Dispersion relations and differential equations for Feynman Integrals

(the two-loop massive sunrise integral and the kite)

Lorenzo Tancredi

TTP - KIT, Karlsruhe

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Based on collaboration with *E. Remiddi*

[arXiv:1311.3342], [arXiv:1602.01481]

Introduction

1- Our understanding of SM is mainly based on **perturbative calculations**

2- **High precision** requires computation of complicated Feynman diagrams

$\Downarrow$

More and more **loops** and **external legs**

3- In the last 15 years impressive progress mainly due to:

a- **Automation** of NLO calculations *(one-loop, many legs)*

b- New method for multiloop calculations: **Differential equations**

c- "(Re-)Discovery" of **Multiple Polylogarithms**

Dimensionally regularised Feynman Integrals fulfil differential equations!

[Kotikov '90, Remiddi '97, Gehrmann-Remiddi '00,...]

$\Downarrow$

Direct consequence of **Integration-by-parts (IBPs)** identities in d-dimensions!

Suppose we consider a "family" of Feynman integrals

$$\mathcal{I}(\sigma_1, ..., \sigma_s; \alpha_1, ..., \alpha_n) = \int \prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^d} \frac{S_1^{\sigma_1} \ldots S_s^{\sigma_s}}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}}$$

where

$D_n = (q_n^2 + m_n^2)\,,$ are the **propagators**

$S_n = k_i \cdot p_j\,,$ are **scalar products** among internal and external momenta

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By use of the IBPs

$$\int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \left(\frac{\partial}{\partial k_j^\mu}\, v_\mu\, \frac{S_1^{\sigma_1} \ldots S_s^{\sigma_s}}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}} \right) = 0\,, \qquad v^\mu = k_j^\mu\,, p_k^\mu$$

They are all reduced to N master integrals, $m_i(d; x_k)$ with $i = 1, \ldots, N$.

$$\Downarrow$$

Differentiating any of the masters w.r.t. the external invariants x_k, and using again the **IBPs** we end up with a system of N coupled differential equations

$$\frac{\partial}{\partial x_k} m_i(d; x_k) = \sum_{j=1}^{N} c_{ij}(d; x_k)\, m_j(d; x_k)\,.$$

The basis of masters is not unique and the **complexity** of the *differential equations* depends a lot on which integrals we choose as masters!

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- It is very well known that playing with basis of MIs, we can very often find a basis where deqs decouple as $d \to 4$, i.e. $\epsilon \to 0$

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If possible integration becomes much simpler as Laurent series in ϵ.

- Originally this was achieved by simple **trial and error**
- A possible way of systematizing it:
  Schouten identities and study IBPs in integers numbers of dimensions
  [E.Remiddi, LT '13; LT '15]

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- When this is possible, one can *(usually)* find a **Canonical Basis**
  [Kotikov, '10; Henn, '13]

$$\frac{\partial}{\partial x_{ij}} \vec{\mathcal{I}}(d; x_{ij}) = (d-4)\, A(x_{ij})\, \vec{\mathcal{I}}(d; x)$$

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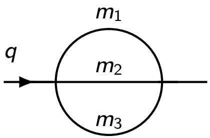
What happens if equations **cannot be decoupled**, not even for $\epsilon \to 0$?

There is only one example that has been studied thoroughly in the literature, the **massive two-loop sunrise integral**

$$S(d; q^2, m_1, m_2, m_3) =$$



It's imaginary part is known to be expressible in terms of **complete elliptic integrals** of the **first**, **second** and **third** kind.

[Broadhurst '90; Bauberger, Berends, Bohm, Buza '95; Remiddi, Laporta '06; Bloch, Kerr, Vanhove '13,'16; Adams, Bogner, Weinzier '12,'15,'16 Previous talk; **Remiddi, LT '13, '16 this talk**]

We would like to get a satisfying representation for the sunrise integral

Main issues:

1- We want control over solution *(analytic result!)*:

a) analytic continuation
b) series expansions
c) precise numerical evaluation

2- What if it appears as **inhomogeneous term** of more complicated graphs?

