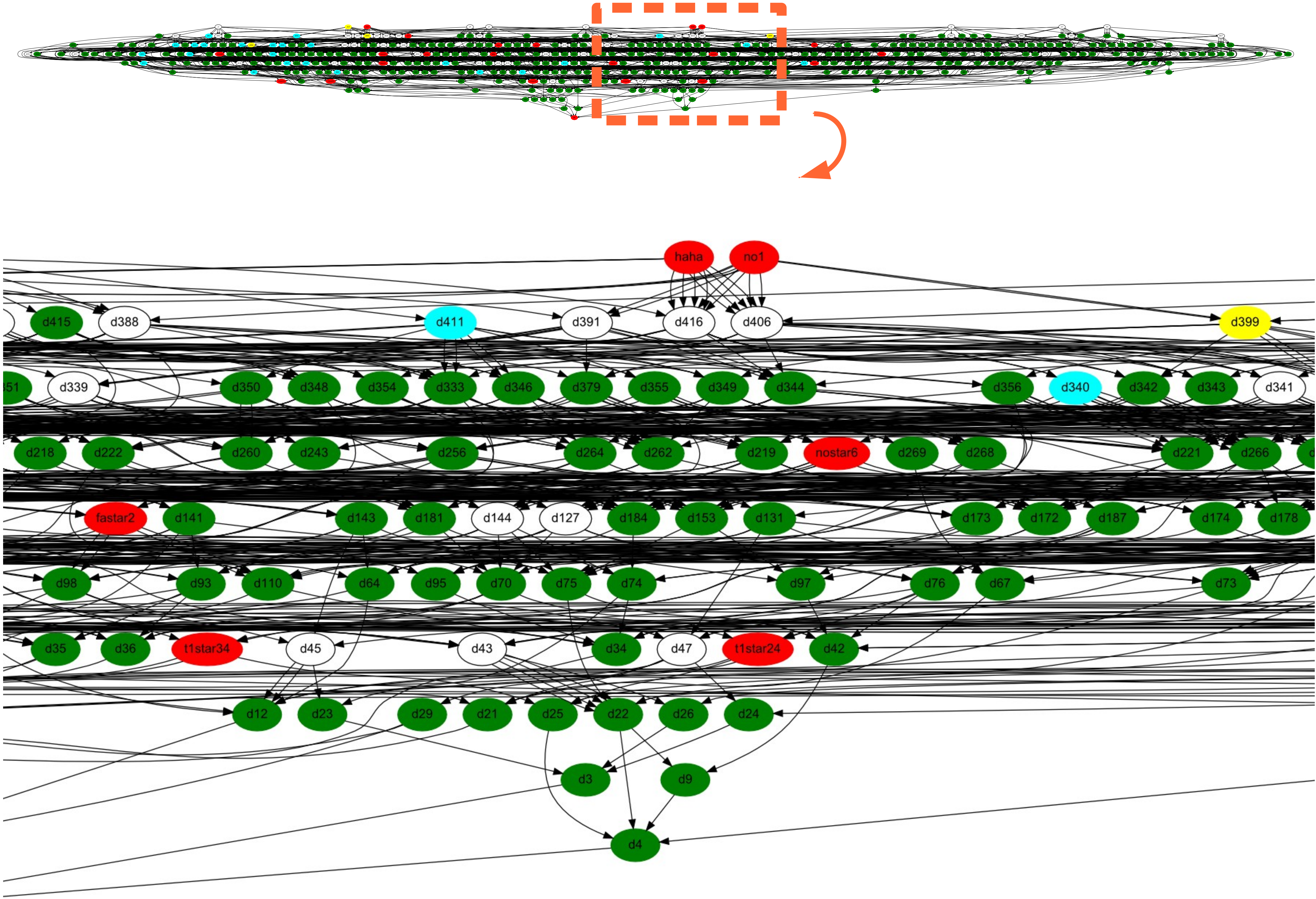


One loop-insertion	(335)	
Carpet integral	(24)	
Triangle rule	(53)	
Diamond rule	(4)	
Special rules	(21)	(pure 4-loop: 9)

Total: 437

- Problem: enormous number of cases!!
- Coding such a reduction by hand is impractical

  ⇨ Automatization


haha
no1
d415
d388
d411
d391
d416
d406
d399
d339
d350
d348
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d333
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d355
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fastar2
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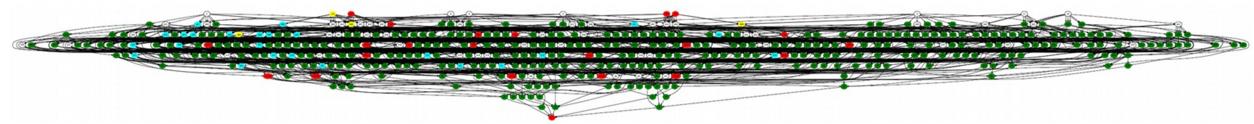
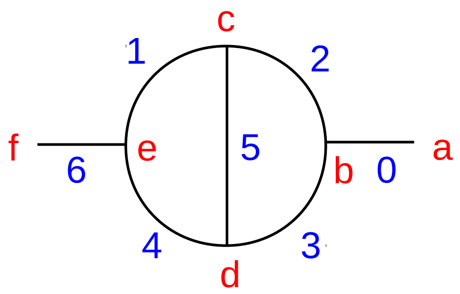
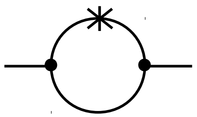
How to handle 

- Each topology as an "undirected graph" in graph theory

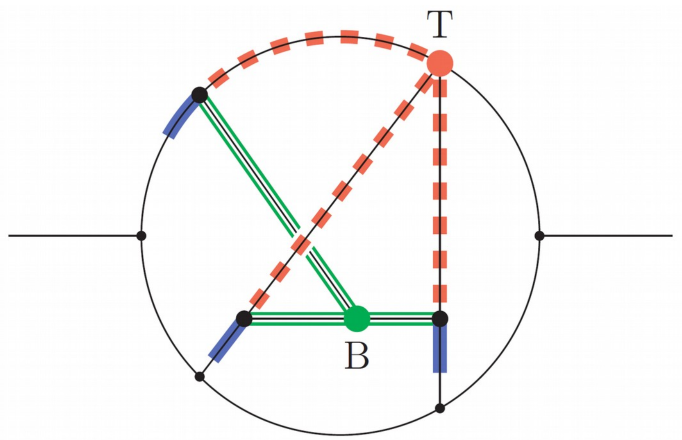


Vertices and edges are labeled
A graph is represented by connections of them
"Stars" as color of edges 

