Leipzig, April 25, 2016

Loops and Legs in Quantum Field Theory (LL2016)

Algorithms to solve coupled differential systems in terms of power series

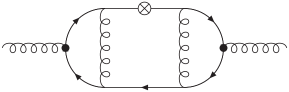
Carsten Schneider

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz

joint with A. Behring, J. Blümlein, A. De Freitas (DESY)
and J. Ablinger (RISC)

Completed project: 3-loop massive ladder and V-diagrams

Ablinger, Behring, Blümlein, De Freitas, von Manteuffel, CS; Comput. Phys. Comm. 202 [arXiv:1509.08324 [hep-ph]]



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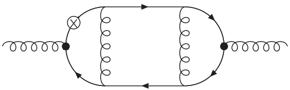

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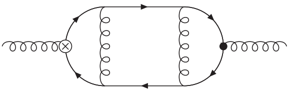

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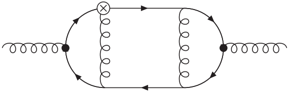

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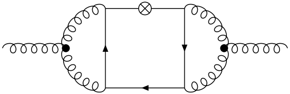

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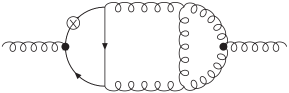

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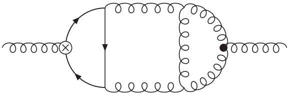

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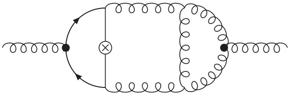

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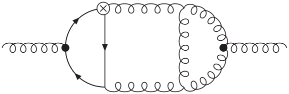

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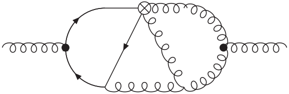

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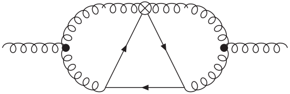

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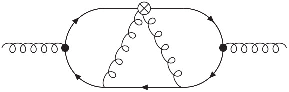

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All diagrams are produced with axodraw (J. Vermaseren)

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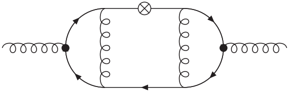
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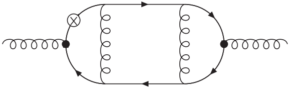
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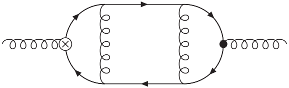
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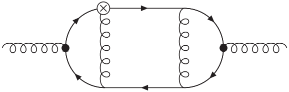
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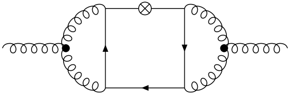
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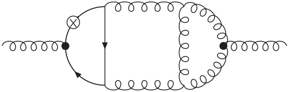
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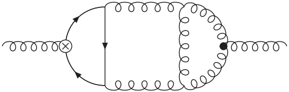
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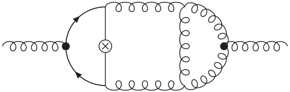
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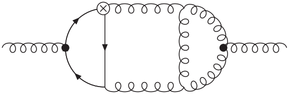
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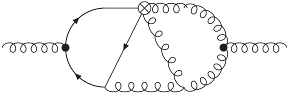
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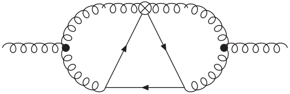
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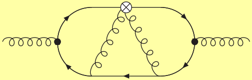
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All diagrams are produced with axodraw (J. Vermaseren)

Used technologies (and mathematical softare)

1. simplification of multi-sums (Sigma.m, EvaluateMultiSums.m)
2. computing recurrences for multi-integrals (Ablinger's MutliIntegrate.m package)
3. special function tools (Ablinger's HarmonicSums.m package)

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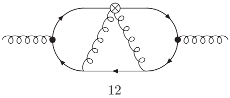
`Example:`



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expression in terms of

harmonic sums (Vermaseren, 1998; Blümlein, Kurth, 1998)

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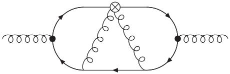

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expression in terms of

harmonic sums (Vermaseren, 1998; Blümlein, Kurth, 1998)

generalized harmonic sums (Moch, Uwer, Weinzierl, 2002; Ablinger, Blümlein, CS 2013)

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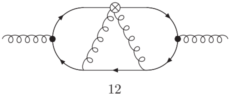

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expression in terms of

harmonic sums (Vermaseren, 1998; Blümlein, Kurth, 1998)
generalized harmonic sums (Moch, Uwer, Weinzierl, 2002; Ablinger, Blümlein, CS 2013)
nested binomial sums (Ablinger, Blümlein, Raab, CS 2014)

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Goal of this talk: present the underlying algorithm and elaborate new ideas

Problem 1: Solving recurrences

Solving recurrences

In[1]:=
$$\mathbf{rec} = -2(N+1)(N+2)^2 I[N] - (N+2)\big(-6N^2-28N-32\big)I[N+1] \\ + \big(-6N^3-50N^2-136N-120\big)I[N+2] \\ - (-N-2)(N+4)(2N+8)I[N+3] == -\frac{4(N+2)}{3(N+3)};$$

Solving recurrences

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In[2]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[3]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, I[N]]$

Solving recurrences

In[1]:= $$\mathbf{rec} = -2(N+1)(N+2)^2 I[N] - (N+2)\big(-6N^2-28N-32\big)I[N+1] + \big(-6N^3-50N^2-136N-120\big)I[N+2] - (-N-2)(N+4)(2N+8)I[N+3] == -\frac{4(N+2)}{3(N+3)};$$