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A possible help in solving these problems comes from a completely orthogonal technique used previously to solve Feynman integrals: **dispersion relations**

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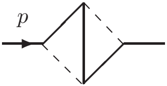
Simplest example to study is the *two-loop electron self-energy in QED*

The Kite Integral

$$\mathcal{I}(n_1, n_2, n_3, n_4, n_5) =$$



$$= \int \mathfrak{D}^d k \, \mathfrak{D}^d l \, \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}}$$

with

$$D_1 = k^2 + m^2 \,, \qquad D_2 = l^2 \,, \qquad D_3 = (k-l)^2 + m^2 \,,$$
$$D_4 = (k-p)^2 \,, \qquad D_5 = (l-p)^2 + m^2 \,,$$

and we put for simplicity

$$m^2 = 1 \,, \qquad u = p^2/m^2 = p^2$$

Note the problem was studied long ago by A. Sabry in 1962 !!!!

There are 8 independent **master integrals** which we choose as follows

$$\begin{aligned} M_1(d;u) &= \mathcal{I}(2,0,2,0,0)\,, & M_2(d;u) &= \mathcal{I}(2,0,2,1,0)\,, \\ M_3(d;u) &= \mathcal{I}(0,2,2,1,0)\,, & M_4(d;u) &= \mathcal{I}(0,2,1,2,0)\,, \\ M_5(d;u) &= \mathcal{I}(2,1,0,1,2)\,, & M_6(d;u) &= \mathcal{I}(1,0,1,0,1)\,, \\ M_7(d;u) &= \mathcal{I}(2,0,1,0,1)\,, & M_8(d;u) &= \mathcal{I}(1,1,1,1,1)\,. \end{aligned} \tag{1}$$

The first 5 Integrals are trivial $\rightarrow$ can be expressed in terms of HPLs

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We choose a **canonical basis** as follows

$$\begin{aligned} f_1(d;u) &= 4\,(d-4)^2\,M_1(d;u)\,, & f_2(d;u) &= (d-4)^2\,u\,M_2(d;u)\,, \\ f_3(d;u) &= (d-4)^2\,u\,M_3(d;u)\,, & f_4(d;u) &= (d-4)^2\,\left[2\,M_3(d;u) + (1-u)M_4(d;u)\right]\,, \\ f_5(d;u) &= (d-4)^2\,u^2\,M_5(d;u)\,, \end{aligned}$$

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The system of differential equations for **simple masters integrals** is

$$\frac{d}{du} f_i(d;u) = (d-4) \sum_{j=1}^{5} A_{ij}(u)\, f_j(d;u)\,, \quad \forall i = 1, ..., 5$$

where the matrix $A(u)$ reads

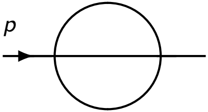
$$A(u) = \frac{1}{u} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 3/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} + \frac{1}{u-1} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1/8 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

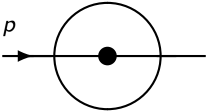
$$\Downarrow$$

Integration in terms of HPLs is straightforward

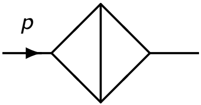
What about the **remaining integrals**?

a) $M_6(d;u)$, $M_7(d;u)$ are **two-loop massive sunrise** with equal masses.

$M_6(d;u) =$ 

$M_7(d;u) =$ 

b) $M_8(d;u)$ is the **Kite**.

$M_8(d;u) =$ 

For **Sunrise** we start from the basis (just trivial rescaling)

$$f_6(d;u) = (d-4)^2\, M_6(d;u)\,,$$
$$f_7(d;u) = (d-4)^2\, M_7(d;u)\,,$$

They satisfy a system of **two coupled differential equations**

$$u\,\frac{d}{du} f_6(d;u) = -f_6(d;u) + 3 f_7(d;u) + (d-2) f_6(d;u)\,,$$

$$\begin{aligned} u(u-1)(u-9)\,\frac{d}{du} f_7(d;u) &= (u-3) f_6(d;u) - (u^2-9) f_7(d;u) \\ &+ (d-2)\left[-\frac{5}{2}(u-3) f_6(d;u) + \frac{u^2+10u-27}{2} f_7(d;u)\right] \\ &+ (d-2)^2 \frac{3\,(u-3)}{2} f_6(d;u) - \frac{u}{2}\, f_1(d)\,. \end{aligned}$$

Homogeneous system is coupled! How can we solve it?

