On a four loop form factor in N=4

arXiv:1301.4165 with B. Kniehl, O. Tarasov and G. Yang
arXiv:1508.03717 with B. Kniehl and G. Yang
and in progress

Rutger Boels
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The wrong way to compute something interesting

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(see Smirnov / Von Manteuffel talks for better ideas…)

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Workshop motivation

Loops and Legs in Quantum Field Theory

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- its a four loop computation

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- N=4 is ‘hardest part’ of QCD

Personal motivation

- formulation of gauge theory through Lagrangians → in principle calculable quantities in perturbation theory

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$$\mathcal{L} = \mathcal{L}(A_\mu, \psi^I, \phi^{[IJ]})$$

$\Longleftrightarrow$

AdS/CFT

IIB superstring on AdS5 x S5 background

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↳ universal function in IR divergences

On the virtues of virtuous N=4?

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? what about the non-planar corrections to the cusp?

Cusp anomalous dimension: color dependence

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3	C_A^3
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$$\gamma_{\text{cusp}} = \sum_l g^{2l} \gamma_{\text{cusp}}^{(l)} = a_1 g^2 C_A + a_2 g^4 C_A^2 + a_3 g^6 C_A^3 + g^8 \left(a_4^P C_A^4 + a_4^{NP} d_{44} \right) + \mathcal{O}(g^9) \,,$$

$$C_A = N_c \qquad d_{44} = N_c^4 + 36 N_c^2$$

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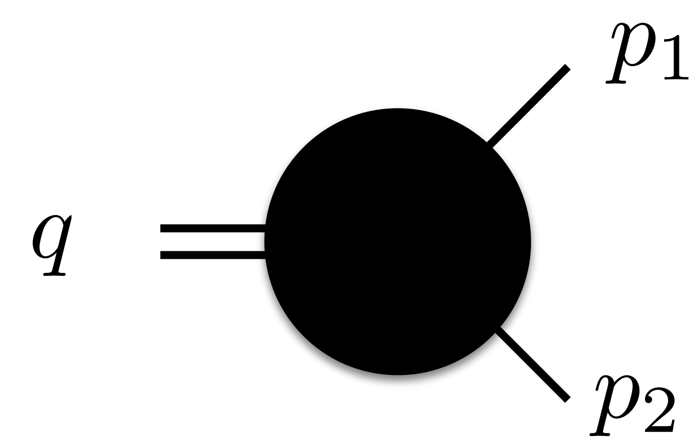

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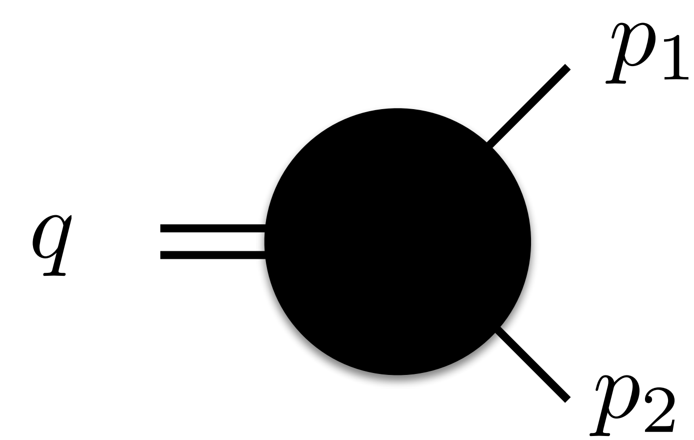

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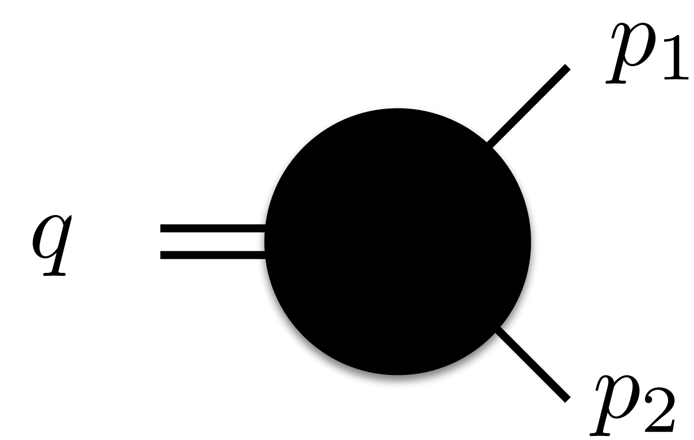