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```
Sigma - A summation package by Carsten Schneider © RISC-Linz
```

In[3]:= $\mathbf{recSol} = \mathbf{SolveRecurrence[rec, I[N]]}$

Out[3]= $$\Big\{\Big\{0, \frac{-1}{N+1}\Big\}, \Big\{0, \frac{-N}{N+1}\Big\}, \Big\{0, \frac{1}{(N+1)^2} + \frac{1}{N+1}\sum_{i=1}^{N}\frac{1}{i}\Big\}, \Big\{1, \frac{2\big(N^2+N-1\big)}{3(N+1)^2} - \frac{2(N+2)}{3(N+1)}\sum_{i=1}^{N}\frac{1}{i}\Big\}\Big\}$$

Solving recurrences

In[1]:= $$\mathbf{rec} = -2(\mathrm{N}+1)(\mathrm{N}+2)^2\mathrm{I}[\mathrm{N}] - (\mathrm{N}+2)\big(-6\mathrm{N}^2-28\mathrm{N}-32\big)\mathrm{I}[\mathrm{N}+1] + \big(-6\mathrm{N}^3-50\mathrm{N}^2-136\mathrm{N}-120\big)\mathrm{I}[\mathrm{N}+2] - (-\mathrm{N}-2)(\mathrm{N}+4)(2\mathrm{N}+8)\mathrm{I}[\mathrm{N}+3] == -\frac{4(\mathrm{N}+2)}{3(\mathrm{N}+3)};$$

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Sigma - A summation package by Carsten Schneider © RISC-Linz

In[3]:= **recSol = SolveRecurrence[rec, I[N]]**

Out[3]= $$\Big\{\Big\{0, \frac{-1}{\mathrm{N}+1}\Big\}, \Big\{0, \frac{-\mathrm{N}}{\mathrm{N}+1}\Big\}, \Big\{0, \frac{1}{(\mathrm{N}+1)^2} + \frac{1}{\mathrm{N}+1}\sum_{i=1}^{\mathrm{N}}\frac{1}{i}\Big\}, \Big\{1, \frac{2\big(\mathrm{N}^2+\mathrm{N}-1\big)}{3(\mathrm{N}+1)^2} - \frac{2(\mathrm{N}+2)}{3(\mathrm{N}+1)}\sum_{i=1}^{\mathrm{N}}\frac{1}{i}\Big\}\Big\}$$

General solution: $$\{c_1\,\tfrac{1}{-N-1} + c_2\,\tfrac{-N}{N+1} + c_3\left(\tfrac{1}{(N+1)^2} + \tfrac{1}{N+1}\right)\sum_{i=1}^{N}\frac{1}{i} + \tfrac{2(N^2+N-1)}{3(N+1)^2} - \tfrac{2(N+2)}{3(N+1)}\sum_{i=1}^{N}\frac{1}{i}\;|\;c_1, c_2, c_3 \in \mathbb{R}\}$$

Solving recurrences

In[1]:= $$\mathbf{rec} = -2(N+1)(N+2)^2 I[N] - (N+2)\big(-6N^2-28N-32\big)I[N+1] + \big(-6N^3-50N^2-136N-120\big)I[N+2] - (-N-2)(N+4)(2N+8)I[N+3] == -\frac{4(N+2)}{3(N+3)};$$

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In[3]:= **recSol = SolveRecurrence[rec, I[N]]**

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$$I(1) = 5, I(2) = \frac{130}{27}, I(3) = \frac{169}{36}$$

Solving recurrences

In[1]:= $$\mathbf{rec} = -2(N+1)(N+2)^2 I[N] - (N+2)\big(-6N^2-28N-32\big)I[N+1] + \big(-6N^3-50N^2-136N-120\big)I[N+2] - (-N-2)(N+4)(2N+8)I[N+3] == -\frac{4(N+2)}{3(N+3)};$$

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In[3]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, \mathbf{I}[\mathbf{N}]]$

Out[3]= $$\Big\{\Big\{0, \frac{-1}{N+1}\Big\}, \Big\{0, \frac{-N}{N+1}\Big\}, \Big\{0, \frac{1}{(N+1)^2} + \frac{1}{N+1}\sum_{i=1}^{N}\frac{1}{i}\Big\}, \Big\{1, \frac{2\big(N^2+N-1\big)}{3(N+1)^2} - \frac{2(N+2)}{3(N+1)}\sum_{i=1}^{N}\frac{1}{i}\Big\}\Big\}$$

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$$I(1) = 5, I(2) = \frac{130}{27}, I(3) = \frac{169}{36}$$

$$\Downarrow c_1 = -\frac{49}{9}, c_2 = -\frac{41}{9}, c_3 = -\frac{2}{3}$$

$$I(N) = \frac{59N^2+120N+49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)}\sum_{i=1}^{N}\frac{1}{i}$$

Solving recurrences

In[1]:= $$\mathbf{rec} = -2(N+1)(N+2)^2 I[N] - (N+2)\big(-6N^2 - 28N - 32\big)I[N+1] + \big(-6N^3 - 50N^2 - 136N - 120\big)I[N+2] - (-N-2)(N+4)(2N+8)I[N+3] == -\frac{4(N+2)}{3(N+3)};$$

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Sigma - A summation package by Carsten Schneider © RISC-Linz

In[3]:= **recSol = SolveRecurrence[rec, I[N]]**

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In[4]:= **sol = FindLinearCombination[recSol, {1, {5, $\frac{130}{27}$, $\frac{169}{36}$}}, N, 3]**

Out[4]= $$\frac{59N^2 + 120N + 49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)}\sum_{i=1}^{N}\frac{1}{i}$$

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In[5]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[6]:= $\mathbf{sol = ReduceToBasis[TransformToSSums[sol], Dynamic \to Automatic]}$

Solving recurrences

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In[3]:= $$\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}, I[N]]$$

Out[3]= $$\Big\{\Big\{0, \frac{-1}{N+1}\Big\}, \Big\{0, \frac{-N}{N+1}\Big\}, \Big\{0, \frac{1}{(N+1)^2} + \frac{1}{N+1}\sum_{i=1}^{N}\frac{1}{i}\Big\}, \Big\{1, \frac{2\big(N^2+N-1\big)}{3(N+1)^2} - \frac{2(N+2)}{3(N+1)}\sum_{i=1}^{N}\frac{1}{i}\Big\}\Big\}$$

In[4]:= $$\mathbf{sol} = \mathbf{FindLinearCombination}\Big[\mathbf{recSol}, \Big\{1, \Big\{5, \frac{130}{27}, \frac{169}{36}\Big\}\Big\}, N, 3\Big]$$

Out[4]= $$\frac{59N^2+120N+49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)}\sum_{i=1}^{N}\frac{1}{i}$$

In[5]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[6]:= $$\mathbf{sol} = \mathbf{ReduceToBasis}[\mathbf{TransformToSSums}[\mathbf{sol}], \mathbf{Dynamic} \to \mathbf{Automatic}]$$

Out[6]= $$\frac{59N^2+120N+49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)}S_1[N]$$

Theorem 1. Given $\langle I(N)\rangle_{N\geq 0}$ specified by the initial values $I(0),\dots,I(d-1)$ and the recurrence

$$a_0(N)I(N)+a_1(N)I(N+1)+\cdots+a_d(N)I(N+d)=r(N)$$

$a_0(N),\dots,a_d(N)$: polynomials in N;
$r(N)$: expression in terms of nested sums and products

`Sigma` computes, if possible, a representation of $I(N)$ in terms of nested sums and products.

Theorem 1. Given $\langle I(N)\rangle_{N\geq 0}$ specified by the initial values $I(0),\dots,I(d-1)$ and the recurrence

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17. S.A. Abramov, M. Bronstein, M. Petkovšek, CS, in preparation

Problem 2: Computing the ε-expansion from a recurrence relation

Computing the ε-expansion from a recurrence relation

$$I(1)=\tfrac{5}{\varepsilon^3}-\tfrac{163}{12\varepsilon^2}+O(\varepsilon^{-1}),\ I(2)=\tfrac{130}{27\varepsilon^3}-\tfrac{695}{54\varepsilon^2}+O(\varepsilon^{-1}),\ I(3)=\tfrac{169}{36\varepsilon^3}-\tfrac{395}{32\varepsilon^2}+O(\varepsilon^{-1})$$

$$+$$

$$\begin{aligned}
&-2(N+1)(N+2)(2+\varepsilon+N)I(N)\\
&\qquad-(N+2)\big(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2\big)I(N+1)\\
&-\big(120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2N+50N^2+4\varepsilon N^2+6N^3\big)I(N+2)\\
&\qquad+(2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I(N+3)\\
&=\frac{1}{\varepsilon^3}\tfrac{-4(N+2)}{3(N+3)}+\frac{1}{\varepsilon^2}\Big[-\tfrac{2(2N+7)S_1}{3(N+3)}-\tfrac{2\left(4N^4+35N^3+101N^2+105N+25\right)}{3(N+1)(N+2)(N+3)^2}\Big]+O(\varepsilon^{-1})
\end{aligned}$$

Computing the ε-expansion from a recurrence relation

$$I(1)=\frac{5}{\varepsilon^3}-\frac{163}{12\varepsilon^2}+O(\varepsilon^{-1}),\ I(2)=\frac{130}{27\varepsilon^3}-\frac{695}{54\varepsilon^2}+O(\varepsilon^{-1}),\ I(3)=\frac{169}{36\varepsilon^3}-\frac{395}{32\varepsilon^2}+O(\varepsilon^{-1})$$

$$+$$

$$\begin{gathered}
-2(N+1)(N+2)(2+\varepsilon+N)I(N)\\
-(N+2)\big(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2\big)I(N+1)\\
-\big(120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2N+50N^2+4\varepsilon N^2+6N^3\big)I(N+2)\\
+(2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I(N+3)\\
=\frac{1}{\varepsilon^3}\frac{-4(N+2)}{3(N+3)}+\frac{1}{\varepsilon^2}\left[-\frac{2(2N+7)S_1}{3(N+3)}-\frac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\right]+O(\varepsilon^{-1})
\end{gathered}$$

$$\downarrow$$

$$\begin{aligned}
I(N)=\varepsilon^{-3}&\left[\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)}{3(N+1)}S_1(N)\right]\\
+\varepsilon^{-2}&\left[\frac{-2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1(N)}{3(N+1)^2}-\frac{S_1(N)^2}{N+1}-\frac{S_2(N)}{N+1}\right]+O(\varepsilon^{-1})
\end{aligned}$$

Computing the ε-expansion from a recurrence relation

$$\begin{aligned}
&\quad a_0(\varepsilon,N)\Big[I(N)\Big]\\
&+a_1(\varepsilon,N)\Big[I(N+1)\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[I(N+d)\Big]\\
&\qquad\qquad = h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}$$

given (in terms of indefinite nested sums and products)

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[I(N+1)\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[I(N+d)\Big]\\
&\qquad\qquad\qquad = h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[I_{-3}(N+1)\varepsilon^{-3}+I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[I(N+d)\Big]\\
&\qquad\qquad\qquad\qquad = h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[I_{-3}(N+1)\varepsilon^{-3}+I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[I_{-3}(N+d)\varepsilon^{-3}+I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

given (in terms of indefinite nested sums and products)

Computing the ε-expansion from a recurrence relation

$$\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[I_{-3}(N+1)\varepsilon^{-3}+I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[I_{-3}(N+d)\varepsilon^{-3}+I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}$$

$\Downarrow$ lowest terms must agree

$$\boxed{a_0(0,N)I_{-3}(N)+a_1(0,N)I_{-3}(N+1)+\dots+a_d(0,N)I_{-3}(N+d)=h_{-3}(N)}$$

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[I_{-3}(N+1)\varepsilon^{-3}+I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[I_{-3}(N+d)\varepsilon^{-3}+I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

$\Downarrow$ lowest terms must agree

$$\boxed{a_0(0,N)I_{-3}(N)+a_1(0,N)I_{-3}(N+1)+\dots+a_d(0,N)I_{-3}(N+d)=h_{-3}(N)}$$

REC solver: Given the initial values $I_{-3}(1),I_{-3}(2),\dots,I_{-3}(d)$, **decide** if $I_{-3}(N)$ can be written in terms of indefinite nested sums and products.

Computing the ε-expansion from a recurrence relation

$$I(1)=\tfrac{5}{\varepsilon^3}-\tfrac{163}{12\varepsilon^2}+O(\varepsilon^{-1}),\ I(2)=\tfrac{130}{27\varepsilon^3}-\tfrac{695}{54\varepsilon^2}+O(\varepsilon^{-1}),\ I(3)=\tfrac{169}{36\varepsilon^3}-\tfrac{395}{32\varepsilon^2}+O(\varepsilon^{-1})$$

$$+$$

$$\begin{gathered}-2(N+1)(N+2)(2+\varepsilon+N)I(N)\\ -(N+2)\big(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2\big)I(N+1)\\ -\big(120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2N+50N^2+4\varepsilon N^2+6N^3\big)I(N+2)\\ +(2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I(N+3)\\ =\frac{1}{\varepsilon^3}\tfrac{-4(N+2)}{3(N+3)}+\frac{1}{\varepsilon^2}\Big[-\tfrac{2(2N+7)S_1}{3(N+3)}-\tfrac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\Big]+O(\varepsilon^{-1})\end{gathered}$$

Solving recurrences

In[1]:= $$\mathbf{rec} = -2(N+1)(N+2)^2 I[N] - (N+2)\big(-6N^2 - 28N - 32\big)I[N+1] + \big(-6N^3 - 50N^2 - 136N - 120\big)I[N+2] - (-N-2)(N+4)(2N+8)I[N+3] == -\frac{4(N+2)}{3(N+3)};$$

In[2]:= $<<$ **Sigma.m**

