Perturbative quantum field theory informs algebraic geometry

David Broadhurst, Open University, UK

Loops and Legs, Leipzig, 29 April 2016

Multi-loop sunrise and vacuum diagrams provide remarkable links between moments of Bessel functions, L-series of modular forms, and logarithmic Mahler measures. I shall report on discoveries, made with **Anton Mellit**, that extend the collection of such findings by considering **determinants** of Feynman integrals, in 2 spacetime dimensions. Our results involving **6-loop sunrise** and **7-loop vacuum** diagrams are rather striking.

1. Introduction: a **period** in various settings
2. On-shell **sunrise** diagrams as **L-series** of **modular** forms
3. **Kloosterman** sums and the 7 **Bessel** function problem
4. **Mahler** measures and **vacuum** diagrams
5. L-series as **determinants** of Feynman integrals (with Anton Mellit)

1 Introduction: a period in various settings

The 4-edge equal-mass **vacuum** diagram in 2 spacetime dimensions is given by the 3-loop Euclidean momentum-space integral

$$V_4 = \frac{1}{\pi^3}\left(\int\prod_{k=1}^{4}\frac{d^2 p_k}{p_k^2+1}\right)\delta^{(2)}(p_1+p_2+p_3+p_4)$$

and by a **Bessel** moment in configuration-space

$$V_4 = 8\int_0^\infty [K_0(t)]^4 t\mathrm{d}t.$$

By integration over Schwinger parameters, the **period** is an **L-series**:

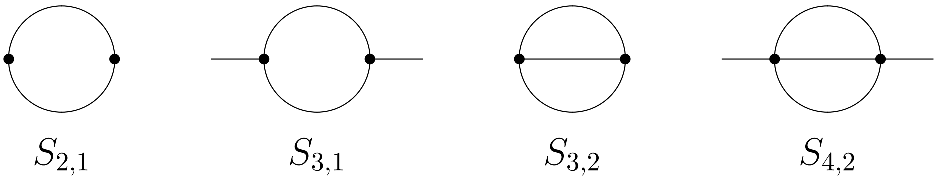
$$\begin{aligned} V_4 &= \int_0^\infty\int_0^\infty\int_0^\infty \left.\frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{(abc+abd+acd+bcd)(a+b+c+d)}\right|_{d=1} \\ &= 7\zeta(3) = 8\sum_{k>0}\frac{1}{(2k-1)^3} = 8\prod_{p>2}\frac{1}{1-p^{-3}}. \end{aligned}$$

It is also possible to prove that

$$\frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\theta_1\int_0^{2\pi}\mathrm{d}\theta_2\int_0^{2\pi}\mathrm{d}\theta_3 \log\left|1+e^{\mathrm{i}\theta_1}+e^{\mathrm{i}\theta_2}+e^{\mathrm{i}\theta_3}\right| = 7\zeta(3)$$

and Fernando Rodriguez Villegas has conjectures for higher-dimensional logarithmic **Mahler measures** evaluating to L-series of **modular forms**.

2 L-series of modular forms



The L-loop on-shell sunrise diagram in $D = 2$ spacetime dimensions is an integral of $L + 2$ Bessel functions. More generally, define an N-Bessel moment at L loops by

$$S_{N,L} := 2^L \int_0^\infty [I_0(t)]^{N-L-1}[K_0(t)]^{L+1} t \mathrm{d}t.$$

Convergence requires that $L < N < 2L + 3$ and $L > 1$ when $N = 2L + 2$. Bailey, Borwein, Broadhurst and Glasser, `arXiv:0801.0891`, proved that

$$S_{1,0} = S_{2,1} = 1, \quad S_{3,1} = \frac{2\pi}{3\sqrt{3}}, \quad S_{3,2} = \frac{4\,\mathrm{Cl}_2(\pi/3)}{\sqrt{3}} = 3\prod_p \frac{1}{1 - (\frac{p}{3})p^{-2}},$$

$$S_{4,2} = \frac{\pi^2}{4}, \quad S_{4,3} = 7\zeta(3), \quad S_{5,2} = \frac{\sqrt{3}}{120\pi}\,\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right).$$

