Five-loop massive tadpoles

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recent work with Thomas Luthe,
Andreas Maier, Peter Marquard

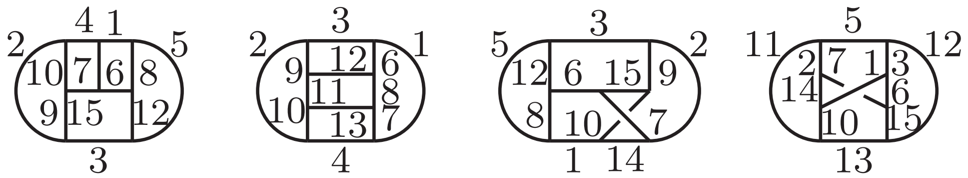
and earlier work with
J. Möller, C. Studerus

Loops&Legs 2016, Leipzig

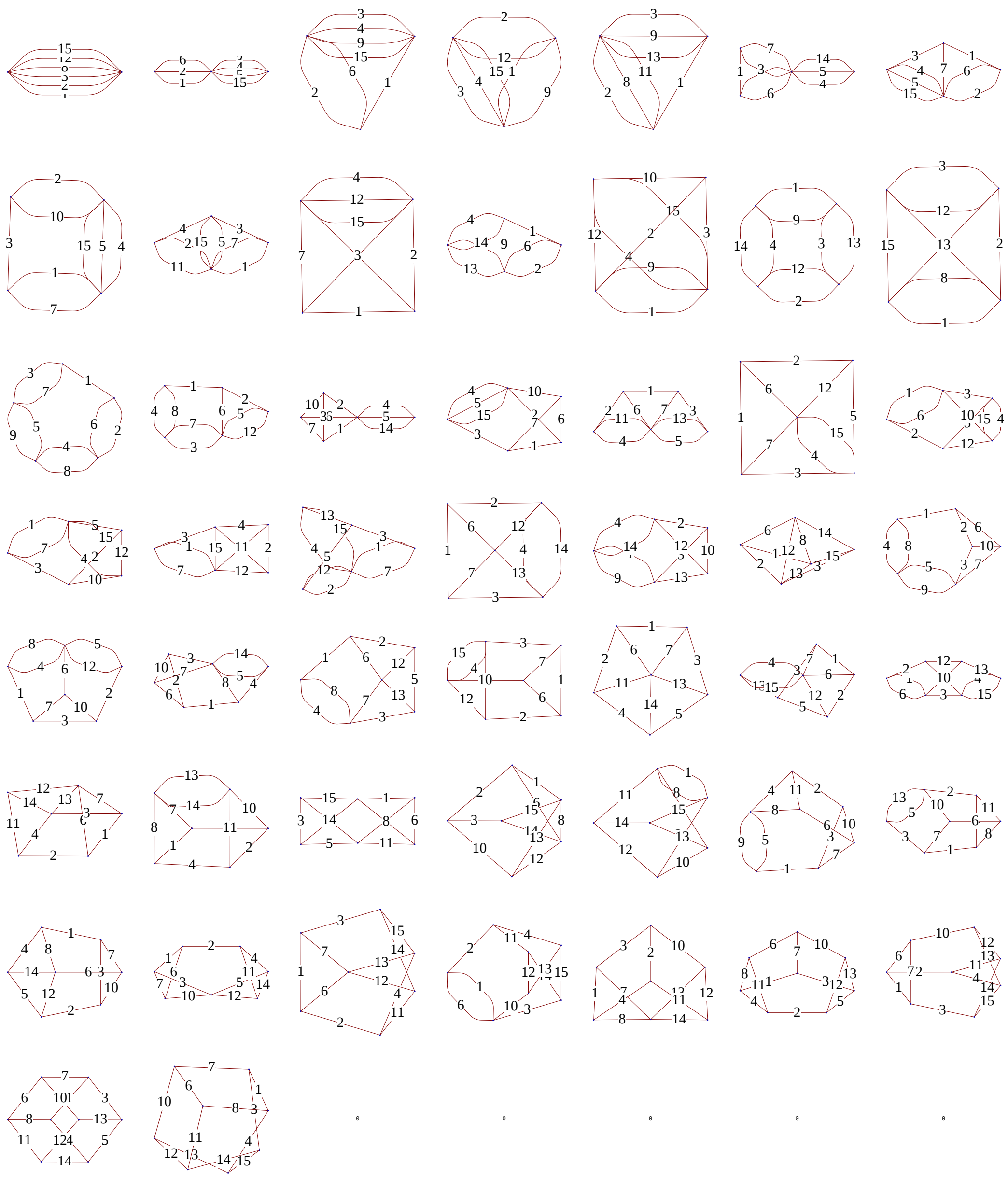
Motivation

- pressure of hot QCD
  - ▷ phenomenology needs physical NLO [Braaten/Nieto 96; KLRS 04]
  - ▷ hardest building block: 5-loop tadpoles(m,0) at $d = 3 - 2\epsilon$
- QCD beta function and anomalous dimensions
  - ▷ known since ∼20 years at 4-loop accuracy [vRitbergen/Vermaseren/Larin 97]
  - ▷ 5-loop needed e.g. for improved ρ-parameter, α_s decoupling relations, ... [4loop: 06]
  - ▷ 5-loop results appear since ∼8 years
    mainly from Karlsruhe (β^{QED}, γ_m, $\beta^{QCD}|_{N_c=3}$) [Baikov/Chetyrkin/Kühn/Rittinger 08-16]
- moments
  - ▷ many problems allow for asymptotic expansions
  - ▷ mapping on tadpoles, often for price of many dots
- in this talk: focus on master integrals