```
Sigma - A summation package by Carsten Schneider © RISC-Linz
```

In[3]:= **recSol = SolveRecurrence[rec, I[N]]**

Out[3]= $$\Big\{\Big\{0, \frac{-1}{N+1}\Big\}, \Big\{0, \frac{-N}{N+1}\Big\}, \Big\{0, \frac{1}{(N+1)^2} + \frac{1}{N+1}\sum_{i=1}^{N}\frac{1}{i}\Big\}, \Big\{1, \frac{2\big(N^2+N-1\big)}{3(N+1)^2} - \frac{2(N+2)}{3(N+1)}\sum_{i=1}^{N}\frac{1}{i}\Big\}\Big\}$$

General solution: $$\{c_1\,\frac{1}{-N-1} + c_2\,\frac{-N}{N+1} + c_3\left(\frac{1}{(N+1)^2} + \frac{1}{N+1}\right)\sum_{i=1}^{N}\frac{1}{i} + \frac{2(N^2+N-1)}{3(N+1)^2} - \frac{2(N+2)}{3(N+1)}\sum_{i=1}^{N}\frac{1}{i}\,|\,c_1, c_2, c_3 \in \mathbb{R}\}$$

$$I(1) = 5, I(2) = \frac{130}{27}, I(3) = \frac{169}{36}$$

$$\Downarrow c_1 = -\frac{49}{9}, c_2 = -\frac{41}{9}, c_3 = -\frac{2}{3}$$

$$I(N) = \frac{59N^2+120N+49}{9(N+1)^2} - \frac{2(N+3)}{3(N+1)}\sum_{i=1}^{N}\frac{1}{i}$$

Computing the ε-expansion from a recurrence relation

$$I(1)=\tfrac{5}{\varepsilon^3}-\tfrac{163}{12\varepsilon^2}+O(\varepsilon^{-1}),\ I(2)=\tfrac{130}{27\varepsilon^3}-\tfrac{695}{54\varepsilon^2}+O(\varepsilon^{-1}),\ I(3)=\tfrac{169}{36\varepsilon^3}-\tfrac{395}{32\varepsilon^2}+O(\varepsilon^{-1})$$

$$+$$

$$\begin{aligned}
&-2(N+1)(N+2)(2+\varepsilon+N)I(N)\\
&\qquad-(N+2)\big(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2\big)I(N+1)\\
&-\big(120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2N+50N^2+4\varepsilon N^2+6N^3\big)I(N+2)\\
&\qquad+(2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I(N+3)\\
&=\frac{1}{\varepsilon^3}\tfrac{-4(N+2)}{3(N+3)}+\frac{1}{\varepsilon^2}\Big[-\tfrac{2(2N+7)S_1}{3(N+3)}-\tfrac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\Big]+O(\varepsilon^{-1})
\end{aligned}$$

$$\downarrow$$

$$I(N)=\varepsilon^{-3}\Big[\tfrac{59N^2+120N+49}{9(N+1)^2}-\tfrac{2(N+3)}{3(N+1)}S_1(N)\Big]$$

Computing the ε-expansion from a recurrence relation

$$\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[I_{-3}(N+1)\varepsilon^{-3}+I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[I_{-3}(N+d)\varepsilon^{-3}+I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}$$

$\Downarrow$ lowest terms must agree

$$\boxed{a_0(0,N)I_{-3}(N)+a_1(0,N)I_{-3}(N+1)+\dots+a_d(0,N)I_{-3}(N+d)=h_{-3}(N)}$$

REC solver: Given the initial values $I_{-3}(1),I_{-3}(2),\dots,I_{-3}(d)$, **decide** if $I_{-3}(N)$ can be written in terms of indefinite nested sums and products.

Computing the ε-expansion from a recurrence relation

$$\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[I_{-3}(N+1)\varepsilon^{-3}+I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[I_{-3}(N+d)\varepsilon^{-3}+I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}$$

$\Downarrow$ lowest terms must agree

$$\boxed{a_0(0,N)I_{-3}(N)+a_1(0,N)I_{-3}(N+1)+\dots+a_d(0,N)I_{-3}(N+d)=h_{-3}(N)}$$

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-3}(N)\varepsilon^{-3}+I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
+&a_1(\varepsilon,N)\Big[I_{-3}(N+1)\varepsilon^{-3}+I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
+&\\
&\vdots\\
+&a_d(\varepsilon,N)\Big[I_{-3}(N+d)\varepsilon^{-3}+I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad=h_{-3}(N)\varepsilon^{-3}+h_{-2}(N)\varepsilon^{-2}+h_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

$\Downarrow$ lowest terms must agree

$$\boxed{a_0(0,N)I_{-3}(N)+a_1(0,N)I_{-3}(N+1)+\dots+a_d(0,N)I_{-3}(N+d)=h_{-3}(N)}$$

Computing the ε-expansion from a recurrence relation

$$
\begin{aligned}
&\quad a_0(\varepsilon,N)\Big[I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad\qquad=h'_{-3}(N)\varepsilon^{-3}+h'_{-2}(N)\varepsilon^{-2}+h'_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}
$$

Computing the ε-expansion from a recurrence relation

$$\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad\qquad\qquad = \underbrace{h'_{-3}(N)}_{\mathbf{=0}}\varepsilon^{-3}+h'_{-2}(N)\varepsilon^{-2}+h'_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}$$

Computing the ε-expansion from a recurrence relation

$$\begin{aligned}
&a_0(\varepsilon,N)\Big[I_{-2}(N)\varepsilon^{-2}+I_{-1}(N)\varepsilon^{-1}+\dots\Big]\\
&+a_1(\varepsilon,N)\Big[I_{-2}(N+1)\varepsilon^{-2}+I_{-1}(N+1)\varepsilon^{-1}+\dots\Big]\\
&+\\
&\ \vdots\\
&+a_d(\varepsilon,N)\Big[I_{-2}(N+d)\varepsilon^{-2}+I_{-1}(N+d)\varepsilon^{-1}+\dots\Big]\\
&\qquad\qquad\qquad\qquad=h'_{-2}(N)\varepsilon^{-2}+h'_{-1}(N)\varepsilon^{-1}+\dots
\end{aligned}$$

Now repeat for $I_{-2}(N), I_{-1}(N), \dots$

Blümlein, Klein, CS, Stan, J. Symbol. Comput. 2012; arXiv:1011.2656[cs.SC]
Ablinger, Blümlein, Round, CS, LL2012, arXiv:1210.1685 [cs.SC]

Computing the ε-expansion from a recurrence relation

$$I(1)=\tfrac{5}{\varepsilon^3}-\tfrac{163}{12\varepsilon^2}+O(\varepsilon^{-1}),\ I(2)=\tfrac{130}{27\varepsilon^3}-\tfrac{695}{54\varepsilon^2}+O(\varepsilon^{-1}),\ I(3)=\tfrac{169}{36\varepsilon^3}-\tfrac{395}{32\varepsilon^2}+O(\varepsilon^{-1})$$

In[7]:=
$$\begin{aligned}
\mathbf{recEp} = {}& -2(N+1)(N+2)(2+\varepsilon+N)I[N]\\
&-(N+2)\big(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2\big)I[N+1]\\
&-\big(120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2N+50N^2+4\varepsilon N^2+6N^3\big)I[N+2]\\
&+(2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I[N+3]\\
&== \frac{1}{\varepsilon^3}\frac{-4(N+2)}{3(N+3)}+\frac{1}{\varepsilon^2}\Big[-\frac{2(2N+7)S_1}{3(N+3)}-\\
&\qquad\frac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\Big]+O(\varepsilon^{-1}).
\end{aligned}$$

Computing the ε-expansion from a recurrence relation

$$I(1) = \frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2} + O(\varepsilon^{-1}),\ I(2) = \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2} + O(\varepsilon^{-1}),\ I(3) = \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2} + O(\varepsilon^{-1})$$

In[7]:=
$$\begin{aligned}
\mathbf{recEp} = {} & -2(\mathrm{N}+1)(\mathrm{N}+2)(2+\varepsilon+\mathrm{N})\mathrm{I}[\mathrm{N}] \\
& - (\mathrm{N}+2)\big(-32 - 7\varepsilon + 2\varepsilon^2 - 28\mathrm{N} - 5\varepsilon\mathrm{N} - 6\mathrm{N}^2\big)\mathrm{I}[\mathrm{N}+1] \\