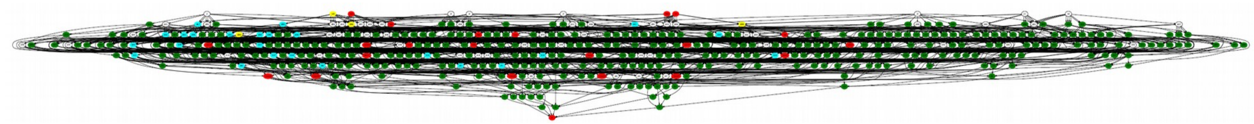
  - Implementation: Python3 with a graph library "igraph"

http://igraph.org

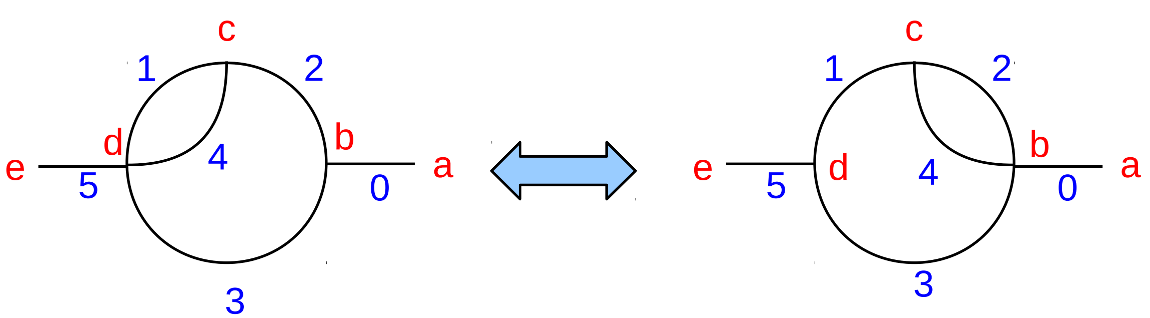
  - Easy to detect one-loop insertion, carpet, triangles, diamonds and tadpoles



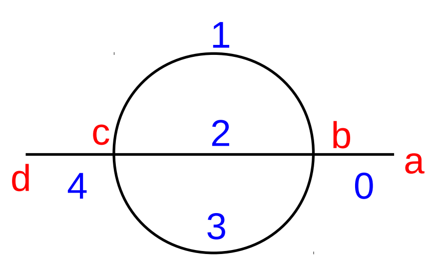
could be difficult by human eyes (diamond example)

How to handle 

- Graph isomorphism
  - Detects equivalent graphs and finds mappings among them

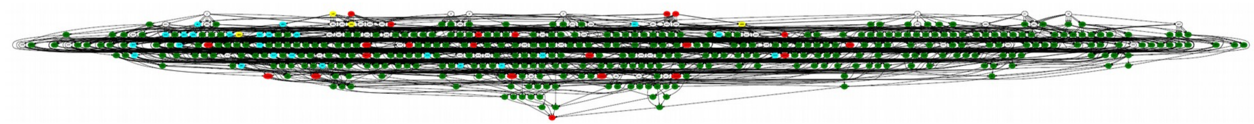


- Graph automorphism
  - Finds symmetry / mappings in each graph

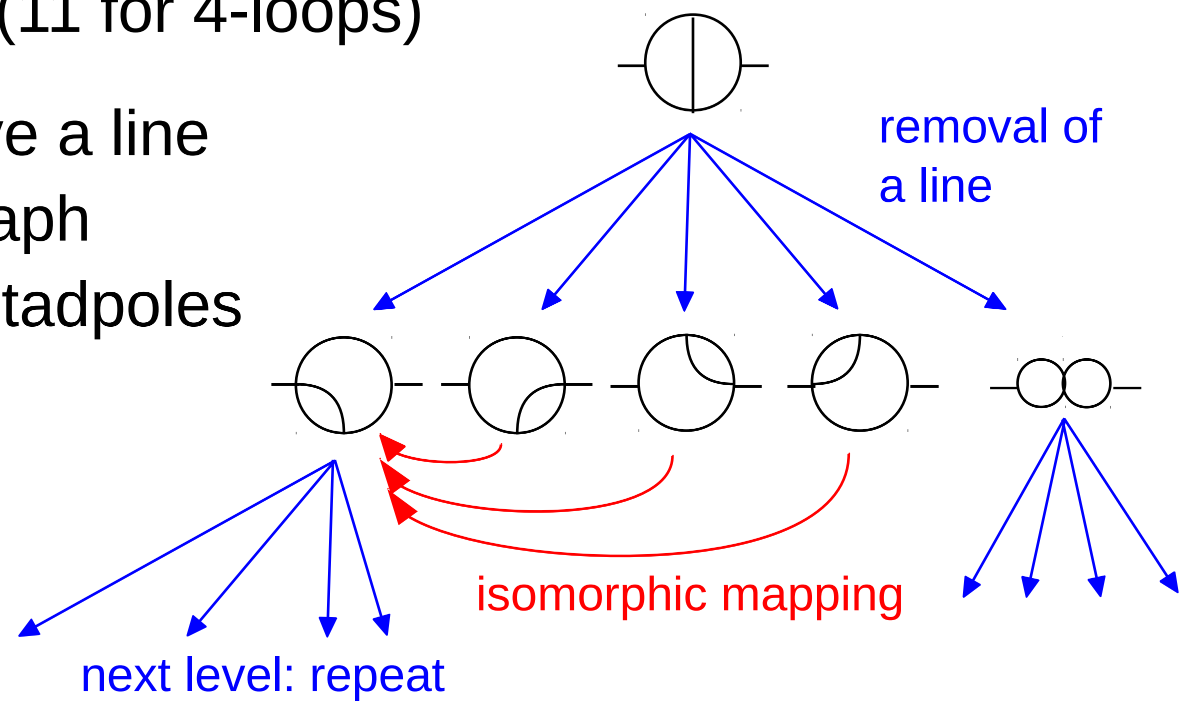


$1 \leftrightarrow 2 \leftrightarrow 3$

How to handle

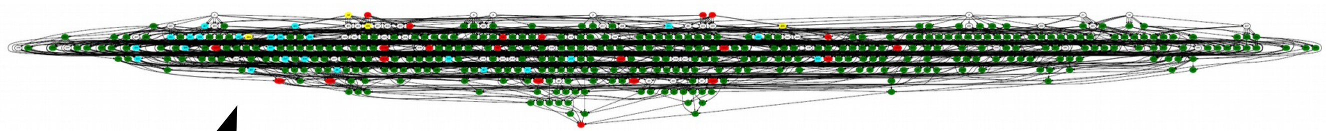


- Input : top-level topologies (11 for 4-loops)
- From each topology, remove a line in all possible ways with graph isomorphism and dropping tadpoles



