Sunrise integrals and elliptic polylogarithms

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joint work with Luise Adams and Stefan Weinzierl

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J. Math. Phys. 56, 072303, (2015), arXiv:1504.03255 [hep-ph],

J. Math. Phys. 55, 102301 (2014), arXiv:1405.5640 [hep-ph],

J. Math. Phys. 54, 052303 (2013), arXiv:1302.7004 [hep-ph].

Loops and Legs 2016, Leipzig

Motivation:

Multiple polylogarithms

$$\mathrm{Li}_{(s_1, ..., s_k)}(z_1, ..., z_k) = \sum_{n_1 > n_2 > ... > n_k \geq 1} \frac{z_1^{n_1} ... z_k^{n_k}}{n_1^{s_1} ... n_k^{s_k}}, \; s_i \geq 1, \; |z_i| < 1$$

are very useful in the computation of Feynman integrals due to their double nature as **nested sums** and **iterated integrals**.

Many Feynman integrals can be expressed in terms of these functions, but apparently **not all of them**. (Bauberger, Böhm, Weiglein, Berends, Buza 1994, Caron-Huot, Larsen 2012, Nandan, Paulos, Spradlin, Volovich 2013)

The **massive sunrise** may be the simplest example, where multiple polylogarithms are **not sufficient**.

Which functions can we use?

So far: Bessel functions, Lauricella functions of type C, integrals over elliptic integrals, elliptic polylogarithms

Motivation:

We apply a class of **elliptic generalisations**, which allow us to use one of the basic advantages of (multiple) polylogarithms.

As an example recall the classical polylogarithms: $\mathrm{Li}_n(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^n}$
They can be written as integrals:

$$\begin{aligned}
\mathrm{Li}_n(z) &= \int_0^z \frac{dx}{x} \mathrm{Li}_{n-1}(x) \\
&= \int_0^z \frac{dx_n}{x_n} \dots \int_0^{x_3} \frac{dx_2}{x_2} \int_0^{x_2} \frac{dx_1}{1 - x_1}
\end{aligned}$$

In the **differential equations approach** we usually have to integrate over known Feynman integrals:

$$\int (\text{certain differential forms}) \quad \cdot (\text{generalized polylog}) \quad \in \{\text{generalized polylog}\}$$

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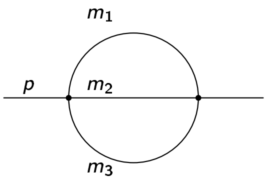
In terms of integrals:

$$\begin{aligned} \mathrm{Li}_n(z) &= \int_0^z \frac{dx}{x} \mathrm{Li}_{n-1}(x) \\ &= \int_0^z \frac{dx_n}{x_n} \dots \int_0^{x_3} \frac{dx_2}{x_2} \int_0^{x_2} \frac{dx_1}{1-x_1} \end{aligned}$$

In the **differential equations approach** we usually integrate over known Feynman integrals:

$$\begin{array}{ccc} \int (\text{certain differential forms}) & \cdot (\text{generalized polylog}) & \in \{\text{generalized polylog}\} \\ \in & \in & \in \\ \left\{ a\frac{dx}{x} + b\frac{dx}{x-1} + c\frac{dx}{x+1} \right\} & \{\text{harmonic polylog}\} & \{\text{harmonic polylog}\} \end{array}$$



We compute the massive sunrise integral

$$S(D,\ t) = \int \frac{d^D k_1 d^D k_2}{\left(i\pi^{D/2}\right)^2} \frac{1}{\left(-k_1^2 + m_1^2\right)\left(-k_2^2 + m_2^2\right)\left(-\left(p - k_1 - k_2\right)^2 + m_3^2\right)}$$

in $D = 2 - 2\epsilon$ and $D = 4 - 2\epsilon$ dimensions:

$$\begin{aligned} S(2-2\epsilon,\ t) &= S^{(0)}(2,\ t) + S^{(1)}(2,\ t)\epsilon + \mathcal{O}\left(\epsilon^2\right), \\ S(4-2\epsilon,\ t) &= S^{(-2)}(4,\ t)\epsilon^{-2} + S^{(-1)}(4,\ t)\epsilon^{-1} + S^{(0)}(4,\ t) + \mathcal{O}(\epsilon) \end{aligned}$$

where

$$t = p^2 \leq 0,\ 0 < m_1 \leq m_2 \leq m_3 < m_1 + m_2.$$

Furthermore we derive an algorithm to compute $S(2-2\epsilon,\ t)$ to **arbitrary order** in the equal-mass case $m_1 = m_2 = m_3$.