  - ▷ basic building block for 5-loop problems
  - ▷ interested in methods that allow to choose d in the end
  - ▷ main progress via refinement of Laporta approach
  - ▷ (problem-specific) reduction not covered here

Classification

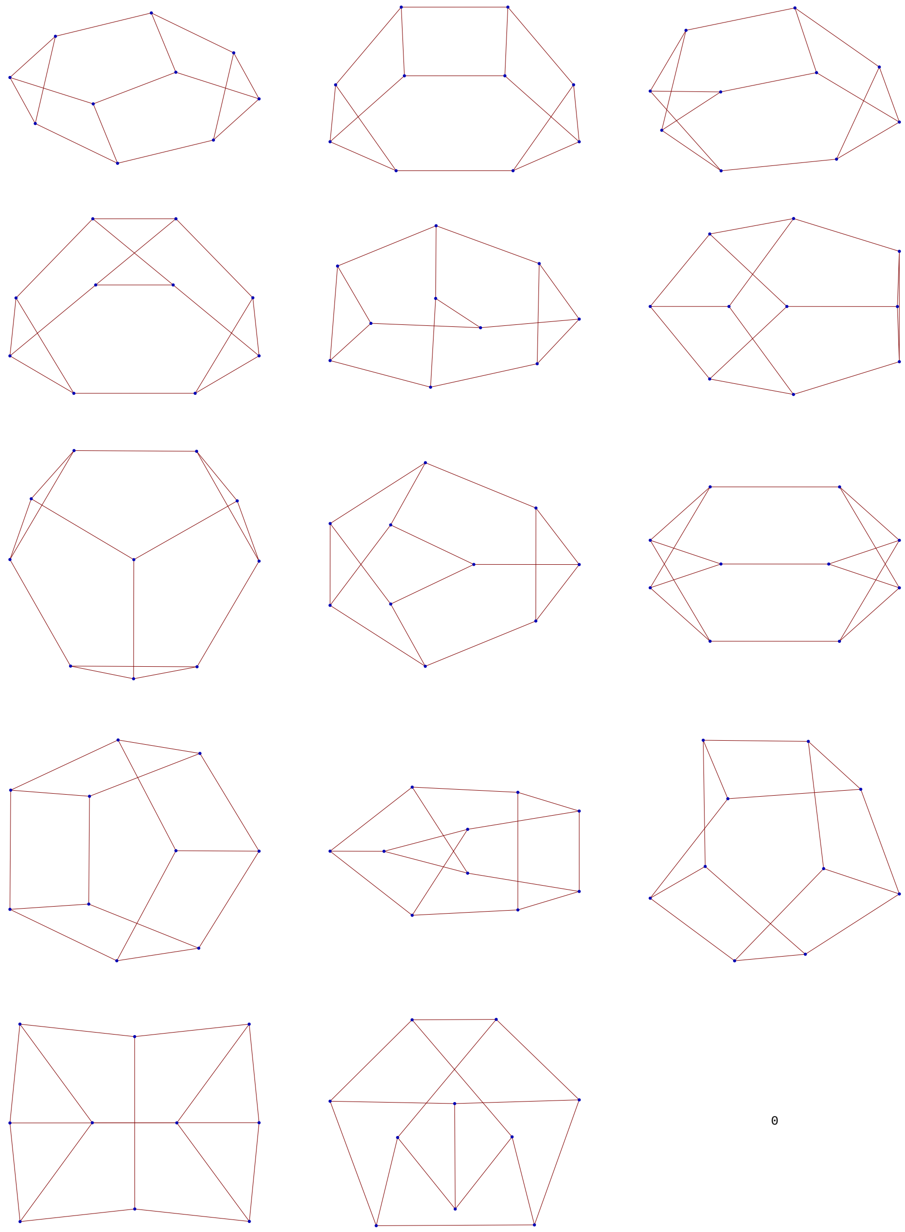
- consider fully massive 5-loop tadpoles
  - ▷ Euclidean space-time
  - ▷ same mass in all propagators $\Rightarrow 1/(q_i^2+1)$
- the 5-loop integral family needs 15 propagators / lines
  - ▷ $q_i \in \{k_1, k_2, k_3, k_4, k_5, k_{13}, k_{14}, k_{15}, k_{23}, k_{24}, k_{25}, k_{35}, k_{45}, k_{124}, k_{34}\}$
    where $k_{a\ldots bc} = k_a + \ldots + k_b - k_c$
  - ▷ trivalent graphs have 12 lines: 
- classification: label sectors by binary rep
  - ▷ identify unique graphs
  - ▷ find all isometries and corresponding momentum shifts
  - ▷ choose largest representative from each class
- normalization: divide out [1-loop tadpole]$^{\#loops}$
  - ▷ recall that in 4d, [1-loop tadpole]$\sim 1/\epsilon$

