QCD running in 5 loops

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in collaboration to

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Loops & Legs, 25.04.2016

RENOLMALIZATION-GROUP (all started in 1953)

Stückelberg and Petermann; Gell-Mann and Low; Bogoliubov and Shirkov

and after only 50 years

⇓

Nobel Prise in Physics in 2004!

$$\beta_0 = \frac{\mathbf{33 - 2N_F}}{\mathbf{12}}$$

QCD Running Today:
(after 20 Years since the first 4-loop results)

- **quark mass anomalous dimension is known**
  /Baikov, K.Ch. and Kühn (2013)/

- **the QCD β-function is just finished /this talk/**

β_{QCD} is expressed completely through Z-factors appearing in the (renormalized) QCD Lagrangian

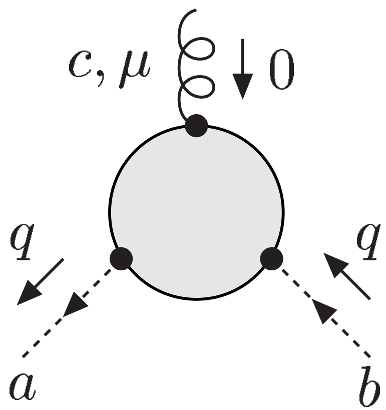
$$\begin{aligned}\mathcal{L}_R^{QCD} &= -\frac{1}{4} Z_3 \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 - \frac{1}{2} g\, Z_1^{3g} \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) \left(A_\mu \times A_\nu\right)^a - \frac{1}{4} g^2\, Z_1^{4g} \left(A_\mu \times A_\nu\right)^2 \\ &+ Z_3^c\, \partial_\nu \bar{c} \left(\partial_\nu c\right) + g\, Z_1^{ccg}\, \partial^\mu \bar{c} \left(A_\mu \times c\right) + Z_2\, \bar{\psi} \mathrm{i} \partial\!\!\!/ \psi - Z_{\bar{\psi}\psi} m_f\, \bar{\psi}\psi + g Z_1^{\psi\psi g}\, \bar{\psi} A\!\!\!/ \psi\end{aligned}$$

Minimal (and simplest) set of Z-factors to compute β: $\boxed{Z_3, Z_3^c, Z_1^{ccg}}$

Let us concentrate on Z_1^{ccg} and consider vertex function

$$\Gamma(a_s, q^2) = Z_1^{ccg} + Z_1^{ccg}\, \delta\Gamma(a_s^0, q^2)$$

of the scalar quark current



$$= q_\mu\, \Gamma(a_s^0, q^2) \equiv q_\mu\, \left(1 + \delta\Gamma(a_s^0, q^2)\right)$$

Suppose we want to compute L-loop contribution to Z_1^{ccg}. There are (at least) 4 ways to do it:

1. set 1 of 2 ext. momenta to zero $\rightarrow$ (the poles of) L-loop p-integrals (massless propagators) to be computed. That is how first 2-loop RG calculations in QCD were done /D. R.T Jones, 1974/.

2. set **all ext.** momenta to zero and introduce an universal mass to **all** propagators (including gluon!) $\rightarrow$ (the poles of) L-loop m-integrals (massive tadpoles) to be computed.

That is how the first 4-loop calculation of the QCD β-function was done /van Ritbergen, T., Vermaseren, J. and Larin, S. (1997).

NOTES:

- **This truly outstanding, ahead of its time, calculation was done long before "Laporta approach" started to flower.**
- **It was checked and confirmed only significantly later / M. Czakon, (2004) /**

3. set all ext. momenta to zero and introduce a mass into only one (*but properly chosen to avoid IR singularities*) propagator $\rightarrow$ (L-1)-loop p-integrals (including their finite part) to be computed /A. Vladimirov (1978)/ That is how the first 3-loop calculation of the QCD β-function was done /Tarasov, O., Vladimirov, A. and Zharkov, A. (1980)/.