In $D = 2$ dimensions:

Equal mass case: Second order differential equation (Broadhurst, Fleischer, Tarasov 1993);

Solutions Groote, Pivovarov 2000, Laporta, Remiddi 2004, Bloch, Vanhove 2013 ...

Arbitrary masses:

- Coupled system of **four** equations of **first order** (Caffo, Czyz, Laporta, Remiddi 1998)
- **One** differential equation of **second order** (Müller-Stach, Weinzierl, Zayadeh 2012)

$$\left(p_2(t)\frac{d^2}{dt^2} + p_1(t)\frac{d}{dt} + p_0(t)\right) S^{(0)}(2,\ t) = p_3(t)$$

$p_0(t),\ p_1(t),\ p_2(t)$: polynomials in t and the m_i^2 ; $p_3(t)$: also involving $\ln(m_i^2),\ i = 1,\ 2,\ 3$.

Standard Ansatz:

$$S^{(0)}(2,\ t) = C_1\psi_1(t) + C_2\psi_2(t) + \int_0^t dt_1 \frac{p_3(t_1)}{p_0(t_1)W(t_1)} \left(-\psi_1(t)\psi_2(t_1) + \psi_2(t)\psi_1(t_1)\right)$$

$\psi_1,\ \psi_2$: solutions of the homogeneous equation; $C_1,\ C_2$: constants; $W(t)$: Wronski determinant.

Underlying geometry:
Second Symanzik polynomial:

$$\mathcal{F} = -x_1 x_2 x_3 t + \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right)\left(x_1 x_2 + x_2 x_3 + x_1 x_3\right).$$

The variety $\mathcal{F} = 0$ intersects the integration domain at **three points**

$P_1 = [1:0:0],\ P_2 = [0:1:0],\ P_3 = [0:0:1]$.

Coosing one of these as **origin** defines an **elliptic curve**.

Transform to Weierstrass normal form $y^2 z - x^3 - g_2(t) x z^2 - g_3(t) z^3 = 0$.

For $z = 1$ define e_1, e_2, e_3 by $y^2 = 4(x - e_1)(x - e_2)(x - e_3)$ with $e_1 + e_2 + e_3 = 0$.

$\Rightarrow$ Two **period integrals** of the elliptic curve are

$$\psi_1 = 2\int_{e_2}^{e_3} \frac{dx}{y} = \frac{4}{D^{\frac{1}{4}}} K(k), \quad \psi_2 = 2\int_{e_1}^{e_3} \frac{dx}{y} = \frac{4i}{D^{\frac{1}{4}}} K(k')$$

with the **complete elliptic integral of the first kind** $K(x) = \int_0^1 dt \frac{1}{\sqrt{(1-t^2)(1-x^2 t^2)}}$,

and modulus $k = \sqrt{\frac{e_3 - e_2}{e_1 - e_2}}$, $k' = \sqrt{1 - k^2} = \sqrt{\frac{e_1 - e_3}{e_1 - e_3}}$.

The **period integrals** ψ_1, ψ_2 are **solutions** of the **homogeneous** differential equation.

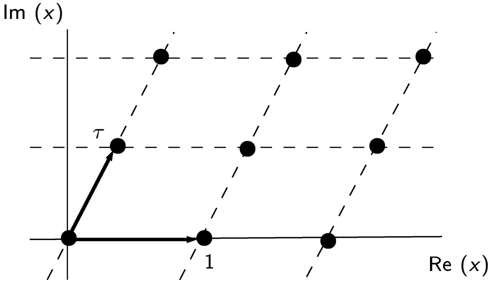
The constants C_1, C_2 are determined from a simple property of ψ_1, ψ_2 and the limit of $S^{(0)}(2, t)$ at $t = 0$ (Davydychev, Tausk 1996).