& - \big(120 + 3\varepsilon - 14\varepsilon^2 - \varepsilon^3 + 136\mathrm{N} + 13\varepsilon\mathrm{N} - 4\varepsilon^2\mathrm{N} + 50\mathrm{N}^2 + 4\varepsilon\mathrm{N}^2 + 6\mathrm{N}^3\big)\mathrm{I}[\mathrm{N}+2] \\
& + (2-\varepsilon+\mathrm{N})(4+\varepsilon+\mathrm{N})(8+\varepsilon+2\mathrm{N})\mathrm{I}[\mathrm{N}+3] \\
& == \frac{1}{\varepsilon^3}\frac{-4(\mathrm{N}+2)}{3(\mathrm{N}+3)} + \frac{1}{\varepsilon^2}\Big[-\frac{2(2\mathrm{N}+7)\mathrm{S}_1}{3(\mathrm{N}+3)} - \\
& \qquad \frac{2\big(4\mathrm{N}^4 + 35\mathrm{N}^3 + 101\mathrm{N}^2 + 105\mathrm{N} + 25\big)}{3(\mathrm{N}+1)(\mathrm{N}+2)(\mathrm{N}+3)^2}\Big] + \mathrm{O}(\varepsilon^{-1}).
\end{aligned}$$

In[8]:=
$$\begin{aligned}
& \mathbf{GenerateExpansion}[\mathbf{recEp}[[1]], \\
& \qquad \{\mathbf{Coefficient}[\mathbf{recEp}[[2]], \varepsilon^{-3}], \mathbf{Coefficient}[\mathbf{recEp}[[2]], \varepsilon^{-2}]\}, \mathrm{I}[\mathrm{N}], \\
& \{\varepsilon, -3, -2\}, \Big\{\Big\{5, \frac{130}{27}, \frac{169}{36}\Big\}, \Big\{-\frac{163}{12}, -\frac{695}{54}, -\frac{395}{32}\Big\}\Big\}, \mathbf{MinInitialValue} \to 1]
\end{aligned}$$

Computing the ε-expansion from a recurrence relation

$$I(1)=\frac{5}{\varepsilon^3}-\frac{163}{12\varepsilon^2}+O(\varepsilon^{-1}),\ I(2)=\frac{130}{27\varepsilon^3}-\frac{695}{54\varepsilon^2}+O(\varepsilon^{-1}),\ I(3)=\frac{169}{36\varepsilon^3}-\frac{395}{32\varepsilon^2}+O(\varepsilon^{-1})$$

In[7]:=
$$\begin{aligned}\mathbf{recEp} = &-2(\mathrm{N}+1)(\mathrm{N}+2)(2+\varepsilon+\mathrm{N})\mathrm{I}[\mathrm{N}]\\ &-(\mathrm{N}+2)\big(-32-7\varepsilon+2\varepsilon^2-28\mathrm{N}-5\varepsilon\mathrm{N}-6\mathrm{N}^2\big)\mathrm{I}[\mathrm{N}+1]\\ &-\big(120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136\mathrm{N}+13\varepsilon\mathrm{N}-4\varepsilon^2\mathrm{N}+50\mathrm{N}^2+4\varepsilon\mathrm{N}^2+6\mathrm{N}^3\big)\mathrm{I}[\mathrm{N}+2]\\ &+(2-\varepsilon+\mathrm{N})(4+\varepsilon+\mathrm{N})(8+\varepsilon+2\mathrm{N})\mathrm{I}[\mathrm{N}+3]\\ &== \frac{1}{\varepsilon^3}\frac{-4(\mathrm{N}+2)}{3(\mathrm{N}+3)}+\frac{1}{\varepsilon^2}\Big[-\frac{2(2\mathrm{N}+7)\mathrm{S}_1}{3(\mathrm{N}+3)}-\\ &\qquad\frac{2\big(4\mathrm{N}^4+35\mathrm{N}^3+101\mathrm{N}^2+105\mathrm{N}+25\big)}{3(\mathrm{N}+1)(\mathrm{N}+2)(\mathrm{N}+3)^2}\Big]+\mathrm{O}(\varepsilon^{-1}).\end{aligned}$$

In[8]:=
$$\begin{aligned}&\mathbf{GenerateExpansion}[\mathbf{recEp}[[1]],\\ &\quad\{\mathbf{Coefficient}[\mathbf{recEp}[[2]],\varepsilon^{-3}],\mathbf{Coefficient}[\mathbf{recEp}[[2]],\varepsilon^{-2}]\},\mathrm{I}[\mathrm{N}],\\ &\{\varepsilon,-3,-2\},\Big\{\Big\{5,\frac{130}{27},\frac{169}{36}\Big\},\Big\{-\frac{163}{12},-\frac{695}{54},-\frac{395}{32}\Big\}\Big\},\mathbf{MinInitialValue}\to 1]\end{aligned}$$

Out[8]=
$$\begin{aligned}&\Big\{\frac{59\mathrm{N}^2+120\mathrm{N}+49}{9(\mathrm{N}+1)^2}-\frac{2(\mathrm{N}+3)\mathrm{S}_1[\mathrm{N}]}{3(\mathrm{N}+1)},\\ &-\frac{2(20\mathrm{N}^3+58\mathrm{N}^2+57\mathrm{N}+22)}{3(\mathrm{N}+1)^3}+\frac{2(\mathrm{N}+2)(2\mathrm{N}-1)\mathrm{S}_1[\mathrm{N}]}{3(\mathrm{N}+1)^2}-\frac{\mathrm{S}_1[\mathrm{N}]^2}{\mathrm{N}+1}-\frac{\mathrm{S}_2[\mathrm{N}]}{\mathrm{N}+1}\Big\}\end{aligned}$$

Theorem 2. Given $\langle I(N) \rangle_{N \geq 0}$ with

$$I(N) = I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + \cdots + I_l(N)\varepsilon^l + O(\varepsilon^{l+1})$$

specified by the initial values $I(0), \dots, I(d-1)$ and the recurrence

$$\begin{aligned} a_0(\varepsilon, N)I(N) + a_1(\varepsilon, N)I(N+1) + \cdots + a_d(\varepsilon, N)I(N+d) \\ = \varepsilon^{-3} r_{-3}(N) + \varepsilon^{-2} r_{-2}(N) + \cdots + \varepsilon^l r_l(N) + O(\varepsilon^{l+1}) \end{aligned}$$

$a_0(\varepsilon, N), \dots, a_d(\varepsilon, N)$: polynomials in N and ε;
$r_i(N)$: nested sums and products.

`Sigma` computes, if possible, a representation of $I_{-3}(N), \dots, I_l(N)$ in terms of nested sums and products.

Theorem 2. Given $\langle I(N)\rangle_{N\geq 0}$ with

$$I(N) = I_{-3}(N)\varepsilon^{-3} + I_{-2}(N)\varepsilon^{-2} + \cdots + I_l(N)\varepsilon^l + O(\varepsilon^{l+1})$$

specified by the initial values $I(0), \ldots, I(d-1)$ and the recurrence

$$\begin{aligned} a_0(\varepsilon,N)I(N) + a_1(\varepsilon,N)I(N+1) + \cdots + a_d(\varepsilon,N)I(N+d)& \\ = \varepsilon^{-3}r_{-3}(N) + \varepsilon^{-2}r_{-2}(N) + \cdots + \varepsilon^l r_l(N) + O(\varepsilon^{l+1})& \end{aligned}$$

$a_0(\varepsilon,N), \ldots, a_d(\varepsilon,N)$: polynomials in N and ε;
$r_i(N)$: nested sums and products.

`Sigma` computes, if possible, a representation of $I_{-3}(N), \ldots, I_l(N)$ in terms of nested sums and products.

Problem 3: Solving a coupled recurrence system

Solving a coupled recurrence system

$$A_0 \begin{pmatrix} I_1(N) \\ I_2(N) \\ I_3(N) \end{pmatrix} + A_1 \begin{pmatrix} I_1(N+1) \\ I_2(N+1) \\ I_3(N+1) \end{pmatrix} = \begin{pmatrix} r_1(N) \\ r_2(N) \\ r_3(N) \end{pmatrix}$$

with

$$A_0 = \begin{pmatrix} N+1 & 0 & 0 \\ \varepsilon(3\varepsilon+2) & -2(3\varepsilon+1) & -2(-1+\varepsilon-2N) \\ -\varepsilon(3\varepsilon+2) & 2(3+3\varepsilon+2N) & 2(\varepsilon+1) \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -2-\varepsilon-N & 2 & 0 \\ -2\varepsilon(3\varepsilon+2) & 2(5\varepsilon+2) & 4(-1+\varepsilon-N) \\ 0 & -2(4+\varepsilon+2N) & 0 \end{pmatrix}$$

and

$$r_1(N) = (-\frac{4(N+3)}{3(N+2)}\varepsilon^{-3} + \left(\frac{2}{3}\frac{6N^3+29N^2+45N+21}{(N+1)(N+2)^2} - \frac{2(2N+3)S_1(N)}{3(N+2)}\right)\varepsilon^{-2} + O(\varepsilon^{-1})$$

$$r_2(N) = -\frac{8}{3}\varepsilon^{-3} + \left(\frac{4(3N+1)}{3(N+1)} - \frac{8S_1(N)}{3}\right)\varepsilon^{-2} + O(\varepsilon^{-1}),$$

$$r_3(N) = \frac{8}{3}\varepsilon^{-3} + \left(\frac{-4(3N+1)}{3(N+1)} + \frac{8S_1(N)}{3}\right)\varepsilon^{-2} + O(\varepsilon^{-1})$$

Solving a coupled recurrence system

$$A_0 \begin{pmatrix} I_1(N) \\ I_2(N) \\ I_3(N) \end{pmatrix} + A_1 \begin{pmatrix} I_1(N+1) \\ I_2(N+1) \\ I_3(N+1) \end{pmatrix} = \begin{pmatrix} r_1(N) \\ r_2(N) \\ r_3(N) \end{pmatrix}$$