Problems: difficult to automatize; not applicable to all diagrams.

4. the same as 3. but IR singularities are removed recursively with so-called R^*-operation /K.Ch. V. Smirnov (1984)/. Features: applicable for every possible diagram, automatization is possible but not simple (due to involved structure of **UV** and **IR** subtractions)

Essentially we have two "big" ways: m-(massive) and p-(massless) ones.

Comparison:

m-way: more complicated IBP-relations, one more squared propagator, straightforward renormalization NO IR singularities (all props are massive)

p-way: both UV and IR subtractions are significantly more sophisticated (basically because even global gauge invariance is broken by singling out a massive line)

At 5-loop level we have employed the 4-th way (until recently the only feasible)

with the use of the following tools:

- global solution of the combinatorics of R^* operation (rather involved and problem specific)
- the Baikov's way of doing reduction with the help of $1/D$ expansion of the corresponding coefficient functions in front of masters (analytically known from **three(!)** independent calculations /K.Ch, P. Baikov (2010); R. Lee, V. Smirnov (2012); E. Panzer (2013)/
- Effective implementation of the $1/D$ expansion would hardly be feasible without excellent possibilities for dealing with gigantic data streams offered by **T-FORM**, as being developed by /J. A. M. Vermaseren, M. Tentyukov, T. Ueda, J. Kuipers . . . /
- properly administrated cluster of workstations (see more details below)

QCD β-function in FIVE loops: result

$$\mu^2 \frac{\partial}{\partial \mu^2} a_s = \beta(a_s)\, a_s, \quad a_s \equiv \frac{\alpha_s}{\pi}, \quad \beta(a_s) = \sum_{i \geq 0} \beta_i\, a_s^{i+1}$$

$$\begin{aligned}
4^5\, \beta_4 = \; & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 \\
+ \quad & n_f \left[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5 \right] \\
+ \quad & n_f^2 \left[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5 \right] \\
+ \quad & n_f^3 \boxed{\left[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5 \right]} + n_f^4 \boxed{\left[\frac{1205}{2916} - \frac{152}{81}\zeta_3 \right]}
\end{aligned}$$

n_f^4 **term is in FULL AGREEMENT with the 20 years old result by John Gracey (in the framework of the conformal bootstrap method of A. Vasiliev, Yu. Pis'mak and J. Honkonen (1981))**

n_f^3 **term is in FULL AGREEMENT with the the very recent calculation by Th. Luthe, A. Maier, P. Marquard and Y. Schröder (made within "massive way", see York's talk on Friday)**

QCD β-function in FIVE loops: Zeta's

In general any 5-loop beta in any theory will have the following "transcendental structure" (an obvious outcome of our knowledge of the corresponding masters)

1 and 2 loops: rational

3 loops: rationals + ζ_3

4 loops: rationals + ζ_3 + ζ_4 + ζ_5

5 loops: rationals + ζ_3 + ζ_4 + ζ_5 + ζ_5 +ζ_7

$$\beta_0 = \frac{1}{4}\left\{11 - \frac{2}{3}n_f,\right\}, \quad \beta_1 = \frac{1}{4^2}\left\{102 - \frac{38}{3}n_f\right\}, \quad \beta_2 = \frac{1}{4^3}\left\{\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right\},$$

$$\beta_3 = \frac{1}{4^4}\left\{\left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right)n_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right)n_f^2 + \frac{1093}{729}n_f^3\right\}$$

The one-loop-delayed appearance of zeta's (well-known at 3 and 4 loops) shows itself also at 5 loops. Any explanation is missing, indeed!