$\Rightarrow$ We obtain $S^{(0)}(2, t)$ as an **integral over** a combination of complete **elliptic integrals** of the first and second type (Adams, C.B., Weinzierl 2013).

Can we use a generalization of polylogarithms instead?

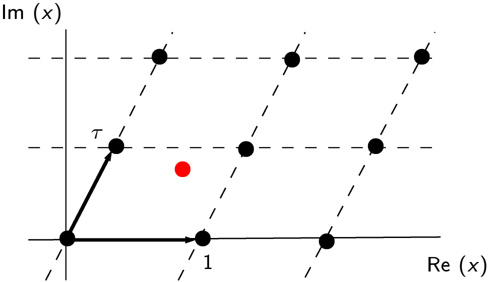
Important step by Bloch and Vanhove (2013) for the **equal mass case**:

New result in terms of an **elliptic dilogarithm**.



Consider the **lattice** $L = \mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{C}$ with $\text{Im}(\tau) > 0$.

Elliptic functions f with respect to L: $f(x) = f(x + \lambda)$ for $\lambda \in L$.



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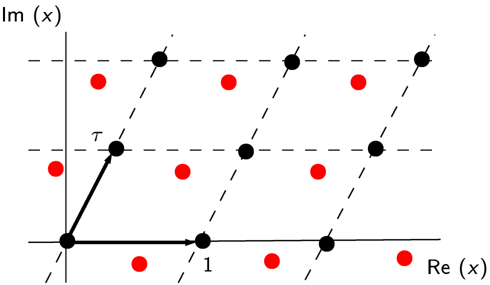

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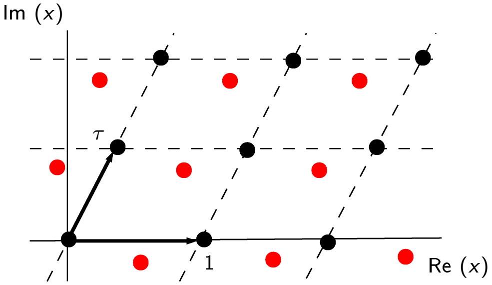

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Elliptic functions f with respect to L: $f(x) = f(x + \lambda)$ for $\lambda \in L$.

Let $\tau = \frac{\psi_1}{\psi_2}$ with ψ_1, ψ_2 the **periods of an elliptic curve** E.

$\Rightarrow$ E is isomorphic to a cell of L. $\Rightarrow$Consider f as a function **defined on** E.

Change variables to $z = e^{2\pi i x} \in \mathbb{C}^\star$

$\Rightarrow$Ellipticity $f(x) = f(x+\lambda)$ means $\tilde{f}(z) = \tilde{f}(z \cdot q_\lambda)$, $q_\lambda \in e^{2\pi i \lambda}$ for $\lambda \in L$.

Particularly: $q = e^{2\pi i \tau}$.

Basic concept: For some function g **construct** an elliptic function of the type

$$f(z, q) = \sum_{n \in \mathbb{Z}} g(z \cdot q^n)$$

E.g. Brown, Levin 2011 consider **elliptic polylogarithms** $\sum_{n\in\mathbb{Z}} u^n \mathrm{Li}_m(z \cdot q^n)$,

elliptic multiple polylogarithms and a framework of **iterated integrals**

(Also see previous definitions in Bloch 1977, Beilinson, Levin 1994, Levin 1997, Levin, Racinet 2007, ...)