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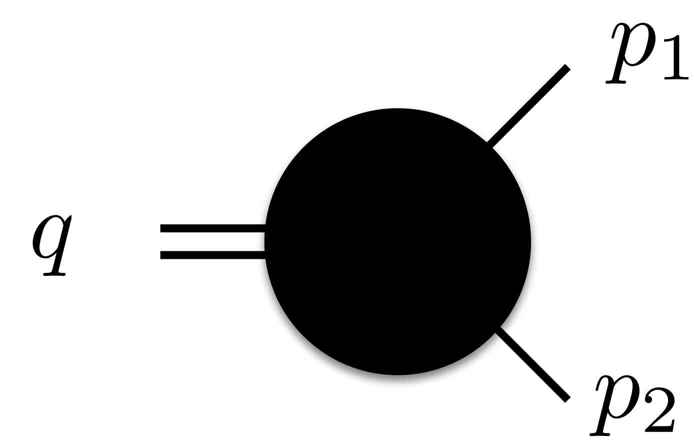

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- must cancel out in total cross-sections: imposes severe restrictions on observables (long story)

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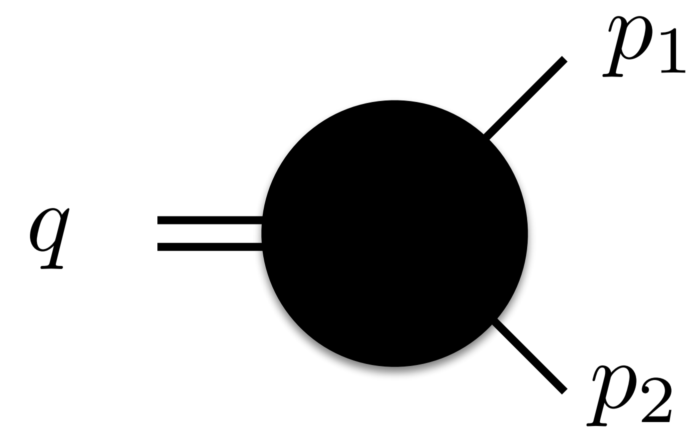

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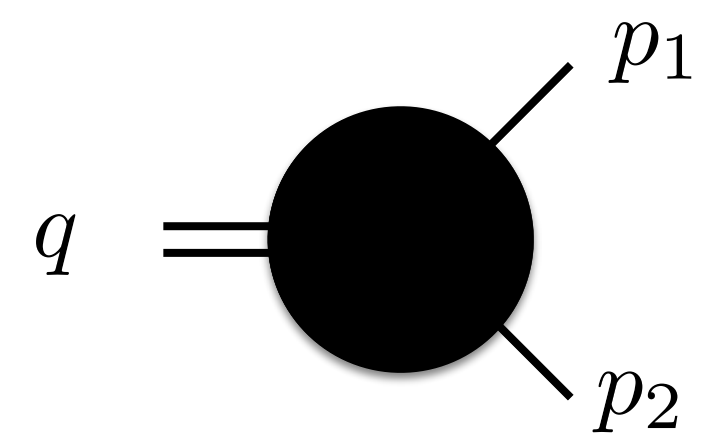