/David, Dirk, AU...! We do need your intuition!/

$$
\begin{aligned}
4^5\,\beta_4 = {} & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 \\
+ \quad & n_f\left[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5\right] \\
+ \quad & n_f^2\left[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5\right] \\
+ \quad & n_f^3\,\boxed{\left[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5\right]} + n_f^4\,\boxed{\left[\frac{1205}{2916} - \frac{152}{81}\zeta_3\right]}
\end{aligned}
$$

QCD β-function in FIVE loops: Numerics

$$
\begin{aligned}
-\beta &= a_s\,(2.75 - 0.166667\,n_f) + a_s^2\,(6.375 - 0.791667\,n_f) \\
&+ a_s^3\,(22.3203 - 4.36892\,n_f + 0.0940394\,n_f^2) \\
&+ a_s^4\,(114.23 - 27.1339\,n_f + 1.58238\,n_f^2 + 0.0058567\,n_f^3) \\
&+ a_s^5\,\underbrace{(\mathbf{524.56 - 181.8\,n_f + 17.16\,n_f^2 - 0.22586\,n_f^3 - 0.0017993 n_f^4})}_{\beta_4} \quad (1)
\end{aligned}
$$

It is instructive to compare β_4 with a (20 years old!) prediction by J. Ellis, I. Jack, D.R.T. Jones, M. Karliner, M.A. Samuel, "Asymptotic Pade approximant predictions: Up to five loops in QCD and SQCD", Phys. Rev. (1998).

$$
\beta_4^{\mathbf{APAP}} = \mathbf{740 - 213\,n_f + 20\,n_f^2} - \mathbf{0.0486\,n_f^3} - \boxed{0.0017993\,n_f^4}
$$

QCD β-function in FIVE loops: Numerics

Unfortunately, this strikingly good agreement does not always survive for fixed values of n_f:

n_f	0	1	2	3	4	5	6
$\beta_4^{\rm exact}$	525	360	228	127	57	15	0.27
$\beta_4^{\rm APAP}$	741	548	395	281	205	169	170

due to severe cancellations between different powers of n_f

Very similar picture for the quark mass anomalous dimension:

$$\gamma_4^{exact} = 559.71 - 143.6\, n_f + 7.4824\, n_f^2 + 0.1083\, n_f^3 - 0.00008535\, n_f^4$$

$$\gamma_4^{APAP} = 530 - 143\, n_f + 6.67\, n_f^2 + 0.037\, n_f^3 - \boxed{0.00008535\, n_f^4}$$

QCD β-function in FIVE loops: Apparent Convergence

It is instructive to consider the properly normalized $\overline{\beta} \equiv \frac{\beta}{-\beta_0}$:

$$\overline{\beta}(n_f = 0) \;=\; 1. + 2.32\, a_s + 8.12\, a_s^2 + 41.54\, a_s^3 + 190.75\, a_s^4$$

$$\overline{\beta}(n_f = 1) \;=\; 1 + 2.16\, a_s + 6.99\, a_s^2 + 34.33\, a_s^3 + 139.23\, a_s^4$$

$$\overline{\beta}(n_f = 2) \;=\; 1 + 1.983\, a_s + 5.776\, a_s^2 + 27.45\, a_s^3 + 94.24\, a_s^4$$

$$\overline{\beta}(n_f = 3) \;=\; 1 + 1.78\, a_s + 4.47\, a_s^2 + 20.99\, a_s^3 + 56.59\, a_s^4$$

$$\overline{\beta}(n_f = 4) \;=\; 1 + 1.54\, a_s + 3.05\, a_s^2 + 15.07\, a_s^3 + 27.33\, a_s^4$$

$$\overline{\beta}(n_f = 5) \;=\; 1 + 1.261\, a_s + 1.47\, a_s^2 + 9.83\, a_s^3 + 7.88\, a_s^4$$

$$\overline{\beta}(n_f = 6) \;=\; 1 + 0.93\, a_s - 0.29\, a_s^2 + 5.52\, a_s^3 + 0.15\, a_s^4$$

We see very modest growth of the coefficients, that is (apparent) convergence us better than one would expect (from comparison with other examples)