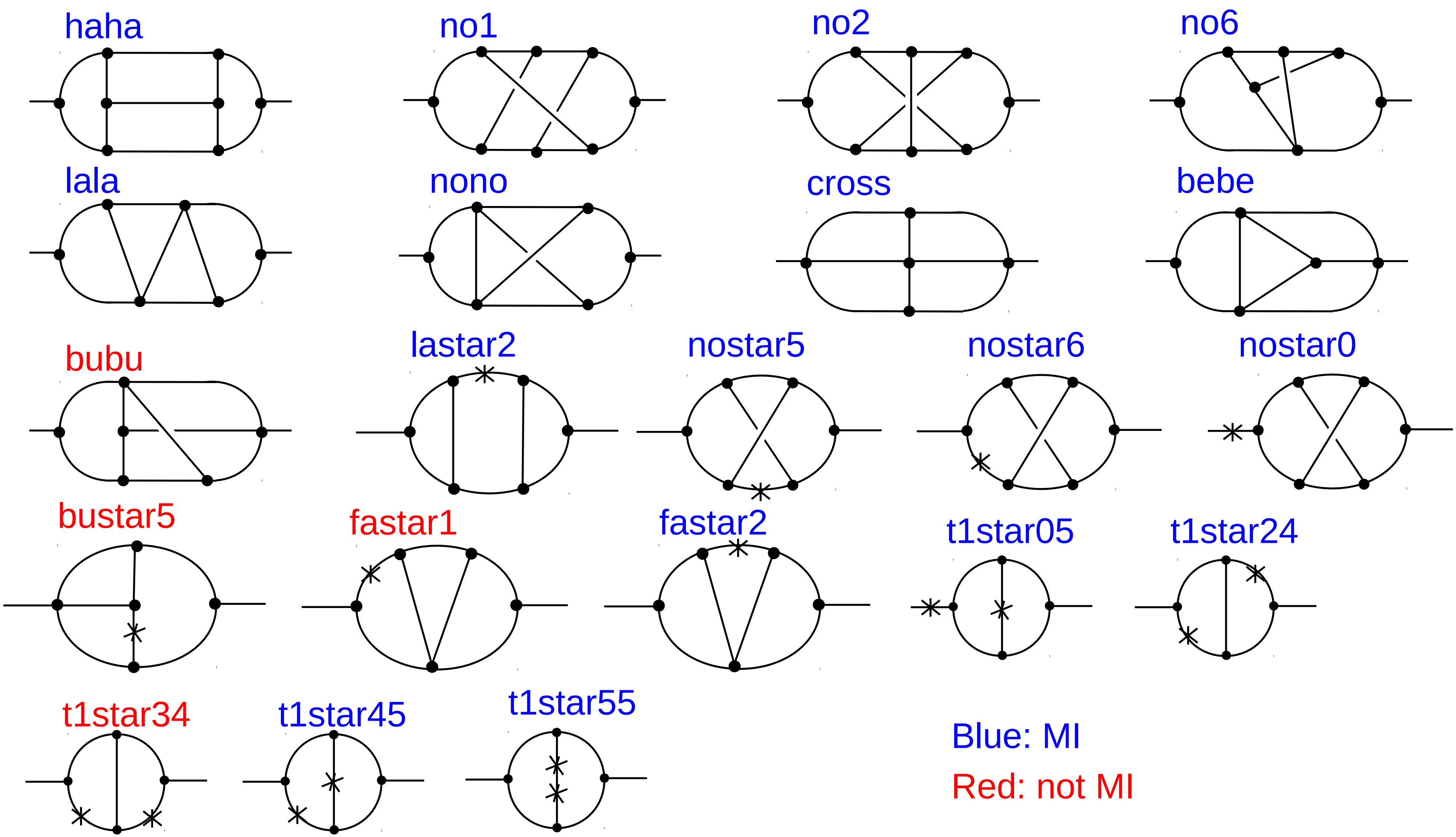
- For each topology, the next action is decided (one-loop, carpet, triangle, diamond, otherwise special rule needed)
  - Irreducible numerators (dot products) are chosen such as they do not interfere with the next action

Code generation

- In the end, we get 
- Code generation from
  - Adequate subroutines are called at each topology
  - Symmetries from graph automorphism
  - Rewriting propagators and irreducible numerators at all transitions from a topology to another
  - Python program with 2813 lines generates FORM code with 39405 lines (+ auxiliary routines) for 4-loops (as of 28.4.2016)
- Works even for 5-loops
  (64 top-topologies, 6570 topology in total, but 284 special rules need to be implemented)

Manual reduction rules

- 21 topologies require special rules, manually constructed



Blue: MI

Red: not MI

Finding manual reduction rules

- Shift an index by 1 (or -1 or more) in IBPs in all possible ways

$$n_1 \to n_1 \pm 1,\ n_2 \to n_2 \pm 1,\ n_3 \to n_3 \pm 1,\ \ldots$$

  - Combined with the original IBPs ⇨ new set of equations

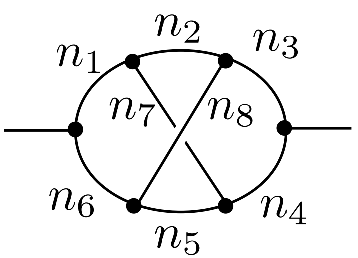
- Eliminate "complicated" integrals that increase indices for propagators (and/or decrease those for irreducible numerators) from the system of equations as possible

- In general, this is not complete, but helps a lot. Human eyes for finding "good" rules for reducing integrals

  less number of terms, decrease of complexity, short coefficients, no spurious poles, etc.