We also conjectured and checked to 1000 digits that

$$S_{5,3} = \frac{4\pi}{\sqrt{15}}S_{5,2}, \quad S_{6,4} = \frac{4\pi^2}{3}S_{6,2}, \quad S_{8,5} = \frac{18\pi^2}{7}S_{8,3}.$$

2.1 Mini-guide to modular forms

For $|q| < 1$, we define the Dedekind eta function by

$$\eta(q) := q^{1/24} \prod_{n>0} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24}.$$

Then for $q = \exp(2\pi \mathrm{i} z)$ with z in the upper half of the complex plane,

$$\eta(\exp(2\pi \mathrm{i} z)) = (\mathrm{i}/z)^{1/2} \eta(\exp(-2\pi \mathrm{i}/z)).$$

If $f(z) = (\sqrt{-N}/z)^w f(-N/z)$, we say that f is a modular form of **modular weight** w and **level** N. Do not confuse this modular weight with the weight of polylogs. Example with weight 12 and level 1:

$$\eta^{24} = \sum_{n>0} A(n) q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 \ldots$$

Its Fourier coefficients are multiplicative: $A(mn) = A(m)A(n)$ for $\gcd(m, n) = 1$, and are determined by $A(p)$ at the primes p:

$$L(s) := \sum_{n>0} \frac{A(n)}{n^s} = \prod_p \frac{1}{1 - A(p)p^{-s} + p^{11-2s}}.$$

Moreover, we can analytically continue to values inside the critical strip:

$$\Lambda(s) := \frac{\Gamma(s)}{(2\pi)^s} L(s) = \sum_{n>0} A(n) \int_1^\infty \mathrm{d}x \left(x^{s-1} + x^{11-s}\right) e^{-2\pi n x} = \Lambda(12 - s).$$

2.2 A modular form of weight 3 for 5 Bessel functions

$$\begin{aligned}
\eta_n &= \eta(q^n) = q^{n/24} \prod_{k>0} (1 - q^{nk}) \\
f_{3,15} &= (\eta_3 \eta_5)^3 + (\eta_1 \eta_{15})^3 = \sum_{n>0} A_n q^n \\
L_{3,15}(s) &= \sum_{n>0} \frac{A_n}{n^s}
\end{aligned}$$

then $L_{3,15}(s)$ is the L-series of a modular form with weight 3 and level 15. I discovered that

$$S_{5,2} = 3L_{3,15}(2), \quad S_{5,3} = \frac{48}{5}\zeta(2)L_{3,15}(1),$$

where $S_{5,3}$ is the the on-shell 3-loop sunrise diagram. The modular form was identified by counts of zeros of the denominator of

$$S_{5,3} = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{(abc + ab + bc + ca)(a + b + c) + ab + bc + ca}$$

in finite fields. The Feynman integral $S_{5,3}$ gives

$$L_{3,15}(2) = \frac{\sqrt{3}}{360\pi}\,\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right).$$

2.3 A modular form of weight 4 for 6 Bessel functions

Let $L_{4,6}(s)$ be the Dirichlet L-function defined by the modular form

$$f_{4,6} = (\eta_1\eta_2\eta_3\eta_6)^2$$

with weight 4 and level 6. I discovered and checked to 1000 digits that

$$S_{6,2} = 6L_{4,6}(2), \quad S_{6,3} = 12L_{4,6}(3), \quad S_{6,4} = 48\zeta(2)L_{4,6}(2),$$

where $S_{6,4}$ is the on-shell 4-loop sunrise diagram.

2.4 A modular form of weight 6 for 8 Bessel functions

Let $L_{6,6}(s)$ be the Dirichlet L-function defined by the modular form

$$f_{6,6} = \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3$$

with weight 6 and level 6. I discovered and checked to 1000 digits that

$$S_{8,3} = 8L_{6,6}(3), \quad S_{8,4} = 36L_{4,6}(4), \quad S_{8,5} = 216L_{4,6}(5),$$

and **recently** evaluated the **6-loop sunrise** diagram, $S_{8,6} = (2\pi)^2 S_{8,4}$.