uncoupling algorithms $\downarrow$ `Gerhold's OreSys.m (using Zürcher's algorithm)`

$$\begin{aligned} a_0(\varepsilon,N)I_1(N)+a_1(\varepsilon,N)I_1(N+1)+a_2(\varepsilon,N)I_1(N+2)+a_3(\varepsilon,N)I_1(N+3) \\ = h_{-3}\varepsilon^{-3}+h_{-2}\varepsilon^{-2}+h_{-1}\varepsilon^{-1}+O(\varepsilon^{0}) \end{aligned}$$

$$\begin{aligned} I_2(N) &= \text{LinearCombination}(I_1(N), I_1(N+1), I_1(N+2)) \\ I_3(N) &= \text{LinearCombination}(I_1(N), I_1(N+1), I_1(N+2)) \end{aligned}$$

Computing the ε-expansion from a recurrence relation

$$I(1)=\frac{5}{\varepsilon^3}-\frac{163}{12\varepsilon^2}+O(\varepsilon^{-1}),\ I(2)=\frac{130}{27\varepsilon^3}-\frac{695}{54\varepsilon^2}+O(\varepsilon^{-1}),\ I(3)=\frac{169}{36\varepsilon^3}-\frac{395}{32\varepsilon^2}+O(\varepsilon^{-1})$$

$$+$$

$$\begin{gathered}
-2(N+1)(N+2)(2+\varepsilon+N)I(N)\\
-(N+2)\big(-32-7\varepsilon+2\varepsilon^2-28N-5\varepsilon N-6N^2\big)I(N+1)\\
-\big(120+3\varepsilon-14\varepsilon^2-\varepsilon^3+136N+13\varepsilon N-4\varepsilon^2N+50N^2+4\varepsilon N^2+6N^3\big)I(N+2)\\
+(2-\varepsilon+N)(4+\varepsilon+N)(8+\varepsilon+2N)I(N+3)\\
=\frac{1}{\varepsilon^3}\frac{-4(N+2)}{3(N+3)}+\frac{1}{\varepsilon^2}\Big[-\frac{2(2N+7)S_1}{3(N+3)}-\frac{2\big(4N^4+35N^3+101N^2+105N+25\big)}{3(N+1)(N+2)(N+3)^2}\Big]+O(\varepsilon^{-1})
\end{gathered}$$

$$\downarrow$$

$$\begin{aligned}
I(N)=&\ \varepsilon^{-3}\Big[\frac{59N^2+120N+49}{9(N+1)^2}-\frac{2(N+3)}{3(N+1)}S_1(N)\Big]\\
&+\varepsilon^{-2}\Big[\frac{-2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1(N)}{3(N+1)^2}-\frac{S_1(N)^2}{N+1}-\frac{S_2(N)}{N+1}\Big]+O(\varepsilon^{-1})
\end{aligned}$$

Solving a coupled recurrence system

$$A_0 \begin{pmatrix} I_1(N) \\ I_2(N) \\ I_3(N) \end{pmatrix} + A_1 \begin{pmatrix} I_1(N+1) \\ I_2(N+1) \\ I_3(N+1) \end{pmatrix} = \begin{pmatrix} r_1(N) \\ r_2(N) \\ r_3(N) \end{pmatrix}$$

uncoupling algorithms $\downarrow$ `Gerhold's OreSys.m (using Zürcher's algorithm)`

$$I_1(N) = \varepsilon^{-3}\left[\tfrac{59N^2+120N+49}{9(N+1)^2} - \tfrac{2(N+3)}{3(N+1)}S_1(N)\right]$$
$$+\,\varepsilon^{-2}\left[\tfrac{-2(20N^3+58N^2+57N+22)}{3(N+1)^3} + \tfrac{2(N+2)(2N-1)S_1(N)}{3(N+1)^2} - \tfrac{S_1(N)^2}{N+1} - \tfrac{S_2(N)}{N+1}\right] + O(\varepsilon^{-1})$$

$\downarrow$ insert

$$I_2(N) = \text{LinearCombination}(I_1(N), I_1(N+1), I_1(N+2))$$
$$I_3(N) = \text{LinearCombination}(I_1(N), I_1(N+1), I_1(N+2))$$

Solving a coupled recurrence system

In[9]:= **<< OreSys.m**

OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz

In[10]:= **<< SolveCoupledSystem.m**

SolveCoupledSystem by Carsten Schneider © RISC-Linz

In[11]:= **coupledSys =**

$$\mathbf{A_0.\{I_1[N], I_2[N], I_3[N]\} + A_1.\{I_1[N], I_2[N], I_3[N]\} - \{r_1[n], r_2[n], r_3[N]\};}$$

Solving a coupled recurrence system

In[9]:= $<<$ **OreSys.m**

OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz

In[10]:= $<<$ **SolveCoupledSystem.m**

SolveCoupledSystem by Carsten Schneider © RISC-Linz

In[11]:= $\mathbf{coupledSys} =$
$\mathbf{A_0.\{I_1[N], I_2[N], I_3[N]\} + A_1.\{I_1[N], I_2[N], I_3[N]\} - \{r_1[n], r_2[n], r_3[N]\};}$

In[12]:= $\mathbf{SolveCoupledRecSystem[coupledSys, \{I_1[N], I_2[N], I_3[N]\}, \varepsilon, -3,}$
$\mathbf{\{-2, -2, -2\}, \{I_1[N], 1, \{\frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2}, \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2}, \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2}\}\}]}$

Solving a coupled recurrence system

In[9]:= $\mathbf{<< OreSys.m}$

OreSys by Stefan Gerhold (optimized by C. Schneider) © RISC-Linz

In[10]:= $\mathbf{<< SolveCoupledSystem.m}$

SolveCoupledSystem by Carsten Schneider © RISC-Linz

In[11]:= $\mathbf{coupledSys =}$
$\mathbf{A_0.\{I_1[N], I_2[N], I_3[N]\} + A_1.\{I_1[N], I_2[N], I_3[N]\} - \{r_1[n], r_2[n], r_3[N]\};}$

In[12]:= $\mathbf{SolveCoupledRecSystem[coupledSys, \{I_1[N], I_2[N], I_3[N]\}, \varepsilon, -3,}$
$\mathbf{\{-2, -2, -2\}, \{I_1[N], 1, \{\frac{5}{\varepsilon^3} - \frac{163}{12\varepsilon^2}, \frac{130}{27\varepsilon^3} - \frac{695}{54\varepsilon^2}, \frac{169}{36\varepsilon^3} - \frac{395}{32\varepsilon^2}\}\}]}$

Out[12]=
$$\Big\{\frac{1}{\varepsilon^3}\Big(\frac{4\big(3N^2+6N+4\big)}{3(N+1)^2}+\frac{4S_1[N]}{3(N+1)}\Big)+\frac{1}{\varepsilon^2}\Big(-\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1}\Big),$$
$$\frac{4}{3\varepsilon^3}-\frac{2}{\varepsilon^2}, \frac{8}{3\varepsilon^3}+\frac{1}{\varepsilon^2}\Big(-\frac{4\big(4N^2+7N+2\big)}{3(N+1)^2}+\frac{4(N+2)S_1[N]}{3(N+1)}\Big)\Big\}$$

Theorem 3. Given $\langle I_1(N)\rangle_{N\geq 0},\ldots,\langle I_n(N)\rangle_{N\geq 0}$ with

$$I_k(N) \;=\; I_{k,-3}(N)\varepsilon^{-3} + \cdots + I_{k,l}(N)\varepsilon^{l} + \,O(\varepsilon^{l+1}), \quad k \;=\; 1,\ldots,n$$

specified by the initial values $I_k(0),\ldots,I_k(\lambda)$ with $k=1,\ldots,n$ and the recurrence system

$$A_0\begin{pmatrix}I_1(N)\\ \vdots\\ I_n(N)\end{pmatrix}+A_1\begin{pmatrix}I_1(N+1)\\ \vdots\\ I_n(N+1)\end{pmatrix}\cdots+A_d\begin{pmatrix}I_1(N+d)\\ \vdots\\ I_n(N+d)\end{pmatrix}$$
$$=\begin{pmatrix}\varepsilon^{-3}r_{1,-3}(N)+\cdots+\varepsilon^{l_1}r_{1,l_1}(N)+O(\varepsilon^{l_1+1})\\ \vdots\\ \varepsilon^{-3}r_{n,-3}(N)+\cdots+\varepsilon^{l_n}r_{n,l_n}(N)+O(\varepsilon^{l_n+1})\end{pmatrix}$$

$A_0,\ldots,A_d$: $n\times n$ matrices with entries from $\mathbb{K}(\varepsilon,N)$;
$r_{i,j}(N)$: nested sums and products.

`Sigma` computes, if possible, a representation of $I_{k,-3}(N),\ldots,I_{k,l}(N)$ in terms of nested sums and products.

Theorem 3. Given $\langle I_1(N)\rangle_{N\geq 0},\dots,\langle I_n(N)\rangle_{N\geq 0}$ with

$$I_k(N) = I_{k,-3}(N)\varepsilon^{-3} + \cdots + I_{k,l}(N)\varepsilon^{l} + O(\varepsilon^{l+1}), \quad k = 1,\dots,n$$

specified by the initial values $I_k(0),\dots,I_k(\lambda)$ with $k=1,\dots,n$ and the recurrence system

$$A_0\begin{pmatrix}I_1(N)\\ \vdots\\ I_n(N)\end{pmatrix} + A_1\begin{pmatrix}I_1(N+1)\\ \vdots\\ I_n(N+1)\end{pmatrix}\cdots + A_d\begin{pmatrix}I_1(N+d)\\ \vdots\\ I_n(N+d)\end{pmatrix}$$