- Expressions may be very complicated. A rule can easily be 100-1000 lines or more. Use of computer algebra systems. (Don't do that by hand!)

Example: 3-loop NO



$n_9 : p_2 \cdot Q$

- 3 loop momenta and 1 external momentum
  8 propagators and 1 irreducible numerator

  ⇨ 3 × 4 = 12 IBPs

- Shift an index by -1 in the IBPs in all possible ways

positive: $n_1, \ldots, n_8$
negative: n_9

$$n_1 \to n_1 - 1,\ n_2 \to n_2 - 1,\ \ldots, n_9 \to n_9 - 1$$

⇨ 12 × 9 = 108 ⇨ 12 + 108 = 120 combined equations

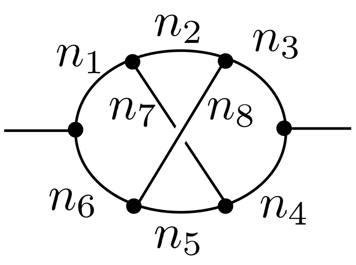
- Eliminate “complicated” integrals from the system of the equations
  A solution is (after overall $n_9 \to n_9 + 1$)

```
id Z(n1?{>=1},n2?{>=1},n3?{>=1},n4?{>=1},n5?{>=1},n6?{>=1},n7?{>=1},n8?{>=1},n9?{<=-1}) =
+Z(n1,n2-1,n3+1,n4,n5,n6,n7,n8,n9+1)*rat(1,2)*rat(n3,-7+n9+n8+n7+n6+n5+n4+n3+n2+n1+4*ep)
+Z(n1,n2,n3+1,n4,n5,n6,n7,n8-1,n9+1)*rat(-1,2)*rat(n3,-7+n9+n8+n7+n6+n5+n4+n3+n2+n1+4*ep)
+Z(n1+1,n2-1,n3,n4,n5,n6,n7,n8,n9+1)*rat(1,2)*rat(n1,-7+n9+n8+n7+n6+n5+n4+n3+n2+n1+4*ep)
+Z(n1+1,n2,n3,n4,n5,n6,n7-1,n8,n9+1)*rat(-1,2)*rat(n1,-7+n9+n8+n7+n6+n5+n4+n3+n2+n1+4*ep)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9+1)*rat(1,2)*rat(-3+n9+n3+2*n2+n1+2*ep,-7+n9+n8+n7+n6+n5+n4+n3+n2+n1+4*ep)
+Z(n1,n2-1,n3,n4,n5,n6,n7,n8,n9+2)*rat(1,2)*rat(1+n9,-7+n9+n8+n7+n6+n5+n4+n3+n2+n1+4*ep)
;
```

Repeated use until $n_9 = 0$ (removal of the irreducible numerator)

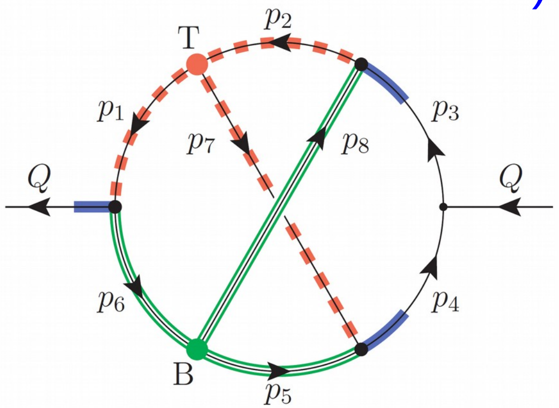
Example: 3-loop NO

- Substitute $n_9 = 0$ to the equations
- Eliminate "complicated" integrals
  (with overall $n_1 \to n_1 - 1$, the 12 IBPs are enough)




```
id Z(n1?{>=2},n2?{>=1},n3?{>=1},n4?{>=1},n5?{>=1},n6?{>=1},n7?{>=1},n8?{>=1},0) =
+Z(n1-1,n2,n3,n4-1,n5,n6,n7+1,n8,0)*rat(-n7,-1+n1)
+Z(n1-1,n2,n3,n4,n5-1,n6,n7+1,n8,0)*rat(n7,-1+n1)
+Z(n1-1,n2+1,n3-1,n4,n5,n6,n7,n8,0)*rat(-n2,-1+n1)
+Z(n1-1,n2+1,n3,n4,n5,n6,n7,n8-1,0)*rat(n2,-1+n1)
+Z(n1-1,n2,n3,n4,n5,n6,n7,n8,0)*rat(-9+2*n8+n7+2*n6+2*n5+n2+n1+4*ep,-1+n1)
+Z(n1,n2,n3,n4,n5,n6-1,n7,n8,0)
;
```

(a variant of the diamond rule)

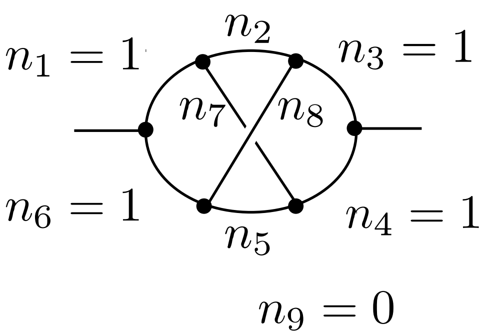


Decrease the complexity until $n_1 = 1$

The symmetry of the topology gives similar rules for n_3, n_4 and n_6

Example: 3-loop NO

- Substitute $n_1 = n_3 = n_4 = n_6 = 1$ and $n_9 = 0$ to the 120 combined equations