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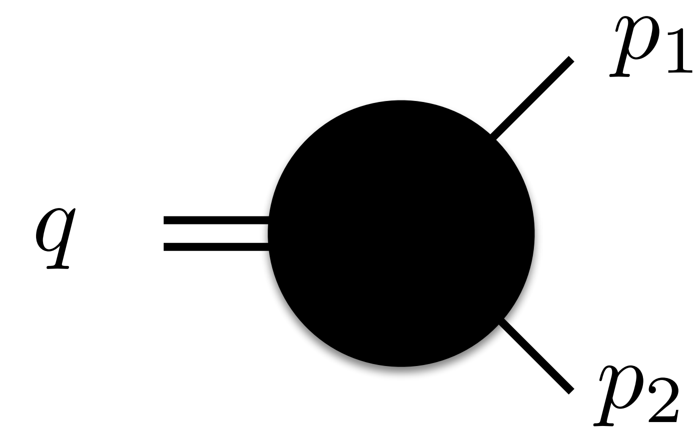

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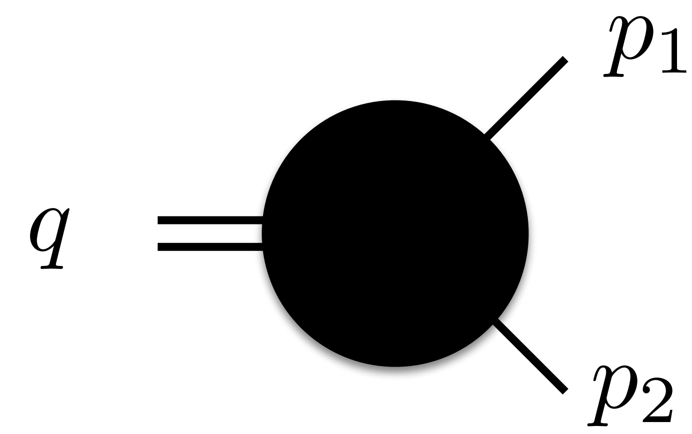

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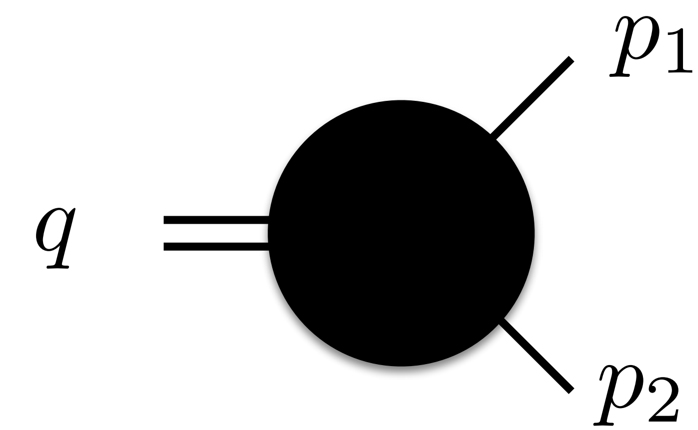

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(see also Gardi's talk!)

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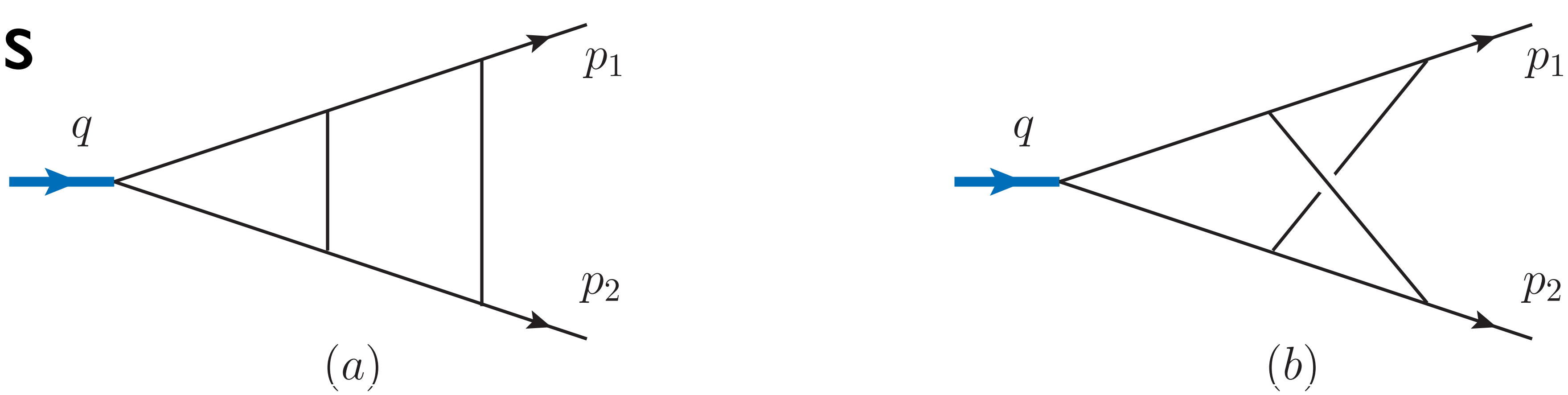