1) Two coupled equations → solution requires finding **two independent** sets of solutions (four functions!!)

2) Note that $f_1(d)$ is tadpole → does not depend on u, **never** develops an imaginary part

3) This implies that **the imaginary part** of the two master integrals must fulfil the **homogeneous** differential equations!!!

⇓

The imaginary part can be computed by different means
→ **Cutkoski-Veltman rules**!

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The **simplest calculation** is in $d = 2$. Imaginary part of the two functions is, for $9 < u < \infty$

$$\frac{1}{\pi}\mathrm{Im}(f_6(d=2;u)) = I(0,u)$$
$$\frac{1}{\pi}\mathrm{Im}(f_7(d=2;u)) = \frac{1}{(u-1)(u-9)}\left[\frac{u^2-6\,u+21}{6}\,I(0,u) - \frac{1}{2}I(2,u)\right],$$

where we introduced the functions

$$I(n,u) = \int_4^{(\sqrt{u}-1)^2} db\,\frac{b^n}{\sqrt{R_4(b,u)}}\,, \qquad R_4(b,u) = b(b-4)((\sqrt{u}-1)^2-b)((\sqrt{u}+1)^2-b)$$

By integration-by-parts one can show that there are **2 master integrals**, we choose

$$I_1^{(9,\infty)}(u) = I(0,u)\,, \qquad I_2^{(9,\infty)}(u) = I(2,u)$$

$\rightarrow$ directly related to the complete elliptic integrals of first and second kind. Also note they are **real** analytic functions for $9 < u < \infty$.

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Given this piece of information, we proceed as follows

1- We want sunrise integrals for $d \approx 4$. Use Tarasov's shifts [Tarasov, '96] to find a basis of master integrals with **"simple"** imaginary parts for $d = 4$

$$\frac{1}{\pi}\mathrm{Im}\, h_6(4;u) = I(0,u)\,,$$
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2- The new masters are **linear combinations** of the previous one, for example

$$h_6(d,u) = \frac{12\,(d-3)\,(3\,d-8)}{(u-1)(u-9)}\,f_6(d,u) + \frac{24\,(d-3)(u+3)}{(u-1)(u-9)}\,f_7(d,u) - \frac{3\,(u-3)}{(u-1)(u-9)}\,f_1(d)$$

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With the new basis the system of differential equations becomes

$$\frac{d}{du}\begin{pmatrix} h_6 \\ h_7 \end{pmatrix} = B(u)\begin{pmatrix} h_6 \\ h_7 \end{pmatrix} + (d-4)\,D(u)\begin{pmatrix} h_6 \\ h_7 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

where the two matrices $B(u), D(u)$ are defined as

$$B(u) = \frac{1}{6\,u(u-1)(u-9)}\begin{pmatrix} 3(3+14u-u^2) & -9 \\ (u+3)(3+75u-15u^2+u^3) & -3(3+14u-u^2) \end{pmatrix}$$

$$D(u) = \frac{1}{6\,u(u-9)(u-1)}\begin{pmatrix} 6\,u(u-1) & 0 \\ (u+3)(9+63u-9u^2+u^3) & 3(u+1)(u-9) \end{pmatrix}$$

Four **regular singular points**: $u = 0, 1, 9, \pm\infty$

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First step to solve problem is find solution of **homogeneous system** in regions

$$\{(-\infty,0)\,,\ (0,1)\,,\ (1,9)\,,\ (9,\infty)\}$$

It is straightforward to check that the two imaginary parts constitute a **first solution** of the system for $9 < u < \infty$. Recalling

$$I_1^{(9,\infty)}(u) = I(0,u)\,, \qquad I_2^{(9,\infty)}(u) = I(2,u)$$

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In order to solve completely the problem we need a second set of solutions.