- Eliminate "complicated" integrals

```
id Z(1,n2?{>=2},1,1,n5?{>=1},1,n7?{>=1},n8?{>=1},0) =
+Z(1,n2-1,1,1,n5-1,1,n7,n8+1,0)*rat(-n8,-1+n2)*rat(-5+n8+2*n7+2*n5+4*ep,-3+n8+n5+n2+2*ep)
+Z(1,n2-1,1,1,n5-1,1,n7+1,n8,0)*rat(-n7,-1+n2)*rat(-7+n8+2*n7+2*n5+2*n2+4*ep,-3+n8+n5+n2+2*ep)
+Z(1,n2-1,1,1,n5,1,n7,n8,0)*rat(-1,-3+n8+n5+n2+2*ep)*rat(30-12*n8+n8^2-15*n7+3*n7*n8+2*n7^2-22*n5
    +4*n5*n8+6*n5*n7+4*n5^2-11*n2+3*n2*n8+2*n2*n7+4*n2*n5+n2^2-32*ep+6*ep*n8+8*ep*n7+12*ep*n5+6*ep*n2
    +8*ep^2,-1+n2)
+Z(1,n2,1,1,n5-1,1,n7,n8,0)*rat(2-n8-n5-2*ep,-3+n8+n5+n2+2*ep)
+Z(1,n2,1,1,n5,1,n7-1,n8,0)
+Z(1,n2,1,1,n5,1,n7,n8-1,0)*rat(7-n8-2*n7-2*n5-2*n2-4*ep,-3+n8+n5+n2+2*ep)
+Z(1,n2-1,1,0,n5,1,n7+1,n8,0)*rat(n7,-1+n2)*rat(-7+n8+2*n7+2*n5+2*n2+4*ep,-3+n8+n5+n2+2*ep)
+Z(1,n2-1,1,1,n5,0,n7,n8+1,0)*rat(n8,-1+n2)*rat(-7+n8+2*n7+2*n5+2*n2+4*ep,-3+n8+n5+n2+2*ep)
+Z(1,n2,0,1,n5,0,n7,n8+1,0)*rat(-2*n8,-3+n8+n5+n2+2*ep)
+Z(1,n2,0,1,n5,1,n7,n8,0)*rat(-9+2*n8+2*n7+3*n5+2*n2+6*ep,-3+n8+n5+n2+2*ep)
+Z(1,n2,1,1,n5,0,n7,n8,0)*rat(-9+2*n8+2*n7+2*n5+3*n2+6*ep,-3+n8+n5+n2+2*ep)
;
```

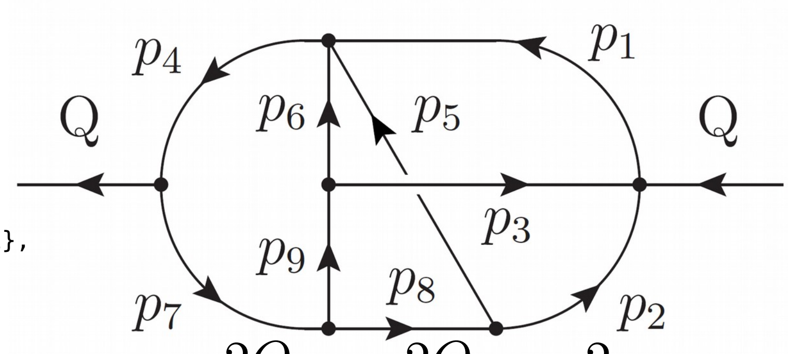
Decrease the complexity until $n_2 = 1$

Similar rules for n_5, n_7 and n_8

➡ The MI: NO(1,1,1,1,1,1,1,1,0)

Example: 4-loop BUBU

- A part of rules (reduction of n11 to 0)



$$2Q\cdot p_2,\, 2Q\cdot p_8,\, 2p_2\cdot p_3$$
$$2p_2\cdot p_9,\, 2p_3\cdot p_8$$