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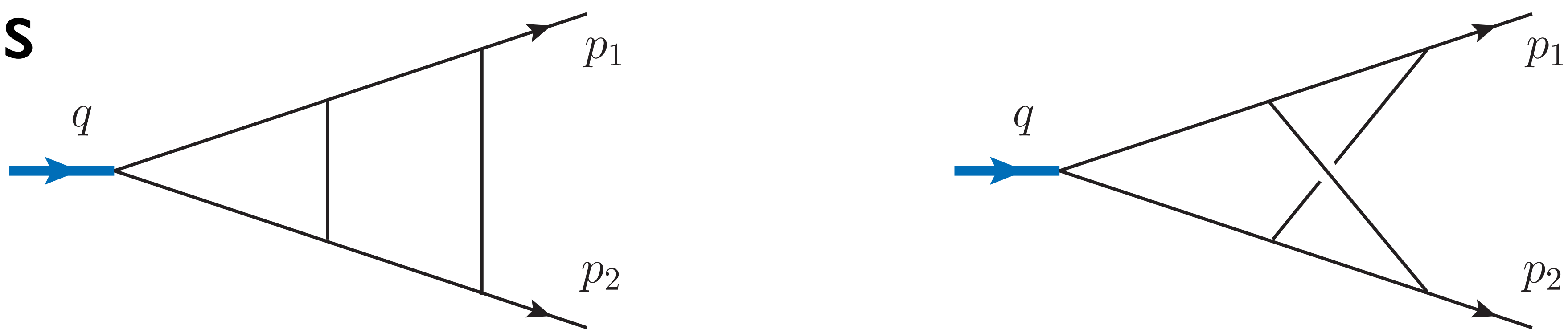

(a) (b)

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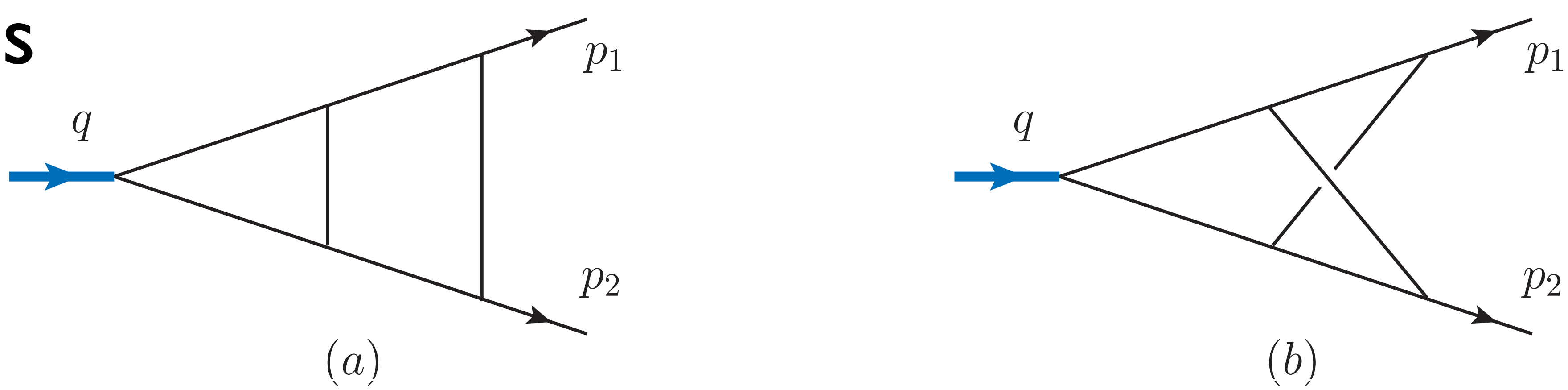