QCD β-function in FIVE loops: Applications

The mass evolution is described by equation $\frac{m(\mu)}{m(\mu_0)} = \frac{c(a_s(\mu))}{c(a_s(\mu_0))}$ where

$$
\begin{aligned}
c(x) &= \exp\left\{\int \frac{dx'}{x'}\frac{\gamma_m(x')}{\beta(x')}\right\} = (x)^{\bar\gamma_0}\left\{1 + (\bar\gamma_1 - \bar\beta_1\bar\gamma_0)x\right.\\
&+ \frac{1}{2}\left[(\bar\gamma_1 - \bar\beta_1\bar\gamma_0)^2 + \bar\gamma_2 + \bar\beta_1{}^2\bar\gamma_0 - \bar\beta_1\bar\gamma_1 - \bar\beta_2\bar\gamma_0\right]x^2\\
&+ \left[\frac{1}{6}(\bar\gamma_1 - \bar\beta_1\bar\gamma_0)^3 + \frac{1}{2}(\bar\gamma_1 - \bar\beta_1\bar\gamma_0)(\bar\gamma_2 + \bar\beta_1{}^2\bar\gamma_0 - \bar\beta_1\bar\gamma_1 - \bar\beta_2\bar\gamma_0)\right.\\
&\left.\left. + \frac{1}{3}\left(\bar\gamma_3 - \bar\beta_1{}^3\bar\gamma_0 + 2\bar\beta_1\bar\beta_2\bar\gamma_0 - \bar\beta_3\bar\gamma_0 + \bar\beta_1\bar\gamma_1 - \bar\beta_2\bar\gamma_1 - \bar\beta_1\bar\gamma_2\right)\right]x^3 + \mathcal{O}(x^4)\right\}
\end{aligned}
$$

$\bar\gamma_i = \gamma_i/\beta_0$, $\bar\beta_i = \beta_i/\beta_0$

Important concept: RGI mass

$$m^{RGI} \equiv m(\mu_0)/c(a_s(\mu_0))$$

is μ and *scheme* independent; in *any* (mass-independent) scheme

The function $c(x)$ is used, e.g, by the **ALPHA** lattice collaboration to find the ($\overline{\text{MS}}$) mass of the strange quark at a lower scale from the RGI mass determined from lattice simulations

Example (setting $a_s(\mu = 2\,\text{GeV}) = \frac{\alpha_s(\mu)}{\pi} = .1$; h counts loops)

$$m_s(2\,\text{GeV}) = \hat{m}_s \cdot (a_s(2\,\text{GeV}))^{\frac{4}{9}} \cdot$$

$$\left(1 + 0.0895\,h^2 + 0.0137\,h^3 + 0.00195\,h^4 + \underbrace{(0.00157 - .000011\,\overline{\beta}_4)}_{0.00096}\,h^5\right)$$

Thus, the 5-loop term in the β-function decreases the overall 5-loop shift from a minute $1.5 \cdot 10^{-3}$ to even smaller $1 \cdot 10^{-3}$

The effects of the 5-loop running on the determination of $\overline{m}_b^2(M_H)$ which appears as a common factor in $H \to b\bar{b}$ decay will be discussed in the talk by J. Kühn

QCD β-function in FIVE loops: effective Hgg vertex in 5 loops

In the heavy top limit the Higgs boson couples directly with gluons via

$$\mathcal{L}_{eff} = -2^{1/4} G_F^{1/2} H C_1(\mu^2/m_t^2, a_s(\mu))\, G^a_{\nu\rho} G^a_{\nu\rho}$$

The effective c.c. $C_1(\mu^2/m_t^2, a_s(\mu))$ is expressible through massive tadpoles and was computed at 4 loops in 1997 **(long before the direct calculation of 4-loop /generic/ massive tadpole started to be technically possible)** / K. Ch., B. Kniehl and M. Steinahuser (1998)/

The trick behind: a low energy theorem:

$$C_1 = -\frac{1}{2} m_t^2 \frac{\partial}{\partial m_t^2} \ln \zeta_g^2, \quad \alpha_s'(\mu) = \zeta_g^2(\mu^2/m_t^2, a_s(\mu))\, \alpha_s(\mu)$$

which connects C_1 with the corresponding "decoupling" constant for α_s.