Classification: 5-loop



- some numerology
  - ▷ for specific momentum list
  - ▷ combinatorics wins!
- $L(L+1)/2 + LE = 15$ scalar products
- $2^{15} = 32768$ possible sectors
- 5151 do not correspond to a Feynman graph
- $1941 + 3625$ zero-sectors
- $22 + 21962$ shifts
- $19 + 48$ unique sectors (16 with 1-loop factors not shown)

Classification: 6-loop

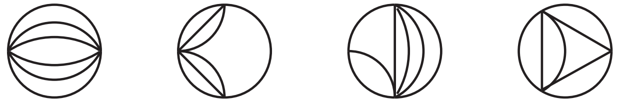


- 21 sc prod
- 2^{21} = 2.1M sectors
- 1.3M no graph
- 178K zeros
- 569K shifts
- 487 unique
  (show 3-conn. cubic graphs)

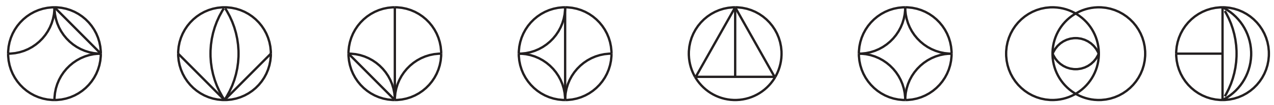
5-loop Sectors

- arrive at 48 unique 5-loop sectors (+19 factorized ones not shown)