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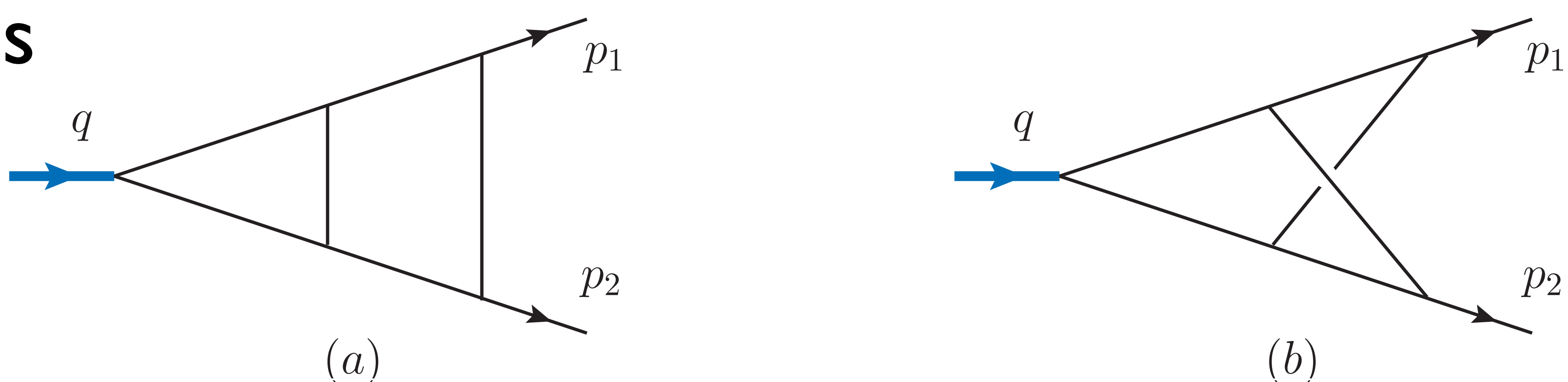