$$= \begin{pmatrix}\varepsilon^{-3}r_{1,-3}(N) + \cdots + \varepsilon^{l_1}r_{1,l_1}(N) + O(\varepsilon^{l_1+1})\\ \vdots\\ \varepsilon^{-3}r_{n,-3}(N) + \cdots + \varepsilon^{l_n}r_{n,l_n}(N) + O(\varepsilon^{l_n+1})\end{pmatrix}$$

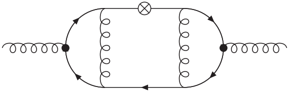
$A_0,\dots,A_d$: $n\times n$ matrices with entries from $\mathbb{K}(\varepsilon,N)$;
$r_{i,j}(N)$: nested sums and products.

`Sigma` computes, if possible, a representation of $I_{k,-3}(N),\dots,I_{k,l}(N)$ in terms of nested sums and products.

Problem 4: Solving a coupled differential system

Completed project: 3-loop massive ladder and V-diagrams

Ablinger, Behring, Blümlein, De Freitas, von Manteuffel, CS; Comput. Phys. Comm. 202 [arXiv:1509.08324 [hep-ph]]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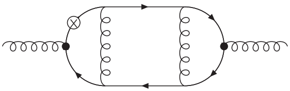
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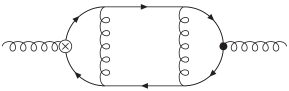
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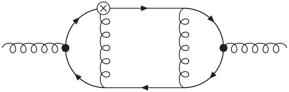
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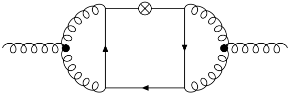
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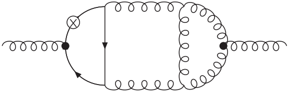
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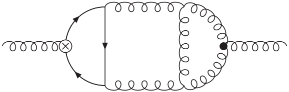
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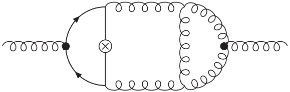
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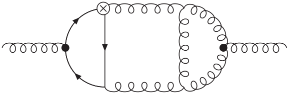
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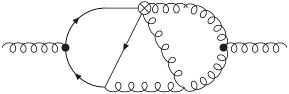
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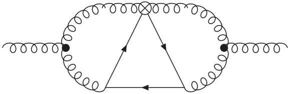
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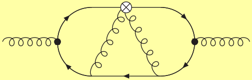
(5 propagators) 12

All diagrams are produced with axodraw (J. Vermaseren)

A coupled differential system for $\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)$

(produced by IBP [extension of `REDUZE_2`, A.v. Manteuffel])

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} = \begin{pmatrix} -\frac{-1-\varepsilon+x}{(x-1)x} & -\frac{2}{(x-1)x} & 0 \\ \frac{\varepsilon(3\varepsilon+2)}{4(x-1)} & -\frac{-2-\varepsilon+3x+3\varepsilon x}{2(x-1)x} & -\frac{\varepsilon+1}{2(x-1)} \\ -\frac{\varepsilon(3\varepsilon+2)(x-2)}{4(x-1)x} & \frac{-2-5\varepsilon+x+3\varepsilon x}{2(x-1)x} & \frac{(-2\varepsilon-x+\varepsilon x)}{2(x-1)x} \end{pmatrix} \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \hat{R}_2(x) \\ -\hat{R}_2(x) \end{pmatrix}$$

where

$$\begin{aligned} \hat{R}_1(x) =& \frac{\hat{B}_4(x)}{(x-1)x}, \\ \hat{R}_2(x) =& \frac{-(\varepsilon+2)^3}{16(\varepsilon+1)(x-1)x}\hat{B}_1(x) + \frac{(\varepsilon+2)(3\varepsilon+4)\left(19\varepsilon^2+36\varepsilon+16\right)}{16\varepsilon(5\varepsilon+6)(x-1)x}\hat{B}_2(x) \\ &+ \frac{(\varepsilon+1)^2(3\varepsilon+4)^2}{2\varepsilon(5\varepsilon+6)x}\hat{B}_3(x) + \frac{-24-50\varepsilon-25\varepsilon^2+8x+14\varepsilon x+6\varepsilon^2 x}{4(5\varepsilon+6)(x-1)x}\hat{B}_4(x) \end{aligned}$$

$\hat{B}_1(x), \hat{B}_2(x), \hat{B}_3(x)$ have been solved with symbolic summation.

Exploit their series representation

$$\hat{I}_1(x) = \sum_{N=0}^{\infty} I_1(N) x^N$$

$$\hat{I}_2(x) = \sum_{N=0}^{\infty} I_2(N) x^N$$

$$\hat{I}_3(x) = \sum_{N=0}^{\infty} I_3(N) x^N$$

DE system → REC system

$$
\begin{aligned}
D_x \hat{I}_1(x) = &- \tfrac{(-\varepsilon+x-1)}{(x-1)x}\hat{I}_1(x) - \tfrac{2}{(x-1)x}\hat{I}_2(x)\\
&+ \tfrac{1}{(x-1)x}\hat{B}_1(x) + \dots
\end{aligned}
$$

$$
\begin{aligned}
D_x \hat{I}_2(x) = &\tfrac{\left(3(\varepsilon+4)^2-22(\varepsilon+4)+40\right)}{4(x-1)}\hat{I}_1(x)\\
&+ \tfrac{(-(\varepsilon+4)(3x-1)+9x-2)}{2(x-1)x}\hat{I}_2(x) - \tfrac{(\varepsilon+1)}{2(x-1)}\hat{I}_3(x)\\
&+ \tfrac{1}{4}\tfrac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x) + \dots
\end{aligned}
$$

$$
\begin{aligned}
D_x \hat{I}_3(x) = &- \tfrac{\left(3(\varepsilon+4)^2(x-2)-22(\varepsilon+4)(x-2)+40x-80\right)}{4(x-1)x}\hat{I}_1(x)\\
&+ \tfrac{((\varepsilon+4)(3x-5)-11x+18)}{2(x-1)x}\hat{I}_2(x) - \tfrac{(-(\varepsilon+4)(x-2)+5x-8)}{2(x-1)x}\hat{I}_3(x)\\
&- \tfrac{1}{4}\tfrac{\left((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\right)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x) + \dots
\end{aligned}
$$

DE system → REC system

$$D_x \hat{I}_1(x) = -\tfrac{(-\varepsilon+x-1)}{(x-1)x}\hat{I}_1(x) - \tfrac{2}{(x-1)x}\hat{I}_2(x) + \tfrac{1}{(x-1)x}\hat{B}_1(x) + \ldots$$

$$\begin{aligned} D_x \hat{I}_2(x) = {} & \frac{\big(3(\varepsilon+4)^2-22(\varepsilon+4)+40\big)}{4(x-1)}\hat{I}_1(x) \\ & + \frac{(-(\varepsilon+4)(3x-1)+9x-2)}{2(x-1)x}\hat{I}_2(x) - \frac{(\varepsilon+1)}{2(x-1)}\hat{I}_3(x) \\ & + \frac{1}{4}\frac{\big((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\big)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x) + \ldots \end{aligned}$$

$$\begin{aligned} D_x \hat{I}_3(x) = {} & -\frac{\big(3(\varepsilon+4)^2(x-2)-22(\varepsilon+4)(x-2)+40x-80\big)}{4(x-1)x}\hat{I}_1(x) \\ & + \frac{((\varepsilon+4)(3x-5)-11x+18)}{2(x-1)x}\hat{I}_2(x) - \frac{(-(\varepsilon+4)(x-2)+5x-8)}{2(x-1)x}\hat{I}_3(x) \\ & - \frac{1}{4}\frac{\big((\varepsilon+4)^2(6x-25)-2(\varepsilon+4)(17x-75)+48x-224\big)}{(5\varepsilon+6)(x-1)x}\hat{B}_1(x) + \ldots \end{aligned}$$

DE system $\rightarrow$ REC system

$$\begin{aligned} D_x\hat{I}_1(x) = &- \tfrac{(-\varepsilon+x-1)}{(x-1)x}\hat{I}_1(x)\\ &- \tfrac{2}{(x-1)x}\hat{I}_2(x)\\ &+ \tfrac{1}{(x-1)x}\hat{B}_1(x) + \dots \end{aligned}$$

DE system $\rightarrow$ REC system

$$\begin{aligned}(x-1)xD_x\hat{I}_1(x) =& -(-\varepsilon+x-1)\hat{I}_1(x)\\ &-2\hat{I}_2(x)\\ &+\hat{B}_1(x)+\dots\end{aligned}$$

DE system → REC system

$$
\begin{aligned}
(x-1)xD_x\sum_{N=0}^{\infty}I_1(N)x^N = & -(-\varepsilon+x-1)\sum_{N=0}^{\infty}I_1(N)x^N \\
& -2\sum_{N=0}^{\infty}I_2(N)x^N \\
& +\sum_{N=0}^{\infty}B_1(N)x^N+\dots
\end{aligned}
$$

DE system → REC system

$$
\begin{aligned}