Generalizing $\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}$ we define (Adams, C.B., Weinzierl 2014)

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m} q^{jk} = \sum_{k=1}^{\infty} y^k \mathrm{Li}_n(q^k x),$$

$$\mathrm{E}_{n;\,m}(x;\ y;\ q) =$$

$$\begin{cases} \frac{1}{i}\left(\frac{1}{2}\mathrm{Li}_2(x) - \frac{1}{2}\mathrm{Li}_2(x^{-1}) + \mathrm{ELi}_{2;\,0}(x;\ y;\ q) - \mathrm{ELi}_{2;\,0}(x^{-1};\ y^{-1};\ q)\right) & ,\ n+m \text{ even}, \\ \frac{1}{2}\mathrm{Li}_2(x) + \frac{1}{2}\mathrm{Li}_2(x^{-1}) + \mathrm{ELi}_{2;\,0}(x;\ y;\ q) + \mathrm{ELi}_{2;\,0}(x^{-1};\ y^{-1};\ q) & ,\ n+m \text{ odd}. \end{cases}$$

With this function, we obtain

$$S^{(0)}(2,\ t) = \frac{\psi_1(q)}{\pi} \sum_{i=1}^{3} \mathrm{E}_{2;\,0}(w_i(q);\ -1;\ -q) \text{ with } q = e^{\pi i \frac{\psi_2}{\psi_1}}.$$

The arguments w_1, w_2, w_3 are obtained from the intersection points P_1, P_2, P_3 by above transformations of the **elliptic curve**.

$\Rightarrow$ Every term in the result can be related to the underlying geometry.

$$
\begin{aligned}
S(2-2\epsilon,\, t) &= S^{(0)}(2,\, t) + S^{(1)}(2,\, t)\epsilon + \mathcal{O}\left(\epsilon^2\right), \\
S(4-2\epsilon,\, t) &= S^{(-2)}(4,\, t)\epsilon^{-2} + S^{(-1)}(4,\, t)\epsilon^{-1} + S^{(0)}(4,\, t) + \mathcal{O}(\epsilon)
\end{aligned}
$$

Using Tarasov's method (1996, 1997), we express $S^{(0)}(4,\, t)$ as linear combination of

$$
S^{(0)}(2,\, t),\; S^{(1)}(2,\, t),\; \frac{\partial}{\partial m_i^2} S^{(0)}(2,\, t),\; \frac{\partial}{\partial m_i^2} S^{(1)}(2,\, t),\; i = 1,\, 2,\, 3.
$$

$S^{(1)}(2,\, t)$ satisfies a differential equation

$$
L_{1,a} L_{1,b} L_2 S^{(1)}(2,\, t) = I_1(t).
$$

This can be solved for $L_2 S^{(1)}(2,\, t)$ and gives

$$
L_2 S^{(1)}(2,\, t) = I_2(t).
$$

Solving this equation, we obtain **results** for $S^{(1)}(2,\, t)$ and $S^{(0)}(4,\, t)$. (Adams, C.B., Weinzierl 2015 a)

In these results, we find the functions $\mathrm{E}_{1;\,0}$, $\mathrm{E}_{2;\,0}$, $\mathrm{E}_{3;\,1}$ and an **additional quadruple sum**.

$\Rightarrow$ Further generalization required

As a further generalization of $\mathrm{ELi}_{n;m}(x;y;q)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^j}{j^n}\frac{y^k}{k^m}q^{jk}$ we define:

$$\mathrm{ELi}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q)$$

$$=\sum_{j_1=1}^{\infty}\cdots\sum_{j_l=1}^{\infty}\sum_{k_1=1}^{\infty}\cdots\sum_{k_l=1}^{\infty}\frac{x_1^{j_1}}{j_1^{n_1}}\cdots\frac{x_l^{j_l}}{j_l^{n_l}}\frac{y_1^{k_1}}{k_1^{m_1}}\cdot\cdot\frac{y_l^{k_l}}{k_l^{m_l}}\frac{q^{j_1k_1+\ldots+j_lk_l}}{\prod_{i=1}^{l-1}(j_ik_i+\ldots+j_lk_l)^{o_i}}$$