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The study of the **imaginary part** gives us again a hint

Consider the new functions defined integrating the same fouth-order polynomial between two other adjacent roots

$$J(n,u) = \int_0^4 db \, \frac{b^n}{\sqrt{-R_4(d,u)}}\,, \qquad R_4(b,u) = b(b-4)((\sqrt{u}-1)^2-b)((\sqrt{u}+1)^2-b)$$

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We find in this way a matrix of solutions, defined for $9 < u < \infty$

$$G^{(9,\infty)}(u) = \begin{pmatrix} I_1^{(9,\infty)}(u) & J_1^{(9,\infty)}(u) \\ I_2^{(9,\infty)}(u) & J_2^{(9,\infty)}(u) \end{pmatrix}$$

such that

$$\frac{d}{du} G^{(9,\infty)}(u) = B(u)\, G^{(9,\infty)}(u)\,.$$

Fundamental property: **the Wronskian** of the solutions is trivial

$$W^{(9,\infty)}(u) = \det G^{(9,\infty)}(u) = I_1^{(9,\infty)}(u) J_2^{(9,\infty)}(u) - J_1^{(9,\infty)}(u) I_2^{(9,\infty)}(u) = \pi$$

This solution is defined only for $9 < u < \infty$, i.e. the functions $I_k^{(9,\infty)}(u)$ and $J_k^{(9,\infty)}(u)$ develop **branch cuts** as $u < 9$.

By properly choosing the **integration boundaries** and the **sign** in of the *fourth-order polynomial* $R_4(b,u)$, find solutions valid in the remaining regions

$$G^{(a,b)}(u)\,, \qquad \text{with} \quad (a,b) = \{(-\infty,0)\,,\ (0,1)\,,\ (1,9)\,,\ (9,\infty)\}$$

a- These provide building blocks for solution valid on the *whole phase-space.*

b- The different solutions can be **matched** in the points $u = 0, 1, 9, \pm\infty$

$$\text{At} \quad u = b \quad \text{we have} \quad G^{(b,c)}(u) = G^{(a,b)}(u)\, M^{(b)}\,,$$

c- At each point **sign imaginary part** fixed by $u \to u + i\,0^+$.

$$M^{(0)} = \begin{pmatrix} 1 & -i/2 \\ 0 & 1 \end{pmatrix}, \qquad M^{(1)} = \begin{pmatrix} 1 & 0 \\ -3\,i & 1 \end{pmatrix}$$

$$M^{(9)} = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}, \qquad M^{(\infty)} = \begin{pmatrix} -2 & 0 \\ 3\,i & -1/2 \end{pmatrix},$$

$$M^{(0)}\, M^{(1)}\, M^{(9)}\, M^{(\infty)} = \mathbf{1}$$

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We now follow **Euler** and perform the rotation

$$\begin{pmatrix} h_6(d;u) \\ h_7(d;u) \end{pmatrix} = G(u) \begin{pmatrix} m_6(d;u) \\ m_7(d;u) \end{pmatrix}$$

Thanks to **trivial Wronskian** we have

$$G^{-1}(u) = \frac{1}{\pi} \begin{pmatrix} J_2(u) & -J_1(u) \\ -I_2(u) & I_1(u) \end{pmatrix} .$$

And the new functions satisfy the equations

$$\frac{d}{du} \begin{pmatrix} m_6(d;u) \\ m_7(d;u) \end{pmatrix} = (d-4)\,\frac{1}{\pi} M(u) \begin{pmatrix} m_6(d;u) \\ m_7(d;u) \end{pmatrix} + \frac{1}{\pi} \begin{pmatrix} -J_1(u) \\ I_1(u) \end{pmatrix}$$

with $M(u) = G^{(-1)}(u)\,D(u)\,G(u)$, which **does not depend on** d !!!!

$M(u)$ contains all information needed for iteration at every order in $(d-4)$!