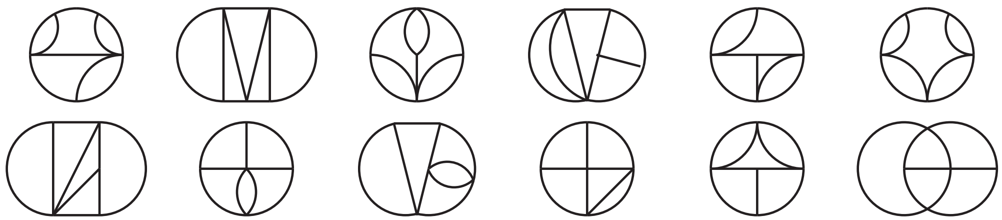
$t = 6, 7:$



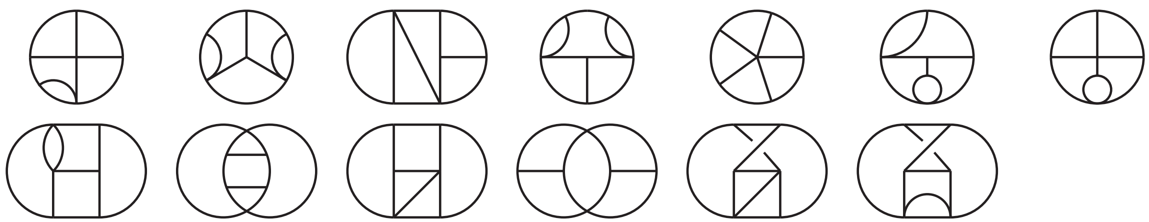
$t = 8:$



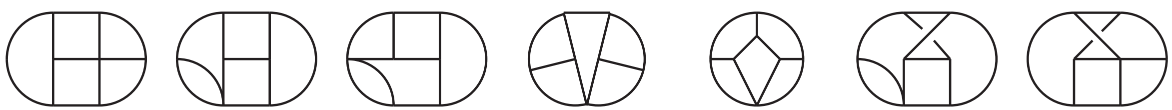
$t = 9:$



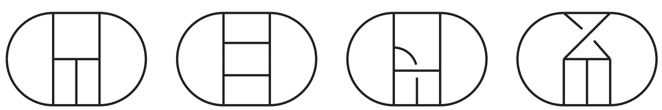
$t = 10:$



$t = 11:$



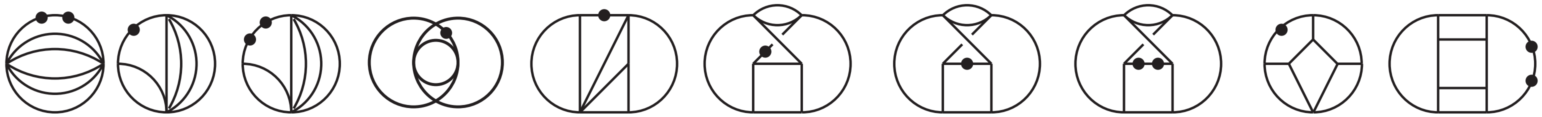
$t = 12:$



▷ recall that at 1/2/3/4-loop there were 1/1/3/10 unique sectors (plus 0/1/2/6 fact)

5-loop Masters

- a (small) IBP reduction reveals that some sectors contain multiple master integrals
  - ▷ need in addition 62 (+3 factorized ones) masters with 'dots'. some examples:



  - ▷ recall that at 1/2/3/4-loop there were 0/0/0/3 masters with 'dots'

- how to evaluate these 48+62 (+19+3) zero-scale master integrals?
  various methods, e.g.
  - ▷ explicit integration in x-space
  - ▷ differential eqs (in mass ratio); solve iteratively with HPLs
  - ▷ explicit solution of low-order difference equations: ${}_PF_{P-1}$ etc.
  - ▷ numerical solution of difference equations via factorial series [Laporta 00]

- Mathematical structure
  - ▷ interested in the coefficients of an ϵ expansion
  - ▷ in many cases, these are from a generic class of functions/numbers
  - ▷ e.g. harmonic polylogarithms HPL(x) [Remiddi/Vermaseren 00]
  - ▷ e.g. harmonic sums $S(N)$ [Vermaseren 98]
  - ▷ relation: $H_{\bar{m}}(1) \to S_{\bar{m}}(\infty)$
  - ▷ if solution numerical: use some PSLQ