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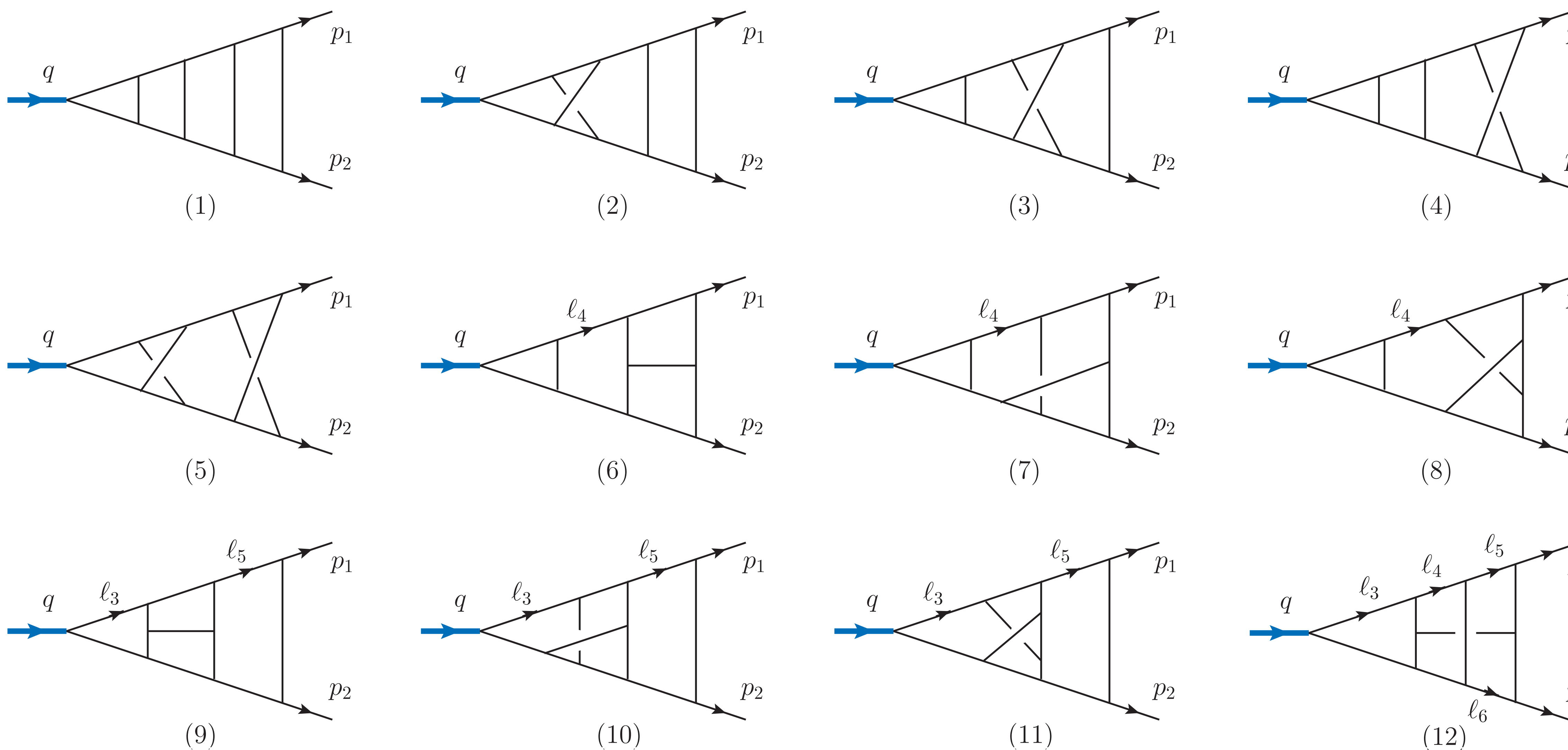