Let's have a closer look at the matrix $M(u)$

$$M_{11}(u) = +\frac{I_1(u)J_2(u)}{u-9} - \frac{(u+1)I_2(u)J_1(u)}{2\,u\,(u-1)} - \frac{(u+3)\,[9+\;u\,(63+(u-9)u)]\,I_1(u)J_1(u)}{6\,u\,(u-1)\,(u-9)}$$

$$M_{12}(u) = -\frac{(u+3)}{6\,u\,(u-1)\,(u-9)}\left\{[9+\;u\,(63+(u-9)u)]\,J_1^2(u) - 3(u+3)J_1(u)\,J_2(u)\right\}$$

$$M_{21}(u) = +\frac{(u+3)}{6\,u\,(u-1)\,(u-9)}\left\{[9+\;u\,(63+(u-9)u)]\,I_1^2(u) - 3(u+3)I_1(u)\,I_2(u)\right\}$$

$$M_{22}(u) = -\frac{I_2(u)J_1(u)}{u-9} + \frac{(u+1)I_1(u)J_2(u)}{2\,u\,(u-1)} + \frac{(u+3)\,[9+\;u\,(63+(u-9)u)]\,I_1(u)J_1(u)}{6\,u\,(u-1)\,(u-9)}$$

It contains rational functions in u and **products of complete elliptic integrals**

With some algebra $M(u)$ can actually be written as a **total differential** !

$$M_{11}(u) = -\frac{d}{d\,u}\left[\left(\frac{(u+3)^2}{6} I_1(u)\,J_1(u)\right) + \frac{\pi}{4}\left(2\ln(u-9) + 2\ln(u-1) - \ln(u)\right)\right],$$

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It looks like a natural **generalization** of a “simple” matrix in d-log form!

We proceed now expanding in $(d-4)$ and trying to integrate the first two non trivial orders

1- At **order zero** we get *(work for* $0 < u < 1$*)*

$$\frac{d}{du}\begin{pmatrix} m_6^{(0)}(u) \\ m_7^{(0)}(u) \end{pmatrix} = \frac{1}{\pi}\begin{pmatrix} -J_1^{(0,1)}(u) \\ I_1^{(0,1)}(u) \end{pmatrix},$$

2- The **solution** is simply

$$m_6^{(0)}(u) = c_6^{(0)} - \frac{1}{\pi}\int_0^u dt\, J_1^{(0,1)}(t)\,, \quad m_7^{(0)}(u) = c_7^{(0)} + \frac{1}{\pi}\int_0^u dt\, I_1^{(0,1)}(t)\,.$$

3- Go back to **physical** functions h_6 and h_7 and fix boundary condition

$$h_6^{(0)}(u) = \frac{1}{\pi}\left[J_1^{(0,1)}(u)\int_0^u dt\, I_1^{(0,1)}(t) - I_1^{(0,1)}(u)\left(\int_0^u dt\, J_1^{(0,1)}(t) - \mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right)\right],$$

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We recall here the definition of the **Clausen function**

$$\mathrm{Cl}_2(x) = -\int_0^x \ln\left|2\sin\frac{y}{2}\right| dy = \frac{i}{2}\left(\mathrm{Li}_2(e^{-ix}) - \mathrm{Li}_2(e^{ix})\right)$$

And it's **generalizations**

$$\mathrm{Ls}_n(\theta) = -\int_0^\theta dy\,\left[\ln\left(2\,\sin\left(\frac{y}{2}\right)\right)\right]^{n-1}$$

It is very simple to use matching matrices defined above to continue this result for $u > 9$ and extract imaginary part.

$$\mathrm{Im}(h_6^{(0)}(u)) = \theta(u-9)\,\pi\, I_1^{(9,\infty)}(u)\,, \quad \mathrm{Im}(h_7^{(0)}(u)) = \theta(u-9)\,\pi\, I_2^{(9,\infty)}(u)\,.$$

Such that the solution can be re-written also as a **dispersion relation**

$$h_6^{(0)}(u) = \int_9^\infty \frac{dt}{t-u-i\,\epsilon}\, I_1^{(9,\infty)}(t)\,,$$

$$h_7^{(0)}(u) = \frac{1}{\sqrt{3}}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + u\left(\frac{5}{6} + \sqrt{3}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) + u^2 \int_9^\infty \frac{dt}{t^2(t-u-i\,\epsilon)}\, I_2^{(9,\infty)}(t)\,, \quad \leftarrow \quad \text{doubly-subtracted!}$$

At order one we must for the first time integrate over the matrix $M(u)$!