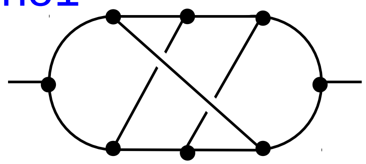
One of difficult topologies to solve, though no MI

```
id Z(n1?{>=1},n2?{>=1},n3?{>=1},n4?{>=2},n5?{>=1},n6?{>=1},n7?{>=1},n8?{>=1},n9?{>=1},
     n10?{<=0},n11?{<=-1},n12?{<=0},n13?{<=0},n14?{<=0}) =
+Z(n1,n2-1,n3,n4-1,n5+1,n6,n7,n8,n9,n10,n11+1,n12,n13,n14)*rat(n5,-1+n4)
+Z(n1,n2,n3,n4-1,n5,n6,n7+1,n8-1,n9,n10,n11+1,n12,n13,n14)*rat(-n7,-1+n4)
+Z(n1,n2,n3,n4-1,n5,n6,n7+1,n8,n9-1,n10,n11+1,n12,n13,n14)*rat(n7,-1+n4)
+Z(n1,n2,n3,n4-1,n5+1,n6,n7,n8-1,n9,n10,n11+1,n12,n13,n14)*rat(-n5,-1+n4)
+Z(n1,n2,n3,n4-1,n5,n6,n7,n8,n9,n10,n11+1,n12,n13,n14)*rat(3-2*n8-n7-n5-n14-n11-2*ep,-1+n4)
+Z(n1,n2,n3,n4,n5,n6,n7-1,n8,n9,n10,n11+1,n12,n13,n14)*rat(-1,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8-1,n9,n10,n11+1,n12,n13,n14)*rat(-1,1)
+Z(n1,n2,n3,n4,n5,n6,n7,n8,n9-1,n10,n11+1,n12,n13,n14)
;
id Z(n1?{>=1},n2?{>=1},n3?{>=1},1,n5?{>=1},n6?{>=1},n7?{>=1},n8?{>=1},n9?{>=1},
     n10?{<=0},n11{<=-1},n12?{<=0},n13?{<=0},n14?{<=0}) =
+Z(n1,n2,n3,1,n5,n6,n7,n8,n9,n10,n11+1,n12+1,n13-1,n14)*rat(n12,-8+n9+n8+n7+n6+n5+n3+n2+n14+n13+n12+n11+n10+n1+5*ep)
+Z(n1,n2,n3,1,n5,n6,n7,n8,n9,n10+1,n11+1,n12,n13-1,n14)*rat(-n10,-8+n9+n8+n7+n6+n5+n3+n2+n14+n13+n12+n11+n10+n1+5*ep)
+Z(n1,n2,n3,1,n5,n6,n7+1,n8-1,n9,n10,n11+1,n12,n13,n14)*rat(-n7,-8+n9+n8+n7+n6+n5+n3+n2+n14+n13+n12+n11+n10+n1+5*ep)
+Z(n1,n2,n3,1,n5,n6,n7+1,n8,n9-1,n10,n11+1,n12,n13,n14)*rat(n7,-8+n9+n8+n7+n6+n5+n3+n2+n14+n13+n12+n11+n10+n1+5*ep)
+Z(n1,n2,n3+1,1,n5,n6-1,n7,n8,n9,n10,n11+1,n12,n13,n14)*rat(-n3,-8+n9+n8+n7+n6+n5+n3+n2+n14+n13+n12+n11+n10+n1+5*ep)
+Z(n1,n2,n3+1,1,n5,n6,n7,n8,n9-1,n10,n11+1,n12,n13,n14)*rat(n3,-8+n9+n8+n7+n6+n5+n3+n2+n14+n13+n12+n11+n10+n1+5*ep)
+Z(n1,n2,n3,1,n5,n6,n7,n8,n9,n10,n11+1,n12,n13,n14)*rat(4+n9-n8-n6-n5-n2-n14-n12-n11-n10-n1-3*ep,
    -8+n9+n8+n7+n6+n5+n3+n2+n14+n13+n12+n11+n10+n1+5*ep)
+Z(n1,n2,n3,1,n5,n6,n7-1,n8,n9,n10,n11+1,n12,n13,n14)*rat(-1,1)
+Z(n1,n2,n3,1,n5,n6,n7-1,n8,n9,n10,n11+1,n12,n13,n14+1)*rat(n14,-8+n9+n8+n7+n6+n5+n3+n2+n14+n13+n12+n11+n10+n1+5*ep)
+Z(n1,n2,n3,1,n5,n6,n7-1,n8,n9,n10,n11+2,n12,n13,n14)*rat(-1-n11,-8+n9+n8+n7+n6+n5+n3+n2+n14+n13+n12+n11+n10+n1+5*ep)
+Z(n1,n2,n3,1,n5,n6,n7,n8-1,n9,n10,n11+1,n12,n13,n14+1)*rat(-n14,-8+n9+n8+n7+n6+n5+n3+n2+n14+n13+n12+n11+n10+n1+5*ep)
+Z(n1,n2,n3,1,n5,n6,n7,n8-1,n9,n10,n11+2,n12,n13,n14)*rat(1+n11,-8+n9+n8+n7+n6+n5+n3+n2+n14+n13+n12+n11+n10+n1+5*ep)
+Z(n1,n2,n3,1,n5,n6,n7,n8,n9-1,n10,n11+1,n12,n13,n14+1)*rat(-n14,-8+n9+n8+n7+n6+n5+n3+n2+n14+n13+n12+n11+n10+n1+5*ep)
+Z(n1,n2,n3,1,n5,n6,n7,n8,n9-1,n10,n11+2,n12,n13,n14)*rat(1+n11,-8+n9+n8+n7+n6+n5+n3+n2+n14+n13+n12+n11+n10+n1+5*ep)
+Z(n1,n2,n3,0,n5,n6,n7,n8,n9,n10,n11+1,n12,n13,n14)
;
```

3274 lines for whole BUBU reduction

Run Forcer: result looks like...

no1

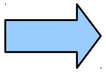


one of the top-level topology

11 propagators

3 irreducible numerators

How to convert Qgraf output to Forcer input?



Jos's talk

```
[no1(2,2,2,2,2,2,2,2,2,2,2,-1,-1,-1)] =

-10/9*num(1+2*ep)^2*num(2+5*ep)*num(3+2*ep)^2*num(3+5*ep)*num(4+5*ep)*num(6+5*ep)*num(7+5*ep)*num(8+5*ep
)*num(9+5*ep)*num(36141384+167650024*ep+369157793*ep^2+504389598*ep^3+470560515*ep^4+312347786*ep^5+1497
70838*ep^6+51734214*ep^7+12600912*ep^8+2060632*ep^9+203648*ep^10+9216*ep^11)*den(1+ep)^2*den(2+ep)^2*den
(2+3*ep)*den(3+ep)^2*den(4+3*ep)*den(5+3*ep)*den(7+3*ep)*den(8+3*ep)*Master(no1)

+2/9*num(1+2*ep)*num(1+4*ep)^2*num(3+2*ep)*num(39337381531008+422506983834144*ep+2126828256064272*ep^2+6
682923999248124*ep^3+14715479582570820*ep^4+24142592323497543*ep^5+30606097414180459*ep^6+30663886432898
386*ep^7+24613240685251374*ep^8+15944111435166437*ep^9+8353498138352567*ep^10+3530865125153640*ep^11+119
5052526807120*ep^12+319518770241334*ep^13+66015276933500*ep^14+10173244132808*ep^15+1101566031376*ep^16+
74820000000*ep^17+2400000000*ep^18)*den(1+ep)^2*den(2+ep)^2*den(2+3*ep)*den(3+ep)^2*den(4+3*ep)*den(5+3*
ep)*den(7+3*ep)*den(8+3*ep)*Master(no6)

-4/9*num(ep)^2*num(1+2*ep)*num(3+2*ep)*num(4970268890523930816+65981799142896214872*ep+41636699974387213
0470*ep^2+1663841711414114631648*ep^3+4730696245772216998455*ep^4+10190116418687033776180*ep^5+172831512
25628577356718*ep^6+23674869565869007631757*ep^7+26649665048245779930527*ep^8+24945440138420091798098*ep
^9+19571616320498135318722*ep^10+12933081950816489701881*ep^11+7214358008133792648788*ep^12+339681000556
8803286172*ep^13+1346680080420400500352*ep^14+447318716827968227680*ep^15+123500240869574621248*ep^16+28
008584917867939712*ep^17+5129371482425778688*ep^18+739826312414941184*ep^19+80910145880457216*ep^20+6306
386119458816*ep^21+312125440000000*ep^22+7372800000000*ep^23)*den(1+ep)^2*den(2+ep)^2*den(3+ep)^2*den(3+
4*ep)*den(4+3*ep)*den(5+3*ep)*den(5+4*ep)*den(7+3*ep)*den(7+4*ep)*den(8+3*ep)*den(9+4*ep)*den(11+4*ep)*d
en(13+4*ep)*Master(lala)
```