3 Kloosterman sums and the 7-Bessel problem

In a finite field $\mathbf{F}_q$, with $q = p^k$, Kloosterman sums are defined by

$$K(a) := \sum_{x \in \mathbf{F}_q^*} \exp\left(\frac{2\pi \mathrm{i}}{p} \mathrm{Trace}\left(x + \frac{a}{x}\right)\right),$$

with p-th roots of unity coming from a trace of Frobenius in $\mathbf{F}_q$ over $\mathbf{F}_p$:

$$\mathrm{Trace}(z) := \sum_{j=0}^{k-1} z^{p^j} = z + z^p + z^{p^2} + \ldots + z^{q/p}.$$

With $K(a) = -g(a) - h(a)$ and $g(a)h(a) = q$, we obtain integer moments

$$c_n(q) := -\frac{1 + S_n(q)}{q^2} \quad \text{with} \quad S_n(q) := \sum_{k=0}^{n} \sum_{a \in \mathbf{F}_q^*} [g(a)]^k [h(a)]^{n-k}.$$

For the 7th moments, Ronald Evans conjectured that, for prime $p > 7$,

$$c_7(p) = \left(\frac{p}{105}\right)\left(|b(p)|^2 - p^2\right) \quad \text{with a sign} \quad \left(\frac{p}{105}\right) = \left(\frac{p}{3}\right)\left(\frac{p}{5}\right)\left(\frac{p}{7}\right)$$

and $b(p)$ is the p-th Hecke eigenvalue for a weight-3 level-525 newform with eigenfield $\mathbf{Q}(\mathrm{i}, \sqrt{6}, \sqrt{14})$, discovered by William Stein, using `Sage`.

3.1 Algorithm for $c_n(q)$, with $q = p^k$ and $n \in [1, N]$

1. Find a monic polynomial, $f(x)$, of degree k, that is irreducible, modulo p, and a primitive polynomial g, of lesser degree, whose powers g^r, modulo f, serve as elements of $\mathbf{F}_q^*$.
2. For $m \in [0, p-1]$, store numerical values of $C[m] = \cos(2\pi m/p)$.
3. For $n \in [1, N]$ and $m \in [1, N]$, store in $U[n, m]$ the integer coefficient of $x^n y^m$ in the expansion of $1/(1 + xy + x^2 q)$.
4. For $m \in [0, k-1]$, store the traces $X[m] = \mathrm{Trace}(x^m) \bmod f(x)$.
5. For $r \in [1, q-1]$, store $T[r]$ computed from data in X, as follows. Set $r = 0$, $t = 1$. While $r < q-1$, add 1 to r, multiply t by g, set $T[r] = \sum_m t_m X[m]$, where t_m is the coefficient of x^m in t.
6. For $a \in [1, q-1]$, store $K[a] = \sum_{0<r<q} C[m(a, r)]$, with $m(a, r) \equiv T[r] + T[s(a, r)] \bmod p$ and $s(a, r) \equiv (a - r) \bmod (q - 1)$.
7. For $m \in [1, N]$, store $V[m] = \sum_{0<a<q} (K[a])^m$.
8. For $n \in [1, N]$, compute $S[n] = \sum_{0<m\le N} U[n, m] V[m]$ and return $c_n(q)$ as the integer nearest to $-(1 + S[n])/q^2$.

This reduces the number of computations of traces from $(p^k - 1)^2$ to k.

3.2 Local factors for the L-series with 7 Bessel functions

I shall suppose that the relevant L-series is of the form

$$L_{5,105}(s) = \sum_{n>0} \frac{A_n}{n^s} = \prod_p \frac{1}{Z_p(p^{-s})}$$

with weight 5, level 105, and with $A_p = c(p)$ for prime p. To determine the local factors, I require that $Z_p(T)$ is a polynomial of degree no greater than 3 that reproduces $c_7(p^n)$ via the Lefschetz formula

$$\log(Z_p(T)) = -\sum_{n>0} \frac{c_7(p^n)}{n} T^n.$$

Then for prime p that does not divide 105, I infer that

$$Z_p(T) = \left(1 - \left(\frac{p}{105}\right) p^2 T\right) \left(1 + \left(\frac{p}{105}\right) (2p^2 - |b(p)|^2)T + p^4 T^2\right).$$

At the bad primes, dividing 105, I obtain quadratic polynomials

$$\begin{aligned} Z_3(T) &= 1 - 10T + (9T)^2, \\ Z_5(T) &= 1 - (25T)^2, \\ Z_7(T) &= 1 + 70T + (49T)^2. \end{aligned}$$