1- The equations are

$$\frac{d}{du}\left(\begin{array}{c} m_6^{(1)}(u) \\ m_7^{(1)}(u) \end{array}\right) = \frac{1}{\pi} M(u) \left(\begin{array}{c} m_6^{(0)}(u) \\ m_7^{(0)}(u) \end{array}\right),$$

2- Repeating the same procedure for $0 < u < 1$ we get

$$\begin{aligned} h_6^{(1)}(u) = & \frac{1}{4\pi} l(u) \left(J_1^{(0,1)}(u) \int_0^u dt\, I_1^{(0,1)}(t) - I_1^{(0,1)}(u) \int_0^u dt\, J_1^{(0,1)}(t) \right) \\ & - \frac{1}{4\pi} \left(J_1^{(0,1)}(u) \int_0^u dt\, I_1^{(0,1)}(t)\, l(t) - I_1^{(0,1)}(u) \int_0^u dt\, J_1^{(0,1)}(t)\, l(t) \right) \\ & - \frac{1}{24\pi} \left[\pi^3 - 6\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) l(u) + 18\,\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) \right] I_1^{(0,1)}(u) \\ & - \frac{1}{2}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) J_1^{(0,1)}(u) \end{aligned}$$

with

$$l(u) = 2\ln(1-u) + 2\ln(9-u) - \ln(u),$$

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1- The equations are

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$$l(u) = 2\ln(1-u) + 2\ln(9-u) - \ln(u),$$

Similarly for the second integral

$$
\begin{aligned}
h_7^{(1)}(u) = &\frac{1}{4\,\pi}\, I(u)\, \left(J_2^{(0,1)}(u) \int_0^u dt\, I_1^{(0,1)}(t) - I_2^{(0,1)}(u) \int_0^u dt\, J_1^{(0,1)}(t) \right) \\
&- \frac{1}{4\,\pi} \left(J_2^{(0,1)}(u) \int_0^u dt\, I_1^{(0,1)}(t)\, I(t) - I_2^{(0,1)}(u) \int_0^u dt\, J_1^{(0,1)}(t)\, I(t) \right) \\
&- \frac{1}{24\,\pi} \left[\pi^3 - 6\, \mathrm{Cl}_2\left(\frac{\pi}{3}\right) I(u) + 18\, \mathrm{Ls}_3\left(\frac{2\,\pi}{3}\right) \right] I_2^{(0,1)}(u) - \frac{1}{2} \mathrm{Cl}_2\left(\frac{\pi}{3}\right) J_2^{(0,1)}(u) \\
&+ \frac{1}{6\,\pi}\, (u+3)^2 \left(J_1^{(0,1)}(u) \int_0^u dt\, I_1^{(0,1)}(t) - I_1^{(0,1)}(u) \int_0^u dt\, J_1^{(0,1)}(t) \right) \\
&+ \frac{1}{6\,\pi}\, (u+3)^2\, I_1^{(0,1)}(u)\, \mathrm{Cl}_2\left(\frac{\pi}{3}\right),
\end{aligned}
$$

It becomes even more interesting if we extract **imaginary part**

$$\frac{1}{\pi}\mathrm{Im}\left(h_6^{(1)}(u)\right) = \theta(u-9)\left[\frac{1}{4}\, I_1^{(9,\infty)}(u)\,\bar{I}(u) - \frac{\pi}{2} J_1^{(9,\infty)}(u)\right]$$

$$\frac{1}{\pi}\mathrm{Im}\left(h_7^{(1)}(u)\right) = \theta(u-9)\left[\frac{1}{4}\, I_2^{(9,\infty)}(u)\,\bar{I}(u) - \frac{\pi}{2} J_2^{(9,\infty)}(u) + \frac{(u+3)^2}{6}\, I_1^{(9,\infty)}(u)\right]\,,$$