(cont'd on next page)

```
+ ( other 17 terms )
```

(cont'd)

```
+1/69984*num(-1+2*ep)^3*num(72419162019449762165096448000000+3199339702731261014302808309760000*ep+6756458
7046942832244281149906944000*ep^2+91350459184391655670944774398730240*ep^3+891289886775817303600596179144
718336*ep^4+66930375969349547572861052984235786624*ep^5+40235415927130336283161325668026920448*ep^6+19871
3617144519275979305008924953882368*ep^7+82091697044731460372708583674670235468 8*ep^8+2873881635171369729
2804388788688381310 08*ep^9+86079527794109515224134940943137287932 80*ep^10+2221477733684447301324246670 01
94857424432*ep^11+4963564313550399839873313962385333772039 2*ep^12+9627197032010951955470647409501905888 4
972*ep^13+1620920042433367369484228940869152451887 10*ep^14+2360110255144947383954778523265243802462 08*ep
^15+29411662512212795213699668738158804546689 7*ep^16+30622594561109334098894115787780494656562 2*ep^17+25
0395537138862785223188155623245534964532*ep^18+1278490968230930766213197239347627354187 74*ep^19-32311698
6586416421659980377524004225109 97*ep^20-1839323952847706865576065109996382637244 34*ep^21-284795600836347
4571865784913534305788703 26*ep^22-3150229191924838122866754344782081602769 10*ep^23-2823055800376111021 51
35831194998484916316 4*ep^24-2129361990228716919075740753830553714327 06*ep^25-1364459935186578669634780 40
015820540166838*ep^26-734900782788793457862107578060975188670 18*ep^27-318984658302192402132855983340890 3
5466304*ep^28-965138248495143989347275710922069398441 4*ep^29-42123656180630018160833079911348461388 4*ep^
30+2017002468644720352265271521884033262822*ep^31+183287476156286668669830509056864470289 9*ep^32+1096707
581456919232171368580075436795092*ep^33+524523702662773366611140017451441153706*ep^34+212149752741671710
6493850325590950347 96*ep^35+742509010290102120707359886108873337 49*ep^36+2272051252243184038118975458520
5484412*ep^37+61032923996908799039999592331232436 84*ep^38+14397363931112773511749107695815768 00*ep^39+29
75285303229757980968392200467472 48*ep^40+536061820919166327089727654950660 48*ep^41+835965908902794893568
4267151014592*ep^42+1117001205642226519475032390006784*ep^43+126129308732809465322496308824064*ep^44+118
1003341832521121415433402777 6*ep^45+8928238489352988841796013096 96*ep^46+52384336126816611070473928704*e
p^47+2238933537865339140484104192*ep^48+620290211369818521600000000*ep^49+83626417685790720000000 0*ep^50)
*den(ep)^6*den(-1+ep)*den(1+ep)^6*den(1+2*ep)^2*den(1+3*ep)^2*den(-2+ep)*den(2+ep)^6*den(2+3*ep)^2*den(3
+ep)^3*den(3+2*ep)^2*den(3+4*ep)*den(4+3*ep)^2*den(5+2*ep)^2*den(5+3*ep)^2*den(5+4*ep)*den(7+3*ep)^2*den
(7+4*ep)*den(8+3*ep)*den(9+4*ep)*den(11+4*ep)*den(13+4*ep)*G10*G20*G30
```

4.2 hours on a desktop PC with “tform -w4”

Exact in ϵ Optionally: ϵ -expansions in intermediate steps

Checks

- Recomputing known results: strong non-trivial checks
  - Reproduced the 4-loop QCD β-function [Ritbergen, Vermaseren, Larin '97; Czakon '04]
  - Checked the gauge invariance, all powers of ξ

    Forcer gives 4-loop propagators and vertices*, exact in ϵ, all powers of ξ

    * with nullifying external momenta except two of them
  - Also with using background field method

with no gauge parameter	10 minutes
with one power	38 minutes
with FULL gauge parameters	8.5 hours

on a decent 24 core machine (6 "tform -w4" jobs)

  - Reproduced low-N NS splitting functions ⇨ Andreas's talk

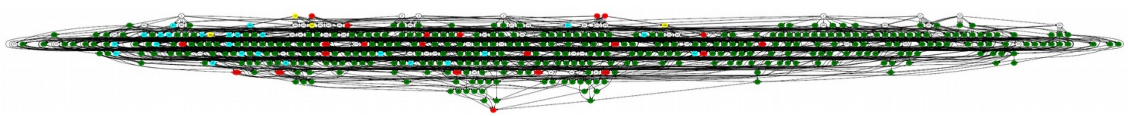
    GLS, N=2,3,4 [Baikov, Chetyrkin, Kühn; Velizhanin]

Summary

Summary

- Forcer: a “4-loop extension” of Mincer
  for massless propagator-type Feynman integrals

  - Highly complicated structure of the program / equations
    ⇒ Automatization: write a program for generating a code

    for 

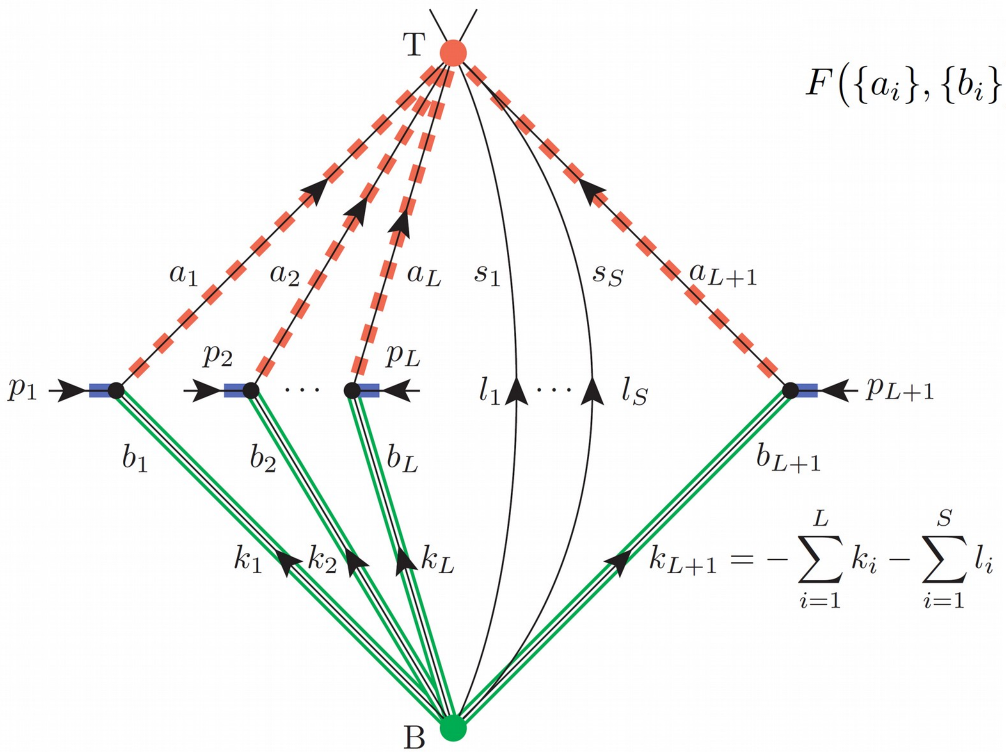
    Manual rules are derived with the aid of computers