Multiplication property:

$$\mathrm{ELi}_{\tilde{n},\tilde{m}}(\tilde{x};\tilde{y};q)\cdot\mathrm{ELi}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q)$$

$$=\mathrm{ELi}_{n_1,\ldots,n_l,\tilde{n};m_1,\ldots,m_l,\tilde{m};2o_1,\ldots,2o_{l-1},0}(x_1,\ldots,x_l,\tilde{x};y_1,\ldots,y_l,\tilde{y};q)$$

Integration property:

$$\int^q\frac{dq'}{q'}\mathrm{ELi}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q')$$

$$=\mathrm{ELi}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}+2}(x_1,\ldots,x_l;y_1,\ldots,y_l;q)$$

$\Rightarrow$We can **multiply** with $\mathrm{ELi}_{n;m}(x;y;q)$ and **integrate** over $\frac{dq}{q}$ **without leaving this class of functions**.

We consider $S(2-2\epsilon,\, t)$ in the equal-mass case $m_1 = m_2 = m_3$.
We factorize

$$S(2-2\epsilon,\, t) = \Gamma\left(1+\epsilon\right)^2 \left(\frac{3\mu^4 \sqrt{t}}{m(t-m^2)(t-9m^2)} \right) \tilde{S}(2-2\epsilon,\, t)$$

and apply the known differential equation for $S(2-2\epsilon,\, t)$.

$\Rightarrow$For the coefficients of $\tilde{S}(2-2\epsilon,\, t) = \tilde{S}^{(0)} + \epsilon \tilde{S}^{(1)} + \epsilon^2 \tilde{S}^{(2)} + ...$ we obtain a recursive relation of the type

$$\tilde{S}^{(j)} = -\frac{\psi_1}{\pi} \int_{q_0}^{q} \frac{dq_1}{q_1} \int_{q_0}^{q_1} \frac{dq_2}{q_2} \Big(\underset{\{\mathrm{ELi}_{n;m}\}}{\underset{\in}{A_j}} + \underset{\{\mathrm{ELi}_{n;m}\}}{\underset{\in}{B}} \cdot \underset{\{\mathrm{ELi}_{n_1,\ldots;m_1,\ldots;2o_1,\ldots}\}}{\underset{\in}{\tilde{S}^{(j-2)}}} \Big)$$

$\Rightarrow$ All coefficients of $S(2-2\epsilon,\, t)$ can be computed in terms of the generalized ELi-functions. (Adams, C.B., Weinzierl 2015 b)

Some results:

For the lowest coefficients of

$$S(2,t) = S^{(0)}(2,t) + \epsilon S^{(1)}(2,t) + \epsilon^2 S^{(2)}(2,t) + \mathcal{O}\left(\epsilon^3\right)$$

we obtain $S^{(j)}(2,t) = \frac{\psi_1}{\pi} E^{(j)}$ with

$$\begin{aligned}
E^{(0)} &= 3\mathrm{E}_{2;0}\left(r_3; -1; -q\right), \\
E^{(1)} &= 3\mathrm{E}_{3;1}\left(r_3; -1; -q\right) + 3\mathrm{E}_{0,1;-2,0;4}\left(r_3, r_3; -1, -1; -q\right) \\
&\quad -9\mathrm{E}_{0,1;-2,0;4}\left(r_3, r_3; -1, 1; -q\right) + 18\mathrm{E}_{0,1;-2,0;4}\left(r_3, -1; -1, 1; -q\right) \\
&\quad + \frac{3}{2i}\Big(-2\mathrm{Li}_{2,1}\left(r_3, 1\right) - 2\mathrm{Li}_3\left(r_3\right) + 2\mathrm{Li}_{2,1}\left(r_3^{-1}, 1\right) \\
&\quad + 2\mathrm{Li}_3\left(r_3^{-1}\right) + 6\mathrm{Li}_1\left(-1\right)\left(\mathrm{Li}_2\left(r_3\right) - \mathrm{Li}_2\left(r_3^{-1}\right)\right)\Big) + L_{1;0} E^{(0)}_{111}
\end{aligned}$$

where $r_3 = e^{\frac{2\pi i}{3}}$.