3.3 Functional equation for 7 Bessel moments

Anton Mellit and I inferred the functional equation

$$\Lambda_{5,105}(s) := \left(\frac{105}{\pi^3}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) L_{5,105}(s) = \Lambda_{5,105}(5-s)$$

and then we were able to use Tim Dokchitser's `computel` to discover that

$$S_{7,4} := 2^4 \int_0^\infty [I_0(t)]^2[K_0(t)]^5 t\mathrm{d}t = 20\zeta(2)L_{5,105}(2).$$

Thus, at last, we have a 7-Bessel result to parallel

$$\begin{aligned}
S_{5,3} &:= 2^3 \int_0^\infty I_0(t)[K_0(t)]^4 t\mathrm{d}t = \frac{48}{5}\zeta(2)L_{3,15}(1)\\
S_{6,4} &:= 2^4 \int_0^\infty I_0(t)[K_0(t)]^5 t\mathrm{d}t = 48\zeta(2)L_{4,6}(2)\\
S_{8,6} &:= 2^6 \int_0^\infty I_0(t)[K_0(t)]^7 t\mathrm{d}t = 864\zeta(2)L_{6,6}(4)\\
S_{5,2} &:= 2^2 \int_0^\infty [I_0(t)]^2[K_0(t)]^3 t\mathrm{d}t = 3L_{3,15}(2)\\
S_{6,3} &:= 2^3 \int_0^\infty [I_0(t)]^2[K_0(t)]^4 t\mathrm{d}t = 12L_{4,6}(3)\\
S_{8,5} &:= 2^5 \int_0^\infty [I_0(t)]^2[K_0(t)]^6 t\mathrm{d}t = 216L_{6,6}(5)
\end{aligned}$$

for the modular forms of weights 3, 4 and 6.

4 Mahler measures and vacuum diagrams

A Laurent polynomial $P(x_1, \ldots, x_n)$ has a logarithmic Mahler measure

$$m(P) := \int_0^1 \mathrm{d}t_1 \ldots \int_0^1 \mathrm{d}t_n \log\left(\left|P\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right)\right|\right).$$

Christopher Deninger conjectured that

$$m\left(1 + x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2}\right) = \frac{15}{4\pi^2} L_{2,15}(2)$$

where the L-series comes from the weight-2 and level-15 modular form

$$f_{2,15} = \eta_1 \eta_3 \eta_5 \eta_{15}.$$

This was proven by Mathew Rogers and Wadim Zudilin `arXiv:1102.1153`. David Boyd conjectured and Anton Mellit proved, `arXiv:1207.4722`, that

$$m\left(1 + x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2} + x_1 x_2 + \frac{1}{x_1 x_2}\right) = \frac{7}{2\pi^2} L_{2,14}(2)$$

where the L-series comes from the weight-2 and level-14 modular form

$$f_{2,14} = \eta_1 \eta_2 \eta_7 \eta_{14}.$$

It is instructive to note that

$$\begin{aligned} m(1+x_1+x_2) &= \frac{\sqrt{3}}{4\pi} S_{3,2} = \frac{\mathrm{Cl}_2(\pi/3)}{\pi} \\ m(1+x_1+x_2+x_3) &= \frac{1}{2\pi^2} S_{4,3} = \frac{7\zeta(3)}{2\pi^2} \end{aligned}$$

evaluate to 2 and 3-loop **vacuum** diagrams, with 3 and 4 Bessel functions. One might expect the 5 and 6-Bessel modular forms,

$$f_{3,15} = (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3, \quad f_{4,6} = (\eta_1\eta_2\eta_3\eta_6)^2,$$

to determine Mahler measures, since I have proven that

$$m(1+x_1+\ldots+x_{N-1}) = -\log(2) - \gamma - \int_0^\infty \mathrm{d}t \log(t) \frac{\mathrm{d}}{\mathrm{d}t}[J_0(t)]^N.$$

This illuminates the conjectures by Fernando Rodriguez Villegas,

$$\begin{aligned} m(1+x_1+x_2+x_3+x_4) &= 6\left(\frac{\sqrt{15}}{2\pi}\right)^5 L_{3,15}(4) \\ m(1+x_1+x_2+x_3+x_4+x_5) &= 3\left(\frac{\sqrt{6}}{\pi}\right)^6 L_{4,6}(5) \end{aligned}$$

which David Bailey has checked to 1000 digits, using my N-Bessel formula.