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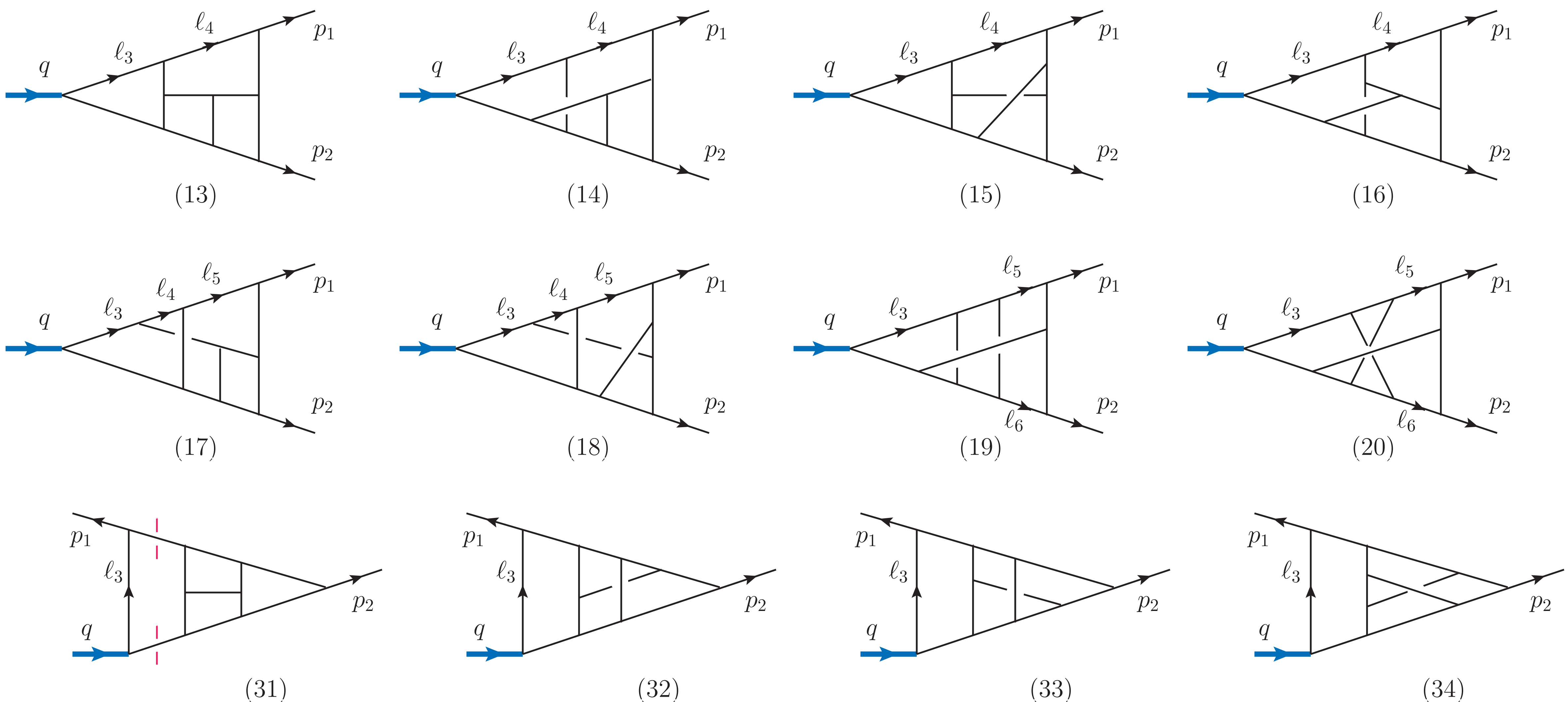

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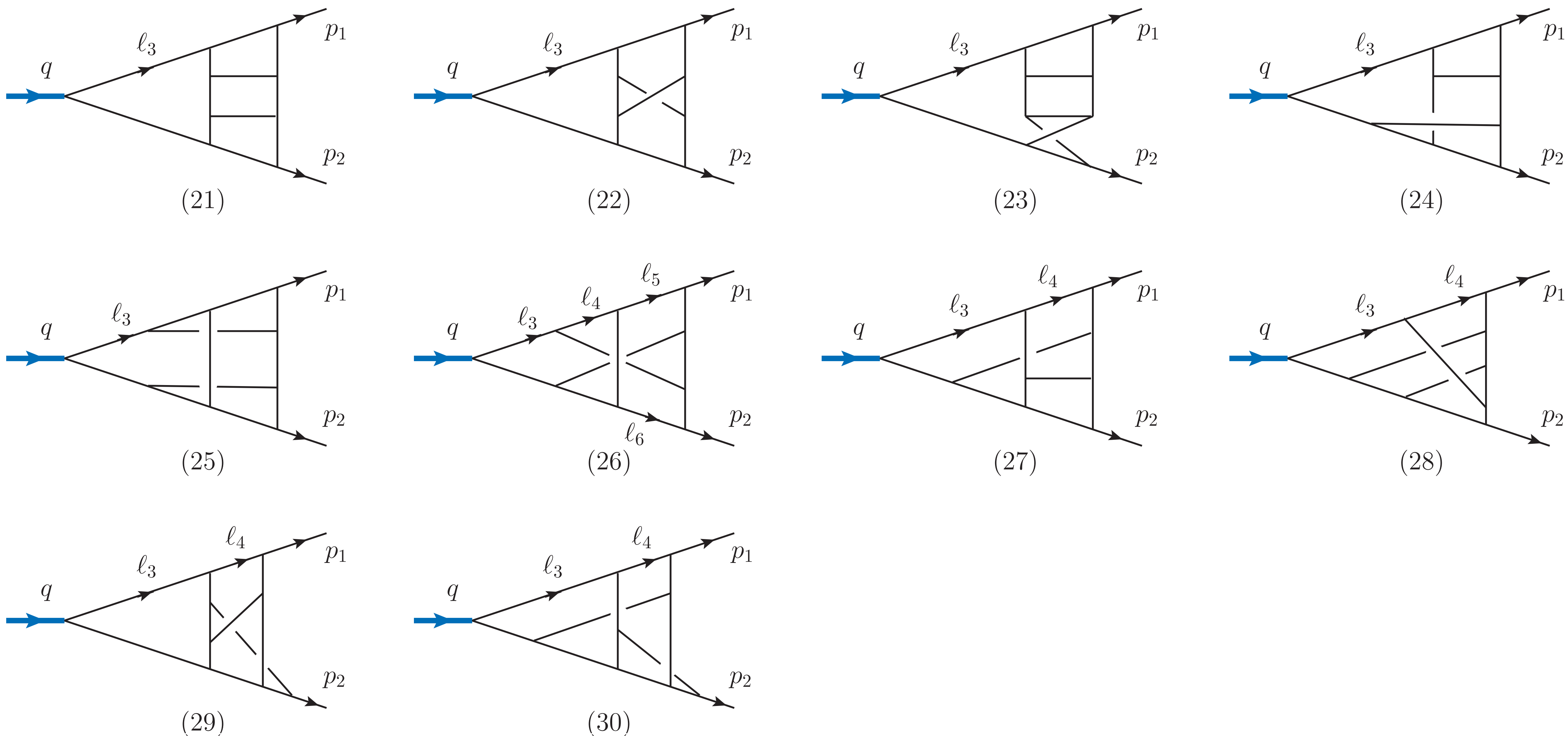