The ϵ^2-coefficient:

$$
\begin{aligned}
E^{(2)}_{111} \quad = \quad & \frac{9}{4}\mathrm{E}_{4;2}\left(r_3;-1;-q\right) + 108\mathrm{E}_{2,0,0,0,0;0,0,0,0,0;4,0,0,0}\left(r_3,r_3,r_3,r_3,r_3;-1,1,1,1,1;-q\right) \\
& +108\mathrm{SE}_{0,0,0,0;0,0,0,0;4,0,0}\left(r_3,r_3,r_3,r_3;1,1,1,1;-q\right)\frac{1}{2i}\left[\mathrm{Li}_2\left(r_3\right)-\mathrm{Li}_2\left(r_3^{-1}\right)\right] \\
& +3\mathrm{E}_{0,1;-2,0;6}\left(r_3,r_3;-1,-1;-q\right) - 9\mathrm{E}_{0,1;-2,0;6}\left(r_3,r_3;-1,1;-q\right) \\
& +18\mathrm{E}_{0,1;-2,0;6}\left(r_3,-1;-1,1;-q\right) \\
& +\frac{27}{2}\mathrm{E}_{0,1,1;-2,0,0;4,0}\left(r_3,r_3,r_3;-1,1,1;-q\right) - 9\mathrm{E}_{0,1,1;-2,0,0;4,0}\left(r_3,r_3,r_3;-1,-1,1;-q\right) \\
& +\frac{3}{2}\mathrm{E}_{0,1,1;-2,0,0;4,0}\left(r_3,r_3,r_3;-1,-1,-1;-q\right) - 54\mathrm{E}_{0,1,1;-2,0,0;4,0}\left(r_3,r_3,-1;-1,1,1;-q\right) \\
& +18\mathrm{E}_{0,1,1;-2,0,0;4,0}\left(r_3,r_3,-1;-1,-1,1;-q\right) + 54\mathrm{E}_{0,1,1;-2,0,0;4,0}\left(r_3,-1,-1;-1,1,1;-q\right) \\
& +\frac{3}{2i}\left\{4\mathrm{Li}_{2,1,1}\left(r_3,1,1\right) - 2\mathrm{Li}_{3,1}\left(r_3,1\right) + \frac{1}{4}\mathrm{Li}_4\left(r_3\right) - 4\mathrm{Li}_{2,1,1}\left(r_3^{-1},1,1\right) + 2\mathrm{Li}_{3,1}\left(r_3^{-1},1\right)\right. \\
& -\frac{1}{4}\mathrm{Li}_4\left(r_3^{-1}\right) + 6\mathrm{Li}_1\left(-1\right)\left[-2\mathrm{Li}_{2,1}\left(r_3,1\right) - \mathrm{Li}_3\left(r_3\right) + 2\mathrm{Li}_{2,1}\left(r_3^{-1},1\right) + \mathrm{Li}_3\left(r_3^{-1}\right)\right] \\
& \left.+18\left(\mathrm{Li}_1\left(-1\right)\right)^2\left[\mathrm{Li}_2\left(r_3\right) - \mathrm{Li}_2\left(r_3^{-1}\right)\right]\right\} + \zeta_2\frac{1}{2i}\left[\mathrm{Li}_2\left(r_3\right) - \mathrm{Li}_2\left(r_3^{-1}\right)\right] \\
& +L_{1;0}E^{(1)}_{111} - \frac{1}{2}\left(L_{1;0}\right)^2 E^{(0)}_{111} + \zeta_2 E^{(0)}_{111}
\end{aligned}
$$