The appearance of the derivative $\frac{\partial}{\partial m_t^2}$ means that the most complicated (that is constant) part of ζ_g^2 **does** not contribute to $C_1 \Longrightarrow$ one could use the corresponding RG equation to find logs at next loop order (**provided we know the β function at the same increased loop order!**)

QCD β-function in FIVE loops: effective Hgg vertex in 5 loops

ζ_g^2 is known since long at 4 loops

/Y. Shröder, M. Steinauser (2006)/
/Baikov, K.Ch. and Kühn (2006)/

with available β_4 we arrive at:

$$C_1 = -\frac{1}{12} a_s(\mu_h) \Big[1 + 2.7500\, a_s(\mu_h) + (9.7951 - 0.6979\, n_l)\, a_s^2(\mu_h)$$
$$+ \left(49.1827 - 7.7743\, n_l - 0.2207\, n_l^2\right) a_s^3(\mu_h)$$
$$+ \left(-662.507 + 137.601\, n_l - 2.53666\, n_l^2 - 0.077522\, n_l^3 + 6\, \Delta\beta_4\right) a_s^4(\mu_h) \Big]$$

where $n_l = n_f - 1$ and,

$$\Delta\beta_4 = \beta_4^{(n_f)} - \beta_4^{(n_f-1)} = 199.179 - 34.99 n_f + 0.667 n_f^2 + 0.0072 n_f^3 = 14.843 (n_f = 6).$$

Finally,

$$C_1 = 1. + 2.75\, a_s + 6.306\, a_s^2 + 9.208\, a_s^3 + (101.49 = 12.427 + 89.058)\, a_s^4$$

QCD β-function in FIVE loops: technical info for experts

Complexity of calculations:

for a given # of loops and a (generic) topology it essentially depends on the deviation from the corresponding scalar graph:

the sum of # of "dots" (that is squared propagators) $+$ maximal power of irreducible scalar products $p_i.p_j$ in the corresponding numerator

For Z_3 (the gluon prop, most demanding part of our calculations) we have:

dots + max. power of $p_i.p_j$ = 3 + 7 = 10 (for the massive way it would be 4+ 7=11)

Required computer time for a "slice" (VERY ROUGHLY):

n_f^4 and n_f^3 slices have required a day and a week correspondingly

all other slices (n_f^2 and n_f^1 and n_f^0) have required 6-7 MONTHS each

QCD β-function in FIVE loops: technical info for experts

Definition of the "Computational time":

the difference between generation of input file with all dias classified according their topologies and with traces, etc. already evaluated and the production of a final file with the results of the corresponding reduction

The use of computer resources:

CPUS:
For $1/D$ reduction it was usually possible for us to employ 15-20 machines, every one with 8 cores running a TFORM with (-w 8 option)

Disk storage:
usually 2-3 Tb per "octet" in most complicated cases up to 6 Tb

Main part of calculations was done without gzip -option

QCD β-function in FIVE loops: THANKS

The continues running of our calculation at such computer & time scales would be certainly impossible without the effective support of our computer gurus:

Peter Marquard, Jens Hoff, David Kunz and Alexander Hasselnuss

$+$

corresponding administrative support due to

Matthias Steinhauser

We do want to say our BIG THANKS to Matthias, Peter, Jens, David, and Alexander!

Conclusions

- R^* + Baikov Algorithm to reduce 4-loop p-integrals + Form (J. Vermaseren, $+\ldots$) + known 4-loop masters $\Longrightarrow$ the 5-loop RG functions are *in principle* doable in *any* model

- The 5-loop quark anomalous dimension γ_m and the QCD β-function have been done (and even partially checked)

- The apparent convergence of PT series for both RG functions is very good $\Longrightarrow$ the phenomenological implications are not not very dramatic.