(x-1)x\sum_{N=1}^{\infty} I_1(N) N x^{N-1} = & -(-\varepsilon + x - 1)\sum_{N=0}^{\infty} I_1(N) x^N \\
& -2\sum_{N=0}^{\infty} I_2(N) x^N \\
& +\sum_{N=0}^{\infty} B_1(N) x^N + \dots
\end{aligned}
$$

DE system $\rightarrow$ REC system

$$
\begin{aligned}
(x-1)x\sum_{N=1}^{\infty} I_1(N)Nx^{N-1} = &-(-\varepsilon+x-1)\sum_{N=0}^{\infty} I_1(N)x^N \\
&-2\sum_{N=0}^{\infty} I_2(N)x^N \\
&+\sum_{N=0}^{\infty} B_1(N)x^N + \dots
\end{aligned}
$$

$\downarrow$ Nth coefficient

$$
NI_1(N-1)-(\varepsilon+N+1)I_1(N)+2I_2(N)=B_1(N)+\dots
$$

DE system → REC system

$$NI_1(N-1)-(\varepsilon+N+1)I_1(N)+2I_2(N)$$
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A coupled system of difference equations

$$\begin{aligned}
&NI_1(N-1)-(\varepsilon+N+1)I_1(N)+2I_2(N)\\
&\qquad=B_1(N)+\dots
\end{aligned}$$

$$\begin{aligned}
&2(\varepsilon+2N+2)I_2(N)-2(3\varepsilon+2N+1)I_2(N-1)\\
&\qquad+\varepsilon(3\varepsilon+2)I_1(N-1)-2(\varepsilon+1)I_3(N-1)\\
&\qquad=(5\varepsilon+4)B_1(N)-\frac{2(\varepsilon+1)(3\varepsilon+4)}{5\varepsilon+6}B_1(N-1)+\dots
\end{aligned}$$

$$\begin{aligned}
&4(\varepsilon-N)I_3(N)-2\varepsilon(3\varepsilon+2)I_1(N)+\varepsilon(3\varepsilon+2)I_1(N-1)\\
&\qquad-2(3\varepsilon+1)I_2(N-1)+2(5\varepsilon+2)I_2(N)\\
&\qquad-2(\varepsilon-2N+1)I_3(N-1)\\
&\qquad=-\frac{2(\varepsilon+1)(3\varepsilon+4)}{5\varepsilon+6}B_1(N-1)+(5\varepsilon+4)B_1(N)+\dots
\end{aligned}$$

A coupled system of difference equations

$$\begin{aligned} &NI_1(N-1)-(\varepsilon+N+1)I_1(N)+2I_2(N)\\ &\qquad=+\,\tfrac{4(N+2)}{3(N+1)}\varepsilon^{-3}+\Big(\tfrac{2(2N+1)}{3(N+1)}S_1(N)-\tfrac{2\left(6N^2+13N+8\right)}{3(N+1)^2}\Big)\varepsilon^{-2}+\dots\end{aligned}$$

$$\begin{aligned} &2(\varepsilon+2N+2)I_2(N)-2(3\varepsilon+2N+1)I_2(N-1)\\ &\qquad+\varepsilon(3\varepsilon+2)I_1(N-1)-2(\varepsilon+1)I_3(N-1)\\ &\qquad=\tfrac{8}{3}\varepsilon^{-3}+\Big(\tfrac{4}{3}S_1(N)^2-\tfrac{4(N+1)}{N}S_1(N)+\tfrac{4}{3}S_2(N)+\zeta_2+6\Big)\varepsilon^{-1}+\dots\end{aligned}$$

$$\begin{aligned} &4(\varepsilon-N)I_3(N)-2\varepsilon(3\varepsilon+2)I_1(N)+\varepsilon(3\varepsilon+2)I_1(N-1)\\ &\qquad-2(3\varepsilon+1)I_2(N-1)+2(5\varepsilon+2)I_2(N)\\ &\qquad-2(\varepsilon-2N+1)I_3(N-1)\\ &\qquad=-\,\tfrac{8}{3}\varepsilon^{-3}-\Big(\tfrac{8}{3}S_1(N)-4\Big)\varepsilon^{-2}\\ &\qquad\quad-\Big(\tfrac{4}{3}S_1(N)^2-\tfrac{4(N+1)}{N}S_1(N)+\tfrac{4}{3}S_2(N)+\zeta_2+6\Big)\varepsilon^{-1}+\dots\end{aligned}$$

Solving a coupled recurrence system

$$A_0 \begin{pmatrix} I_1(N) \\ I_2(N) \\ I_3(N) \end{pmatrix} + A_1 \begin{pmatrix} I_1(N+1) \\ I_2(N+1) \\ I_3(N+1) \end{pmatrix} = \begin{pmatrix} r_1(N) \\ r_2(N) \\ r_3(N) \end{pmatrix}$$

with

$$A_0 = \begin{pmatrix} N+1 & 0 & 0 \\ \varepsilon(3\varepsilon+2) & -2(3\varepsilon+1) & -2(-1+\varepsilon-2N) \\ -\varepsilon(3\varepsilon+2) & 2(3+3\varepsilon+2N) & 2(\varepsilon+1) \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -2-\varepsilon-N & 2 & 0 \\ -2\varepsilon(3\varepsilon+2) & 2(5\varepsilon+2) & 4(-1+\varepsilon-N) \\ 0 & -2(4+\varepsilon+2N) & 0 \end{pmatrix}$$

and

$$r_1(N) = (-\frac{4(N+3)}{3(N+2)}\varepsilon^{-3} + \left(\frac{2}{3}\frac{6N^3+29N^2+45N+21}{(N+1)(N+2)^2} - \frac{2(2N+3)S_1(N)}{3(N+2)}\right)\varepsilon^{-2} + O(\varepsilon^{-1})$$

$$r_2(N) = -\frac{8}{3}\varepsilon^{-3} + \left(\frac{4(3N+1)}{3(N+1)} - \frac{8S_1(N)}{3}\right)\varepsilon^{-2} + O(\varepsilon^{-1}),$$

$$r_3(N) = \frac{8}{3}\varepsilon^{-3} + \left(\frac{-4(3N+1)}{3(N+1)} + \frac{8S_1(N)}{3}\right)\varepsilon^{-2} + O(\varepsilon^{-1})$$

Solving a coupled recurrence system

$$A_0 \begin{pmatrix} I_1(N) \\ I_2(N) \\ I_3(N) \end{pmatrix} + A_1 \begin{pmatrix} I_1(N+1) \\ I_2(N+1) \\ I_3(N+1) \end{pmatrix} = \begin{pmatrix} r_1(N) \\ r_2(N) \\ r_3(N) \end{pmatrix}$$

uncoupling algorithms $\downarrow$ `Gerhold's OreSys.m (using Zürcher's algorithm)`

$$I_1(N) = \varepsilon^{-3}\left[\tfrac{59N^2+120N+49}{9(N+1)^2} - \tfrac{2(N+3)}{3(N+1)} S_1(N)\right]$$
$$+\,\varepsilon^{-2}\left[\tfrac{-2(20N^3+58N^2+57N+22)}{3(N+1)^3} + \tfrac{2(N+2)(2N-1)S_1(N)}{3(N+1)^2} - \tfrac{S_1(N)^2}{N+1} - \tfrac{S_2(N)}{N+1}\right] + O(\varepsilon^{-1})$$

$\downarrow$ insert

$$I_2(N) = \text{LinearCombination}(I_1(N), I_1(N+1), I_1(N+2))$$
$$I_3(N) = \text{LinearCombination}(I_1(N), I_1(N+1), I_1(N+2))$$

Solving a coupled differential system

In[13]:= $\mathbf{coupledDESys = D[\{\hat{I_1}(x), \hat{I}_2(x), \hat{I}_3(x)\}, x] - A.\{\hat{I_1}(x), \hat{I}_2(x), \hat{I}_3(x)\};}$

In[14]:= $\mathbf{rhs = \{\hat{R}_1(x), \hat{R}_2(x), -\hat{R}_2(x)\}}$ in power series representation;

In[15]:= $\mathbf{SolveCoupledDESystem[coupledDESys, \{I_1[x], I_2[x], I_3[x]\}, \varepsilon, -3, \{-2, -2, -2\}, rhs, \ldots]}$

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Out[15]=
$$\Big\{\frac{1}{\varepsilon^3}\Big(\frac{4\big(3N^2+6N+4\big)}{3(N+1)^2}+\frac{4S_1[N]}{3(N+1)}\Big)+\frac{1}{\varepsilon^2}\Big(-\frac{2(20N^3+58N^2+57N+22)}{3(N+1)^3}+\frac{2(N+2)(2N-1)S_1[N]}{3(N+1)^2}-\frac{S_1[N]^2}{N+1}-\frac{S_2[N]}{N+1}\Big),$$
$$\frac{4}{3\varepsilon^3}-\frac{2}{\varepsilon^2},\frac{8}{3\varepsilon^3}+\frac{1}{\varepsilon^2}\Big(-\frac{4\big(4N^2+7N+2\big)}{3(N+1)^2}+\frac{4(N+2)S_1[N]}{3(N+1)}\Big)\Big\}$$

Theorem 4. Given power series $\hat{I}_1(x), \dots, \hat{I}_n(x)$ with

$$\hat{I}_k(x) \quad = \quad \sum_{N=0}^{\infty} \Big(I_{k,-3}(N), \varepsilon^{-3} \; + \; \cdots \; + \; I_{k,l}(N)\,\varepsilon^{l} \; + \; O(\varepsilon^{l+1}) \Big) x^N$$

specified by the initial values $I_{k,j}(0)$ and the differential system

$$A_0 \begin{pmatrix} \hat{I}_1(x) \\ \vdots \\ \hat{I}_n(x) \end{pmatrix} + A_1 D_x \begin{pmatrix} \hat{I}_1(x) \\ \vdots \\ \hat{I}_n(x) \end{pmatrix} \cdots + A_\delta D_x^\delta \begin{pmatrix} \hat{I}_1(x) \\ \vdots \\ \hat{I}_n(x) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{N=0}^{\infty} \Big(r_{1,-3}(N)\,\varepsilon^{-3} + \cdots + r_{1,l_1}(N)\,\varepsilon^{l_1} + O(\varepsilon^{l_1+1}) \Big) x^N \\ \vdots \\ \sum_{N=0}^{\infty} \Big(r_{n,-3}(N)\,\varepsilon^{-3} + \cdots + r_{n,l_n}(N)\,\varepsilon^{l_n} + O(\varepsilon^{l_n+1}) \Big) x^N \end{pmatrix}$$