Evaluation: differential equations

- perform IBP reduction with two masses: M, m
  - ▷ get differential eqn in $z = M/m$
  - ▷ use boundary values at $z = 0$ ($z = 1$)
  - ▷ use symmetry relations like $z \leftrightarrow 1/z$
  - ▷ typically, want the integral at $z = 1$ ($z = 0$)

- simple 3-loop example (basketball type)

$$B_{24}(z) \equiv \frac{1}{J^3}\int_{p_{1..3}}^{(d)} \frac{1}{p_1^2+z^2}\,\frac{1}{p_2^2+z^2}\,\frac{1}{p_3^2+1}\,\frac{1}{(p_1+p_2+p_3)^2+1}$$

$$B_{24}(z) = x^{3d-8}\,B_{24}(1/z) \quad , \quad B_{24}(0) = 2^{d-3}\,\frac{\Gamma(\frac{8-3d}{2})\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{7-2d}{2})\Gamma(\frac{2-d}{2})}$$

satisfies $\Big\{z(1-z^2)\partial_z{}^2 - 2(1-2z^2)(d-3)\partial_z - z(d-3)(3d-8)\Big\}\,B_{24}(z) = (d-2)^2\,z^{d-3}\left(z^{d-2}-1\right)$

- solution standard, via variation of constants, in terms of HPL(z); set $z = 1$ and use algebra of HPL(1) resp. $S(\infty)$

```
B4 = - 2 + ...

     + ep^4 * ( 1141/24 - 112/3*z3 ) + ...

     + ep^7 * ( 418903/192 + 2278/45*pi^4 + 32/5*pi^6 - 3840*s6 - ...) + ...
```

Evaluation: difference equations

- perform IBP reduction with symbolic power x on one line
- derive difference equation for generalized master $I(x) \equiv \int \frac{1}{D_1^x D_2 \dots D_N}$

$$\sum_{j=0}^{R} p_j(x) I(x+j) \quad = \quad F(x)$$

- typically, want $I(1)$; solve the difference equation
  - ▷ explicitly (if 1st order)
  - ▷ numerically (very general setup) [Laporta 00]
- solve via factorial series $I(x) = I_0(x) + \sum_{j=1}^{R} I_j(x)$, where

$$I_j(x) = \mu_j^x \sum_{s=0}^{\infty} a_j(s) \frac{\Gamma(x+1)}{\Gamma(x+1+s-K_j)}$$

- need boundary condition for fixing, say, $a_j(0)$: use decoupling at large x
  - ▷ $I(x) = \int_{k_1} g(k_1)/(k_1^2+1)^x \;\Rightarrow\; I(x) \sim (1)^x x^{-d/2} g(0)$

Evaluation: difference equations

- Advantages
  - ▷ high level of automation
  - ▷ works well with divergent integrals
  - ▷ does not rely on specific function classes
  - ▷ high-precision results for arbitrary ε orders
  - ▷ can expand around any dimension
  - ▷ cross-checks by putting x on different lines

- Problems and limitations
  - ▷ limited use for multi-scale integrals
  - ▷ complexity of coefficients in high-order equations
  - ▷ high orders of recurrence relations
  - ▷ instability of factorial series in numerical evaluation

- Progress and fixes
  - ▷ use coupled equations
  - ▷ reduce recurrence relations
  - ▷ predict instability factors

Choice of basis of transcendentals

- to absorb single powers of π as well as powers of $\ln 3$, def

$$h_n \equiv \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)}{\Gamma(k+1)\Gamma(1/2)} \frac{(3/4)^k}{(2k+1)^n}$$

$$H_n \equiv h_n + h_1 \,\text{Coefficient}\left[1 - \frac{3^{\epsilon/2}\Gamma(1-\epsilon)}{\Gamma^2(1-\epsilon/2)} + \mathcal{O}(\epsilon^n), \epsilon, n-1\right]$$

$$H_1 = h_1 = \frac{2\pi}{3\sqrt{3}} \,, \quad H_2 = h_2 - \frac{1}{2} h_1 \ln 3 \,, \quad etc$$