- Correctness of the program was checked for known results

Backup

Diamond rule

[Ruijl, TU, Vermaseren ’15]



$$F(\{a_i\},\{b_i\}) = \left[\prod_{i=1}^{L}\int d^D k_i\right]\left[\prod_{i=1}^{S}\int d^D l_i\right] \times \left[\prod_{i=1}^{L+1}\frac{k_i^{\mu_1^{(i)}}\dots k_i^{\mu_{N_i}^{(i)}}}{\left[(k_i+p_i)^2+m_i^2\right]^{a_i}(k_i^2)^{b_i}}\right]\left[\prod_{i=1}^{S}\frac{l_i^{\nu_1^{(i)}}\dots l_i^{\nu_{R_i}^{(i)}}}{(l_i^2)^{s_i}}\right]$$

$$(L+S)D+\sum_{i=1}^{L+1}(N_i-a_i-2b_i)+\sum_{i=1}^{S}(R_i-2s_i) = \sum_{i=1}^{L+1} a_i \boldsymbol{A}_i^+\left[\boldsymbol{B}_i^- -(p_i^2+m_i^2)\right]$$

Diamond rule

- Explicit summation formula

$$F(\{a_i\},\{b_i\},\{c_i\}) =$$

$$\sum_{r=1}^{L+1}\Bigg[\Bigg(\prod_{\substack{i=1\\i\neq r}}^{L+1}\sum_{k_i^+=0}^{b_i-1}\Bigg)\Bigg(\prod_{i=1}^{L+1}\sum_{k_i^-=0}^{c_i-1}\Bigg)(-1)^{k^-}\frac{k_r^+(k^+ + k^- - 1)!}{\prod_{i=1}^{L+1}k_i^+!k_i^-!}(E+k^+)_{-k^+-k^-}$$

$$\times\Bigg(\prod_{i=1}^{L+1}(a_i)_{k_i^+ + k_i^-}\Bigg)F\left(\{a_i+k_i^+ + k_i^-\},\{b_i-k_i^+\},\{c_i-k_i^-\}\right)\Bigg]_{k_r^+=b_r}$$

$$+\sum_{r=1}^{L+1}\Bigg[\Bigg(\prod_{i=1}^{L+1}\sum_{k_i^+=0}^{b_i-1}\Bigg)\Bigg(\prod_{\substack{i=1\\i\neq r}}^{L+1}\sum_{k_i^-=0}^{c_i-1}\Bigg)(-1)^{k^-}\frac{k_r^-(k^+ + k^- - 1)!}{\prod_{i=1}^{L+1}k_i^+!k_i^-!}(E+k^+ + 1)_{-k^+-k^-}$$

$$\times\Bigg(\prod_{i=1}^{L+1}(a_i)_{k_i^+ + k_i^-}\Bigg)F\left(\{a_i+k_i^+ + k_i^-\},\{b_i-k_i^+\},\{c_i-k_i^-\}\right)\Bigg]_{k_r^-=c_r}$$

$$E=(L+S)D+\sum_{i=1}^{L+1}(N_i-a_i-2b_i)+\sum_{i=1}^{S}(R_i-2s_i)$$

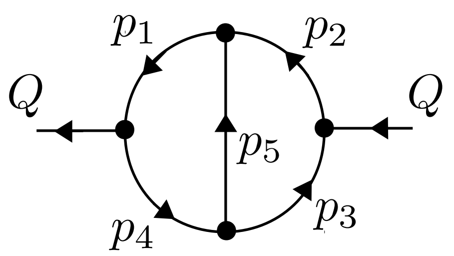
$$k^+=\sum_{i=1}^{L+1}k_i^+ \qquad k^-=\sum_{i=1}^{L+1}k_i^-$$

Avoids spurious poles

Triangle/diamond rule may be not the best

- The triangle rule does not change the total complexity of integrals

```
id Z(n1?{>=1},n2?{>=1},n3?{>=1},n4?{>=1},n5?{>=1}) =
+Z(n1,n2,n3-1,n4+1,n5)*rat(n4,-4+2*n5+n4+n1+2*ep)
+Z(n1,n2,n3,n4+1,n5-1)*rat(-n4,-4+2*n5+n4+n1+2*ep)
+Z(n1+1,n2-1,n3,n4,n5)*rat(n1,-4+2*n5+n4+n1+2*ep)
+Z(n1+1,n2,n3,n4,n5-1)*rat(-n1,-4+2*n5+n4+n1+2*ep)
;
```



- The following rule (a solution of 6+30 equations) decreases the complexity and gives only 3 terms in the RHS

```
id Z(n1?{>=1},n2?{>=1},n3?{>=1},n4?{>=1},n5?{>=1}) =
+Z(n1,n2-1,n3,n4,n5)*rat(-7+2*n5+n4+n3+n2+n1+3*ep,-3+n5+n2+n1+ep)
+Z(n1,n2,n3-1,n4,n5)*rat(-7+2*n5+n4+n3+n2+n1+3*ep,-3+n5+n4+n3+ep)
+Z(n1,n2,n3,n4,n5-1)*rat(3-n5-n4-n1-ep,-3+n5+n2+n1+ep)
    *rat(-4+n5+n4+n1+2*ep,-3+n5+n4+n3+ep)
;
```

- Possibility of more optimizations??