$A_0, \dots, A_d$: $n \times n$ matrices with entries from $\mathbb{K}(\varepsilon, x)$;
$r_{i,j}(N)$: nested sums and products.

`Sigma` computes, if possible, a representation of $I_{k,-3}(N), \dots, I_{k,l}(N)$ in

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SUMMARY: the pure REC approach

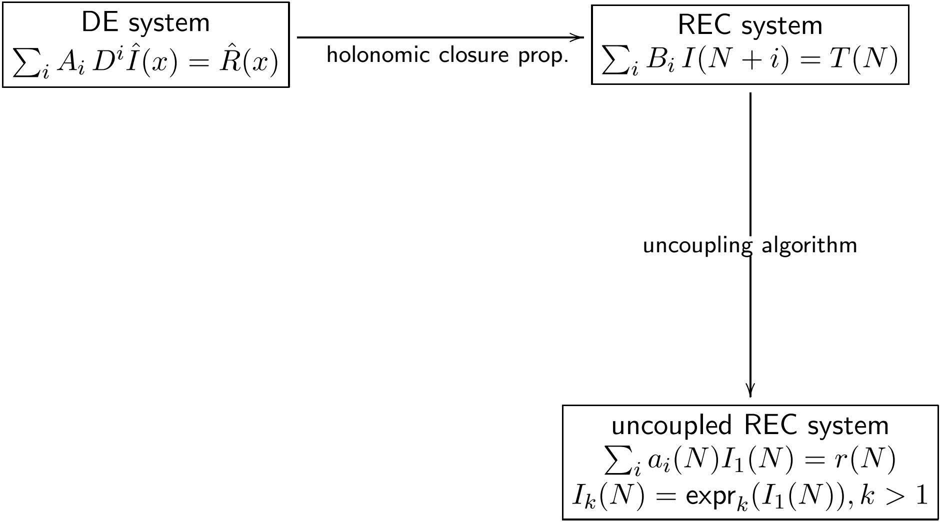
DE system

$$\boxed{\textstyle\sum_i A_i\, D^i \hat{I}(x) = \hat{R}(x)}$$

SUMMARY: the pure REC approach

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{DE system}\\ \sum_i A_i\, D^i \hat{I}(x) = \hat{R}(x)\end{array}} \arrow[r, "\text{holonomic closure prop.}"'] & \boxed{\begin{array}{c}\text{REC system}\\ \sum_i B_i\, I(N+i) = T(N)\end{array}}
\end{tikzcd}$$

SUMMARY: the pure REC approach



DE system

$\sum_i A_i\, D^i \hat{I}(x) = \hat{R}(x)$

holonomic closure prop.

REC system

$\sum_i B_i\, I(N+i) = T(N)$

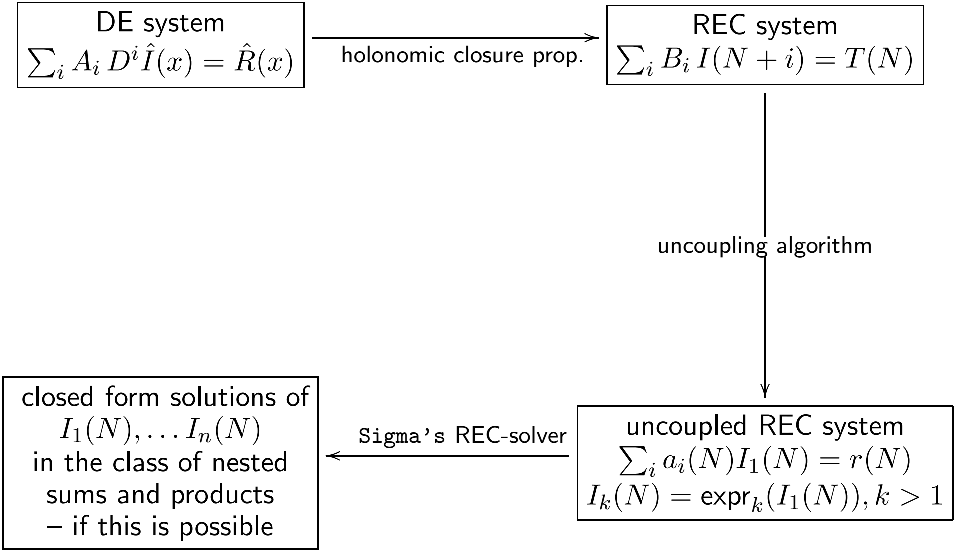
uncoupling algorithm

uncoupled REC system

$\sum_i a_i(N) I_1(N) = r(N)$

$I_k(N) = \mathsf{expr}_k(I_1(N)), k > 1$

SUMMARY: the pure REC approach



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`Sigma's` REC-solver

closed form solutions of
$I_1(N), \dots I_n(N)$
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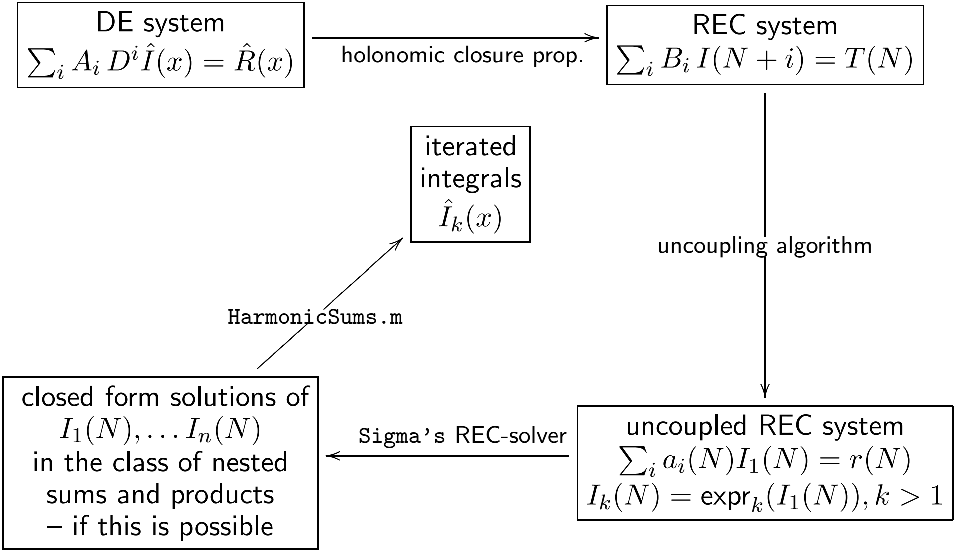

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`HarmonicSums.m`

iterated
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$$
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&+\sum_{N=0}^{\infty} B_1(N)x^N+\dots
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DE system $\to$ REC system

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DE system → REC system

$$(x-1)x\sum_{N=1}^{\infty} I_1(N)Nx^{N-1} = -\left(-\varepsilon+x-1\right)\sum_{N=0}^{\infty} I_1(N)x^N$$

$$-2\sum_{N=0}^{\infty} I_2(N)x^N$$

$$+\sum_{N=0}^{\infty} B_1(N)x^N+\dots$$

$\downarrow$ Nth coefficient

$$NI_1(N-1)-(\varepsilon+N+1)I_1(N)+2I_2(N)=B_1(N)+\dots$$

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Exploit nice output of IBP!

A coupled differential system for $\hat{I}_1(x), \hat{I}_2(x), \hat{I}_3(x)$

(produced by IBP [extension of `REDUZE_2`, A.v. Manteuffel])

$$D_x \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} = \begin{pmatrix} -\frac{-1-\varepsilon+x}{(x-1)x} & -\frac{2}{(x-1)x} & 0 \\ \frac{\varepsilon(3\varepsilon+2)}{4(x-1)} & -\frac{-2-\varepsilon+3x+3\varepsilon x}{2(x-1)x} & -\frac{\varepsilon+1}{2(x-1)} \\ -\frac{\varepsilon(3\varepsilon+2)(x-2)}{4(x-1)x} & \frac{-2-5\varepsilon+x+3\varepsilon x}{2(x-1)x} & \frac{(-2\varepsilon-x+\varepsilon x)}{2(x-1)x} \end{pmatrix} \begin{pmatrix} \hat{I}_1(x) \\ \hat{I}_2(x) \\ \hat{I}_3(x) \end{pmatrix} + \begin{pmatrix} \hat{R}_1(x) \\ \hat{R}_2(x) \\ -\hat{R}_2(x) \end{pmatrix}$$

where

$$\hat{R}_1(x) = \frac{\hat{B}_4(x)}{(x-1)x},$$

$$\hat{R}_2(x) = \frac{-(\varepsilon+2)^3}{16(\varepsilon+1)(x-1)x}\hat{B}_1(x) + \frac{(\varepsilon+2)(3\varepsilon+4)\left(19\varepsilon^2+36\varepsilon+16\right)}{16\varepsilon(5\varepsilon+6)(x-1)x}\hat{B}_2(x)$$
$$+ \frac{(\varepsilon+1)^2(3\varepsilon+4)^2}{2\varepsilon(5\varepsilon+6)x}\hat{B}_3(x) + \frac{-24-50\varepsilon-25\varepsilon^2+8x+14\varepsilon x+6\varepsilon^2 x}{4(5\varepsilon+6)(x-1)x}\hat{B}_4(x)$$

$\hat{B}_1(x), \hat{B}_2(x), \hat{B}_3(x)$ have been solved with symbolic summation.

SUMMARY: the DE-REC approach

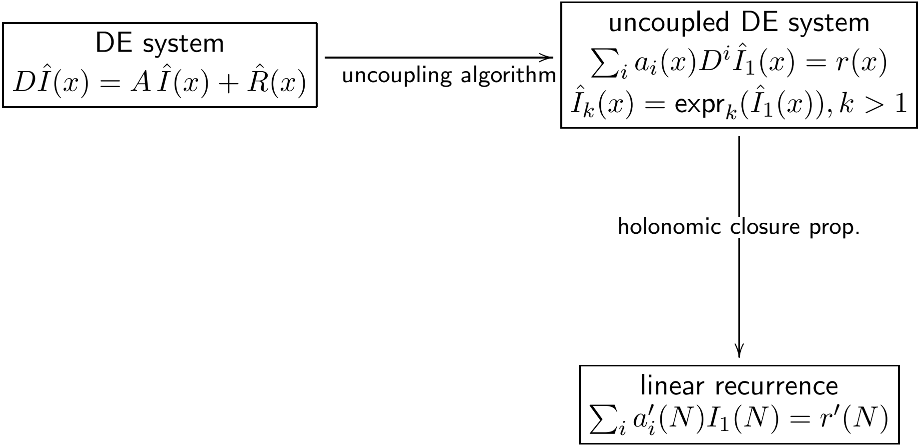
DE system

$$\boxed{D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)}$$

SUMMARY: the DE-REC approach

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{DE system}\\ D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)\end{array}} \arrow[r, "\text{uncoupling algorithm}"'] & \boxed{\begin{array}{c}\text{uncoupled DE system}\\ \sum_i a_i(x) D^i \hat{I}_1(x) = r(x)\\ \hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)), k > 1\end{array}}
\end{tikzcd}$$

SUMMARY: the DE-REC approach



DE system

$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$

uncoupling algorithm

uncoupled DE system

$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$

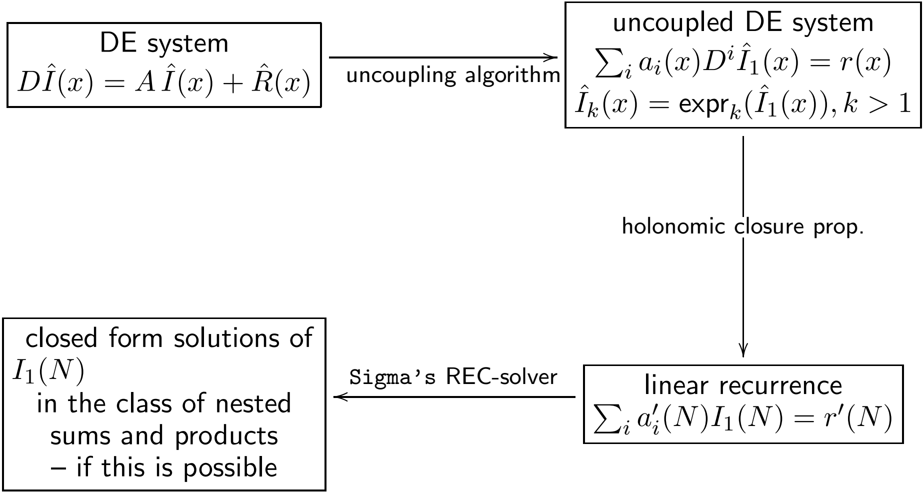
$\hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)), k > 1$

holonomic closure prop.

linear recurrence

$\sum_i a'_i(N) I_1(N) = r'(N)$

SUMMARY: the DE-REC approach



DE system
$D\hat{I}(x) = A\,\hat{I}(x) + \hat{R}(x)$

uncoupling algorithm

uncoupled DE system
$\sum_i a_i(x) D^i \hat{I}_1(x) = r(x)$
$\hat{I}_k(x) = \mathsf{expr}_k(\hat{I}_1(x)), k > 1$

holonomic closure prop.

linear recurrence
$\sum_i a'_i(N) I_1(N) = r'(N)$

`Sigma's` REC-solver

closed form solutions of
$I_1(N)$
in the class of nested
sums and products
– if this is possible

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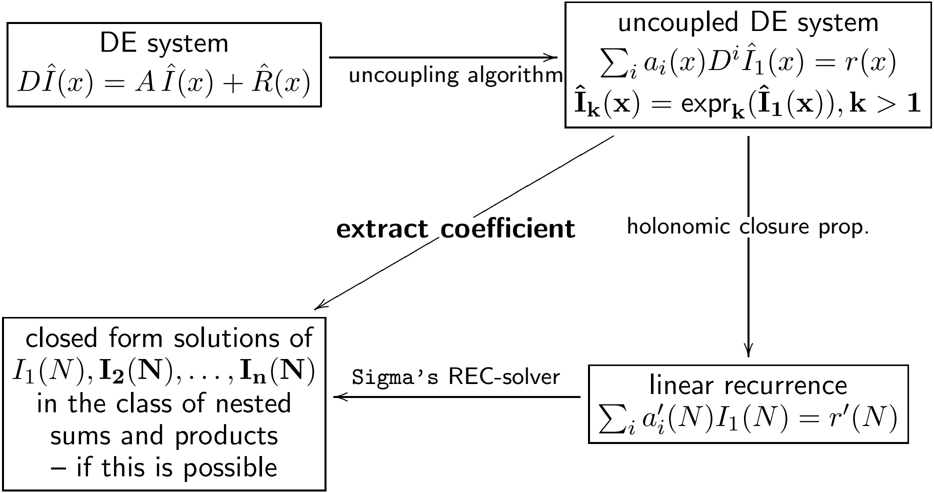

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Problem 4: Extract the Nth coefficient

Extract the Nth coefficient (example 1)

$$\frac{1}{(1-x)x}\hat{F}(x)$$

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$$= \frac{1}{x}\left(\sum_{N=0}^{\infty} x^N\right)\left(\sum_{N=2}^{\infty}\frac{x^N S_{2,1,1}(N)}{(1+N)^2(2+N)}\right)$$

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↓ Cauchy product

$$f(N) = \sum_{k=2}^{N}\frac{S_{2,1,1}(k)}{(1+k)^2(2+k)} + \frac{S_{2,1,1}(N)}{(N+2)^2(N+3)} + \frac{S_1^2(N)}{2(N+1)^2(N+2)^2(N+3)} + \frac{S_2(N)}{2(N+1)^2(N+2)^2(N+3)} + \frac{S_1(N)}{(N+1)^3(N+2)^2(N+3)} + \frac{1}{(N+1)^4(N+2)^2(N+3)}.$$

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↓ symbolic summation

$\boxed{\text{harmonic sum expression}}$

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Method implemented in the package `SumProduction`

(using `Sigma`, `HarmonicSums`, `EvaluateMultiSums`) [arXiv:1509.08324]

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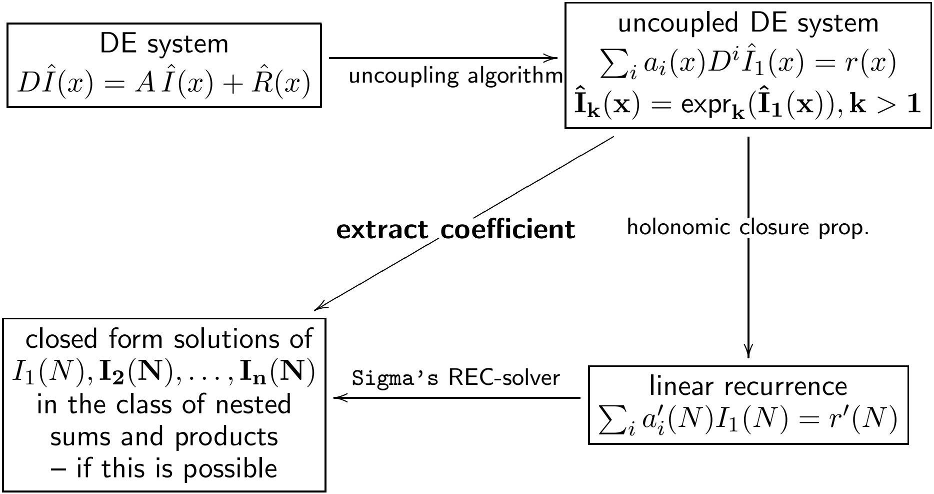

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- More involved massive 3-loop topologies are currently investigated.