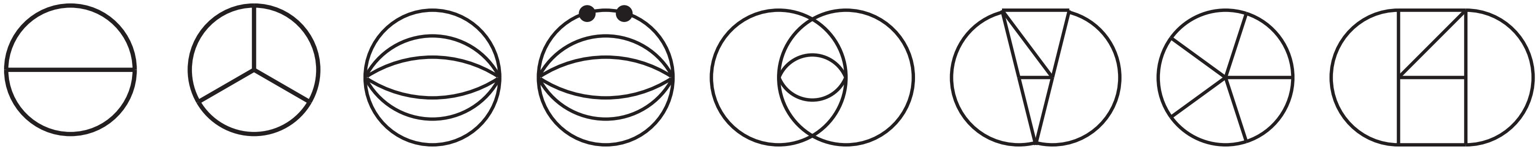
▷ $I_{7.1.1} = +(-\frac{3}{2})\,\epsilon^0 + (-\frac{3}{2})\,\epsilon^1 + (9H_2 - 3)\,\epsilon^2 + (9H_2 - 18H_3 - 6)\,\epsilon^3 + (18H_2 - 18H_3 + 36H_4 - 12)\,\epsilon^4 + \ldots$

- to absorb powers of $\ln 2$, def as elements of the MZV basis

$$A_n \equiv Li_n(\frac{1}{2}) + (-1)^n \frac{\ln^n 2}{n!}\left(1 - \frac{n(n-1)}{2}\frac{\zeta_2}{\ln^2 2}\right)$$

▷ $I_{63.1.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (-2\zeta_3)\,\epsilon^2 + (-16A_4 + 27H_2^2 + \frac{34\zeta_2^2}{5})\,\epsilon^3 + \ldots$

Sample results (4d)



$$I_{28686.1.1} = +(-3)\,\epsilon^0 + (-\frac{3}{2})\,\epsilon^1 + (\frac{13}{24})\,\epsilon^2 + (-\frac{1267}{1440})\,\epsilon^3 + (-\frac{4193}{3456})\,\epsilon^4 + $$
$$+135.95072868792871461956492733702218574897992953584\,\epsilon^5 + \ldots$$

$$I_{28686.1.3} = +(0)\,\epsilon^0 + (\frac{3}{2})\,\epsilon^1 + (-\frac{1}{2})\,\epsilon^2 + (-\frac{443}{360})\,\epsilon^3 + (\frac{95}{216})\,\epsilon^4 +$$
$$-38.292059175062436961881799538284449799148385376441\,\epsilon^5 + \ldots$$

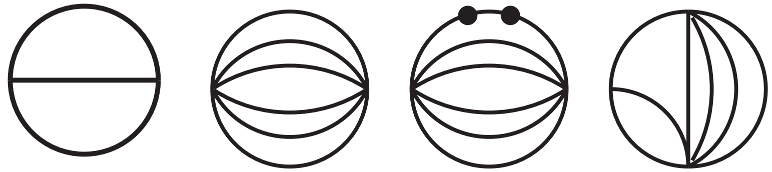
$$I_{30862.1.1} = +(-\frac{3}{5})\,\epsilon^0 + (-\frac{27}{10})\,\epsilon^1 + (-\frac{4\zeta_3}{5} - \frac{421}{60})\,\epsilon^2 + (-\frac{12\zeta_2^2}{25} + \frac{24\zeta_3}{5} + \frac{211}{24})\,\epsilon^3 + (\frac{72\zeta_2^2}{25} - 98\zeta_3 + \frac{32\zeta_5}{5} + \frac{12959}{48})\,\epsilon^4 +$$
$$+1143.1838307558764599466030303839590323268318605888\,\epsilon^5 + \ldots$$

$$I_{30231.1.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (\frac{3\zeta_3}{5})\,\epsilon^2 + (\frac{9\zeta_2^2}{25} + \frac{21\zeta_3}{5} + 3\zeta_5)\,\epsilon^3 + (-36H_2\zeta_3 + \frac{12\zeta_2^3}{7} + \frac{63\zeta_2^2}{25} - \frac{21\zeta_3^2}{5} + 27\zeta_3 - \frac{24\zeta_5}{5})\,\epsilon^4$$
$$-531.32391547725635267943444561495368318398901378435\,\epsilon^5 + \ldots$$