with the short-hand notation

$$
\begin{aligned}
& \mathrm{SE}_{0,0,0,0;0,0,0,0;4,0,0}\left(r_3,r_3,r_3,r_3;1,1,1,1;-q\right) = \mathrm{E}_{0,0,0,0;0,0,0,0;4,0,0}\left(r_3,r_3,r_3,r_3;1,1,1,1;-q\right) \\
& +\tfrac{1}{2i}\left[\mathrm{Li}_0\left(r_3\right) - \mathrm{Li}_0\left(r_3^{-1}\right)\right]\mathrm{E}_{0,0,0;0,0,0;4,0}\left(r_3,r_3,r_3;1,1,1;-q\right) - \\
& \tfrac{1}{4}\left[\mathrm{Li}_0\left(r_3\right) - \mathrm{Li}_0\left(r_3^{-1}\right)\right]^2\mathrm{E}_{0,0;0,0;4}\left(r_3,r_3;1,1;-q\right) \\
& -\tfrac{1}{8i}\left[\mathrm{Li}_0\left(r_3\right) - \mathrm{Li}_0\left(r_3^{-1}\right)\right]^3\tfrac{1}{i}\left[\mathrm{ELi}_{2;2}\left(r_3;1;-q\right) - \mathrm{ELi}_{2;2}\left(r_3^{-1};1;-q\right)\right]
\end{aligned}
$$

Towards a general method:

With the ELi-functions we can compute integrals of the type

$$\int \frac{dq}{q} \, A \, \cdot \, \mathrm{ELi}_{n_1,...,n_l;m_1,...,m_l;2o_1,...,2o_{l-1}} \left(x_1,...,x_l;y_1,...,y_l;q\right)$$

if we can express A in terms of products of functions $\mathrm{ELi}_{n;m}(x;y;q)$.

Bottleneck: Finding such expressions is non-trivial in general.

For example we have

$$\begin{aligned}
\ln\left(\frac{\left(t-m^2\right)\left(t-9m^2\right)}{3m^3\sqrt{t}}\right) \quad = \quad & -\frac{1}{2}\ln\left(-q\right) + 12\mathrm{ELi}_{1;0}\left(-1;1;-q\right) \\
& +\mathrm{ELi}_{1;0}\left(r_3;-1;-q\right) + \mathrm{ELi}_{1;0}\left(r_3^{-1};-1;-q\right) \\
& -3\mathrm{ELi}_{1;0}\left(r_3;1;-q\right) - 3\mathrm{ELi}_{1;0}\left(r_3^{-1};1;-q\right)
\end{aligned}$$

where the relation between t and q is given by

$$q = e^{i\pi\tau}, \tau = \frac{\psi_2}{\psi_1}, \;\; t = -9m^2 \frac{\eta\left(\tau\right)^4 \eta\left(\frac{3\tau}{2}\right)^4 \eta\left(6\tau\right)^4}{\eta\left(\frac{\tau}{2}\right)^4 \eta\left(2\tau\right)^4 \eta\left(3\tau\right)^4}$$

Questions for future work:

- Can we apply the ELi-functions to integrals beyond the sunrise graph? (We are very optimistic!)
- General properties of the ELi-functions?
- Can we understand the geometry beyond the $D = 2$ case?
- Correspondence to the elliptic mulltiple polylogarithms of Brown and Levin (2011)
- Correspondence to the elliptic multiple zeta values in the string theory results of Broedel, Mafra, Matthes, Schlotterer 2014, 2015
- Correspondence to functions used in
  - results of Bloch, Kerr, Vanhove (2016)
  - results of Remiddi, Tancredi (2016)

Conclusions:

An **elliptic curve**, defined by the second Symanzik polynomial $\mathcal{F}$ is very useful in the computation of the sunrise integral.

In $D = 2$ we obtain an **elliptic generalization** $\mathrm{E}_{2;0}$ of the dilogarithm $\mathrm{Li}_2(z)$ with arguments obtained from the elliptic curve.

In $D = 4 - 2\epsilon$ we obtain a result furthermore involving $\mathrm{E}_{1;0}$, $\mathrm{E}_{2;0}$, $\mathrm{E}_{3;1}$.

With a **multi-variable generalization** of these functions we can compute the two-dimensional case with equal masses to **arbitrary order in** ϵ.