It contains only products of **elliptic integrals and logarithms** !!!
We can use them to write equally compact dispersion relations

$$h_6^{(1)}(u) = \int_9^\infty \frac{dt}{t-u-i\,\epsilon}\left(\frac{1}{4}\, I_1^{(9,\infty)}(t)\,\bar{I}(t) - \frac{\pi}{2}\, J_1^{(9,\infty)}(t)\right)$$

$$h_7^{(1)}(u) = c_0 + c_1\, u + u^2 \int_9^\infty \frac{dt}{t^2(t-u-i\,\epsilon)}\left(\frac{1}{4}\, I_2^{(9,\infty)}(t)\,\bar{I}(t) - \frac{\pi}{2}\, J_2^{(9,\infty)}(t) + \frac{(t+3)^2}{6}\, I_1^{(9,\infty)}(t)\right)\,,$$

Simplest graph that contains the **two-loop sunrise** as sub-topology is the Kite integral. It's differential equation in u reads

$$\frac{d}{du} f_8(d;u) = (d-4)\left(\frac{1}{u-1} - \frac{1}{2\,u}\right) f_8(d;u) + \frac{(d-4)^3}{24}\left(1 - \frac{8}{u-1}\right) h_6(d;u) + \frac{(d-4)}{u-1}\left(-\frac{1}{8} f_1(d;u) + 2\, f_3(d;u) + f_4(d;u)\right) + (d-4)\frac{1}{u}\, f_5(d;u)$$

Expand in $(d-4)$ and insert previous results, first non trivial order is $(d-4)^3$

$$\frac{d}{du} f_8^{(3)}(u) = \frac{1}{24}\left(1 - \frac{8}{u-1}\right) h_6^{(0)}(u) + \frac{1}{u-1}\left(\frac{\pi^2}{96} - \frac{1}{16} G(0,1,u)\right) + \frac{1}{8\,u} G(1,1,u)\,.$$

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In order to be able to proceed we include sunrise as **dispersive integral**!

$$h_6^{(0)}(u) = \int_9^\infty \frac{dt}{t-u}\, I_1^{(9,\infty)}(t)$$

The integration in u becomes completely trivial! While the elliptic kernel coming from the **two-loop sunrise** factorises out of the calculation completely!

$$f_8^{(3)}(u) = \frac{1}{8}G(0,1,1,u) - \frac{1}{16}G(1,0,1,u) - \frac{\pi^2}{96}G(1,u) - \frac{1}{24}\int_9^\infty dt\, I_1^{(9,\infty)}(t)\left(1-\frac{8}{t-1}\right)G(t,u)$$

Similarly **simple** result can be obtained **one order higher**

$$
\begin{aligned}
f_8^{(4)}(u) = {} & \frac{\pi^2}{192}\left(G(0,1,u) - 2G(1,1,u)\right) + \left(\frac{\zeta_3}{32} + \frac{\pi}{12}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) G(1,u) - \frac{3}{16} G(0,0,1,1,u) \\
& - \frac{1}{32} G(0,1,0,1,u) + \frac{3}{8} G(0,1,1,1,u) + \frac{1}{32} G(1,0,0,1,u) - \frac{1}{16} G(1,1,0,1,u) \\
& - \frac{1}{96} G(1,u) \int_9^\infty dt\, I_1^{(9,\infty)}(t)\, \bar{I}(t) + \frac{\pi}{48} \int_9^\infty dt\, J_1^{(9,\infty)}(t) \left(1 - \frac{8}{t-1}\right) G(t,u) \\
& - \frac{1}{96} \int_9^\infty dt\, I_1^{(9,\infty)}(t) \left(1 - \frac{8}{t-1}\right) \bar{I}(t) \left(G(t,u) - G(1,u)\right) \\
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$\Downarrow$

MPLs of weight 4 and integrals over **elliptic integrals** and dilogarithms.

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MPLs of weight 4 and integrals over **elliptic integrals** and dilogarithms.

CONCLUSIONS

1- Differential equations is a very powerful tool to compute Feynman integrals

2- If results are MPLs, we only need to **first order, decoupled** diff. equations!

3- General case, coupled (homogeneous) equations **must be solved**
elliptic integrals or more complicated functions might appear

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5- Using **dispersion relations** together with **differential equations** provides a tool to handle *(disentangle)* complexity

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THANKS!