$$I_{32596.1.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (0)\,\epsilon^2 + (0)\,\epsilon^3 + (-14\zeta_7)\,\epsilon^4 +$$ [Wheel: Broadhurst 85]
$$+235.07729596783467131454388080950411779239347239580\,\epsilon^5 + \ldots$$

$$I_{32279.3.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (0)\,\epsilon^2 + (0)\,\epsilon^3 + (-\frac{441\zeta_7}{40})\,\epsilon^4 +$$ [Zigzag: Kazakov 83; Broadhurst/Kreimer 95; Brown/Schnetz 12]
$$+181.78223928612340820790788236018642961198741994209\,\epsilon^5 + \ldots$$

Sample results (3d)



$$
\begin{aligned}
I_{7.1.1} &= +(0)\,\epsilon^{-2} + (\tfrac{1}{4})\,\epsilon^{-1} + (-\tfrac{1}{2} + \ln 2 - \ln 3)\,\epsilon^{0} + \\
&\quad +(3\,\mathrm{Li}_2\left(\tfrac{1}{3}\right) - \zeta_2 + 2\ln^2 3 + 2\ln^2 2 - 4\ln 2\ln 3 + 2\ln 3 - 2\ln 2)\,\epsilon^{1} + \ldots \\
I_{28686.1.1} &= +(0)\,\epsilon^{-2} + (7)\,\epsilon^{-1} + (\tfrac{121}{3} - 36\ln 3)\,\epsilon^{0} + \\
&\quad +(-64\,\mathrm{Li}_2\left(\tfrac{1}{3}\right) + 104\zeta_2 + \tfrac{2794}{9} + 40\ln^2 3 + 64\ln 2\ln 3 - 216\ln 3)\,\epsilon^{1} + \\
&\quad +118.57677574582856746860787632168756206029610839677\,\epsilon^{2} + \ldots \\
I_{28686.1.3} &= +(0)\,\epsilon^{-2} + (-\tfrac{11}{32})\,\epsilon^{-1} + (\tfrac{3\ln 3}{2} - \tfrac{7}{4})\,\epsilon^{0} + \\
&\quad +(\mathrm{Li}_2\left(\tfrac{1}{3}\right) - \tfrac{7\zeta_2}{2} - 9 - \tfrac{5\ln^2 3}{2} - \ln 2\ln 3 + 9\ln 3)\,\epsilon^{1} - \\
&\quad -14.715434500778808291488578821585514439043449944829\,\epsilon^{2} + \ldots \\
I_{30214.1.1} &= +(\tfrac{1}{8})\,\epsilon^{-2} + (-\tfrac{1}{8} + \ln 2 - \ln 3)\,\epsilon^{-1} + \\
&\quad +(3\,\mathrm{Li}_2\left(\tfrac{1}{3}\right) + \tfrac{25}{4}\mathrm{Li}_2\left(\tfrac{1}{6}\right) - \tfrac{33\zeta_2}{8} - 2 + 4\ln^2 3 + \tfrac{57\ln^2 2}{8} - \tfrac{25}{4}\ln 2\ln 5 - \tfrac{7}{4}\ln 2\ln 3 + \ln 3 + 2\ln 2)\,\epsilon^{0} + \\
&\quad +16.186699562838003097806370945032650193751988027833\,\epsilon^{1} + \ldots
\end{aligned}
$$

Application: 5-loop QCD β-function

notation: $a \equiv \frac{C_A\, g^2(\mu)}{16\pi^2}$, $\partial_{\ln \mu^2} a = -a\Big[\epsilon + b_0\, a + b_1\, a^2 + b_2\, a^3 + b_3\, a^4 + b_4\, a^5 + \dots\Big]$