5 L-series as determinants of Feynman integrals

Let M_n be the $n \times n$ matrix with elements

$$(M_n)_{a,b} := \int_0^\infty [I_0(t)]^a [K_0(t)]^{2n+1-a} t^{2b-1} \mathrm{d}t.$$

Then I found, at 1000-digit precision, that

$$\begin{aligned}
\det(M_1) &= \frac{\pi}{\sqrt{3^3}}, \\
\det(M_2) &= \frac{2\pi^3}{\sqrt{3^3\,5^5}}, \\
\det(M_3) &= \frac{2^4\pi^6}{\sqrt{3^3\,5^5\,7^7}}, \\
\det(M_{15}) &= \frac{2^{182}\pi^{120}}{3^{33}\,5^{20}\,7^5\sqrt{11^3\,13^9\,17^{17}\,19^{19}\,23^{23}\,29^{29}\,31^{31}}},
\end{aligned}$$

and conjecture that

$$\det(M_n) = \prod_{j=1}^{n} \frac{(2j)^{n-j}\pi^j}{\sqrt{(2j+1)^{2j+1}}}.$$

Moreover I have a similar conjecture for determinants of moments of even numbers of Bessel functions, for which no square roots occur.

5.1 Mahler measures and L-series from determinants

Recently, Anton Mellit and I discovered the connection of **Mahler** measures to **vacuum integrals** with 4 and 5 loops, obtaining

$$L_{3,15}(4) = \frac{8\pi^2}{45}\det\left(\int_0^\infty K_0^3(t)\begin{bmatrix} K_0^2(t) & t^2K_0^2(t) \\ I_0^2(t) & t^2I_0^2(t)\end{bmatrix} t\,\mathrm{d}t\right),$$

$$L_{4,6}(5) = \frac{4\pi^2}{27}\det\left(\int_0^\infty K_0^4(t)\begin{bmatrix} K_0^2(t) & t^2K_0^2(t) \\ I_0^2(t) & t^2I_0^2(t)\end{bmatrix} t\,\mathrm{d}t\right).$$

5.2 7-loop determinant giving an L-series of weight 6 at $s = 7$

Finally, for the L-series of the **weight-6** level-6 modular form

$$f_{6,6} = \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3$$

of the 8-Bessel problem, we obtained the empirical evaluation

$$L_{6,6}(7) = \frac{128\pi^2}{6075}\det\left(\int_0^\infty K_0^6(t)\begin{bmatrix} K_0^2(t) & t^2(1-2t^2)K_0^2(t) \\ I_0^2(t) & t^2(1-2t^2)I_0^2(t)\end{bmatrix} t\,\mathrm{d}t\right)$$

with the **7-loop vacuum period** $\int_0^\infty K_0^8(t)t\,\mathrm{d}t$ appearing in the determinant for the L-series at $s = 7$, **outside** the critical strip.

Conclusions

1. We have studied on-shell **sunrise** and **vacuum** diagrams whose periods are integrals of $N < 9$ **Bessel** functions.
2. At $N = 5$, 6 or 8, at least two **Feynman periods** are given by L-series of **modular forms** inside the critical strip.
3. At $N = 7$, we found an **L-series with weight 5** and level 105.
4. Thanks to a determination of the **functional equation**, we obtained
$$S_{7,4} := 2^4 \int_0^\infty [I_0(t)]^2 [K_0(t)]^5 t \mathrm{d}t = 20\zeta(2) L_{5,105}(2).$$
5. **Kloosterman** sums yield integers that inform the L-series for $N < 9$.
6. **Mahler measures** are evaluated by the L-series for $N < 7$.
7. **Vacuum diagrams** with $N = 3, 4, 5, 6$ determine L-series **outside** the critical strip, and hence Mahler measures. They enter **determinants** at $N = 5, 6$.
8. Beyond Mahler, we have evaluated the L-series of a **weight-6** modular form at $s = 7$ in terms of a **determinant** that includes the **7-loop** vacuum period $\int_0^\infty K_0^8(t) t \, \mathrm{d}t$ with **8 Bessel** functions.