Additional Slides

The functions $\mathrm{E}_{1;\,0}$, $\mathrm{E}_{2;\,0}$, $\mathrm{E}_{3;\,1}$ defined by

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^j}{j^n}\frac{y^k}{k^m}q^{jk}$$

$$\mathrm{E}_{n;\,m}(x;\,y;\,q) =$$

$$\begin{cases} \frac{1}{i}\left(\frac{1}{2}\mathrm{Li}_2(x) - \frac{1}{2}\mathrm{Li}_2(x^{-1}) + \mathrm{ELi}_{2;\,0}(x;\,y;\,q) - \mathrm{ELi}_{2;\,0}(x^{-1};\,y^{-1};\,q)\right) & ,\, n+m \text{ even}, \\ \frac{1}{2}\mathrm{Li}_2(x) + \frac{1}{2}\mathrm{Li}_2(x^{-1}) + \mathrm{ELi}_{2;\,0}(x;\,y;\,q) + \mathrm{ELi}_{2;\,0}(x^{-1};\,y^{-1};\,q) & ,\, n+m \text{ odd}. \end{cases}$$

can be seen as generalizations of Clausen and Glaisher functions:

$$\mathrm{Cl}_n(\varphi) = \begin{cases} \frac{1}{2i}\left(\mathrm{Li}_n\left(e^{i\varphi}\right) - \mathrm{Li}_n\left(e^{-i\varphi}\right)\right) \\ \frac{1}{2}\left(\mathrm{Li}_n\left(e^{i\varphi}\right) + \mathrm{Li}_n\left(e^{-i\varphi}\right)\right) \end{cases} \quad \mathrm{Gl}_n(\varphi) = \begin{cases} \frac{1}{2}\left(\mathrm{Li}_n\left(e^{i\varphi}\right) + \mathrm{Li}_n\left(e^{-i\varphi}\right)\right) & ,n \text{ even}, \\ \frac{1}{2i}\left(\mathrm{Li}_n\left(e^{i\varphi}\right) - \mathrm{Li}_n\left(e^{-i\varphi}\right)\right) & ,n \text{ odd}, \end{cases}$$

$$\begin{aligned} \lim_{q\to 0}\mathrm{E}_{1;\,0}\left(e^{i\varphi};\,y;\,q\right) &= \mathrm{Cl}_1\left(\varphi\right), \\ \lim_{q\to 0}\mathrm{E}_{2;\,0}\left(e^{i\varphi};\,y;\,q\right) &= \mathrm{Cl}_2\left(\varphi\right), \\ \lim_{q\to 0}\mathrm{E}_{3;\,1}\left(e^{i\varphi};\,y;\,q\right) &= \mathrm{Gl}_3\left(\varphi\right). \end{aligned}$$

Perspective 1: Generalized hypergeometric functions

Berends, Buza, Böhm and Scharf (1994) expressed $S(D, t)$ as a linear combination of type C Lauricella functions

$$F_C\left(\alpha_1 - A_1 D,\, \alpha_2 - A_2 D;\, \beta_1 - B_1 D,\, \beta_2 - B_2 D,\, \beta_3 - B_3 D;\, \frac{m_1^2}{t},\, \frac{m_2^2}{t},\, \frac{m_3^2}{t}\right),$$

with all $\alpha_i, \beta_i \in \mathbb{N}$ and A_i, B_i half-integers. They are defined by

$$F_C(a_1, a_2; b_1, b_2, b_3; x_1, x_2, x_3) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} \frac{(a_1)_{n_1+n_2+n_3}(a_2)_{n_1+n_2+n_3}}{(b_1)_{n_1}(b_2)_{n_2}(b_3)_{n_3}} \frac{x_1^{n_1} x_2^{n_2} x_3^{n_3}}{n_1! n_2! n_3!}$$

Remark: Techniques for the **expansion** of generalized hypergeometric functions today extend to certain Lauricella functions (e.g. Bytev, Kalmykov and Moch 2014), but the expansion of F_C **remains a problem**.
No multiple polylogarithms?

Reminder: Dedekind's eta-function:

$$\eta(\tau) = e^{\frac{i\pi\tau}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n\tau}\right) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} \left(1 - q^{2n}\right)$$