- computed from Z_{cc}, Z_{ccg} and Z_{gg}
- via "p-way" [similar to 3-loop: [Tarasov/Vladimirov/Zharkov 80]]
  - ▷ put mass into one propag, external $p_i = 0$, cut massive line
    $\Rightarrow$ massless $(L-1)$-loop propagators [Chetyrkin/Baikov 10; Lee/Smirnov 12; Panzer 13]
  - ▷ reduction via $1/D$-expansion [Baikov 96ff]
  - ▷ status: b_4 for all n_f, SU(3) only [see talk by K.Chetyrkin]
- via "m-way" [analogous to 4-loop: [vRitbergen/Vermaseren/Larin 97; Czakon 04]]
  - ▷ common mass m in all propags, external $p_i = 0$
    $\Rightarrow$ massive L-loop tadpoles [Luthe/Schröder]
  - ▷ reduction via Laporta-type algorithm `Crusher` [Marquard]
  - ▷ status: b_4 for n_f^4, n_f^3 [Luthe/Maier/Marquard/Schröder]

Application: 5-loop QCD β-function

notation: $a \equiv \frac{C_A\, g^2(\mu)}{16\pi^2}$, $\partial_{\ln \mu^2} a = -a\Big[\epsilon + b_0\, a + b_1\, a^2 + b_2\, a^3 + b_3\, a^4 + b_4\, a^5 + \dots\Big]$

- color: $n_f \equiv \frac{T_F\, N_f}{C_A}$, $c_f \equiv \frac{C_F}{C_A}$, $d_{FF} \equiv \frac{[sTr(T^a T^b T^c T^d)]^2}{N_A\, T_F^2\, C_A^2}$

$$3^1\, b_0 = 11 - 4\, n_f \quad , \quad \dots \quad ,$$

$$3^5\, b_4 = n_f^4 \Big[c_1\, c_f + c_2\Big] + n_f^3 \Big[c_3\, c_f^2 + c_4\, c_f + c_5 + c_6\, d_{FF}\Big] + \dots$$

$$c_1 = -8(107 + 144\zeta_3)\,, \quad c_2 = 4(229 - 480\zeta_3)$$

$$c_3 = -6(4961 - 11424\zeta_3 + 4752\zeta_4)$$

$$c_4 = -48(46 + 1065\zeta_3 - 378\zeta_4)$$

$$c_5 = -3(6231 + 9736\zeta_3 - 3024\zeta_4 - 2880\zeta_5)$$

$$c_6 = 1728(55 - 123\zeta_3 + 36\zeta_4 + 60\zeta_5)$$

▷ the n_f^4 term agrees exactly with known result [Gracey 96]

Application: 5-loop QCD β-function

notation: $a \equiv \frac{C_A\, g^2(\mu)}{16\pi^2}$, $\partial_{\ln \mu^2}\, a = -a\Big[\epsilon + b_0\, a + b_1\, a^2 + b_2\, a^3 + b_3\, a^4 + b_4\, a^5 + \dots\Big]$

- SU(3): $n_f = \frac{N_f}{6}$, $c_f = \frac{4}{9}$, $d_{FF} = \frac{5}{216}$

$$3^1\, b_0 \overset{\mathrm{SU}(3)}{=} 11 - \frac{2}{3} N_f \quad , \quad \dots \quad ,$$

$$3^5\, b_4 \overset{\mathrm{SU}(3)}{=} \Big[\frac{1205}{2916} - \frac{152}{81}\zeta_3\Big] N_f^4 + \Big[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5\Big] N_f^3 + \dots$$

▷ the N_f^3 term agrees exactly with known SU(3) result [see talk by K.Chetyrkin]

Summary

- studied class of fully massive vacuum diagrams
  - ▷ useful for e.g. hot QCD / anomalous dimensions / moments
- classification done at 5-loop level
  - ▷ complete set of (48+62) master integrals identified
- automated numerical evaluation via IBP / difference eqs / factorial series
  - ▷ `C++` implementation and parallelization, using **`Fermat`** for polynomial algebra
  - ▷ substantial fine-tuning of Laporta approach
  - ▷ at 5 loops: difference eqs of order 20, recurrence relations of order 28
- solved 44 of the 48 5-loop master-sectors numerically
  - ▷ ca 300 digits precision, ≥ 10 ε-orders around $d = 4 - 2\varepsilon$ and $d = 3 - 2\varepsilon$
- as an application, checked N_f^3 term of 5-loop QCD β-function
  - ▷ new result: full color dependence