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byproduct → color-kinematic duality exists up to four loops for (some) form factors

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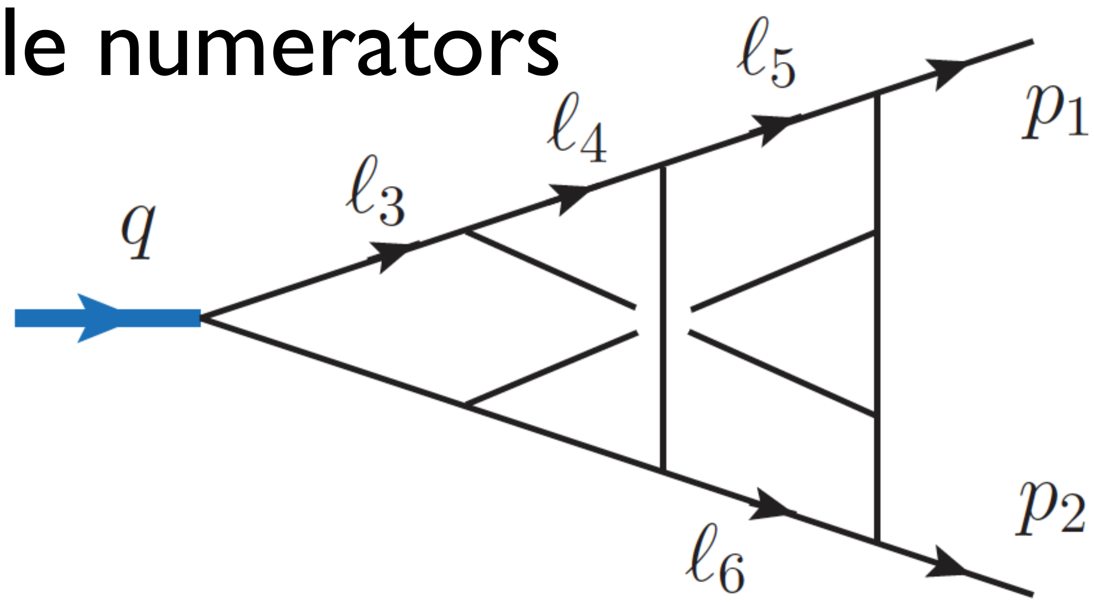

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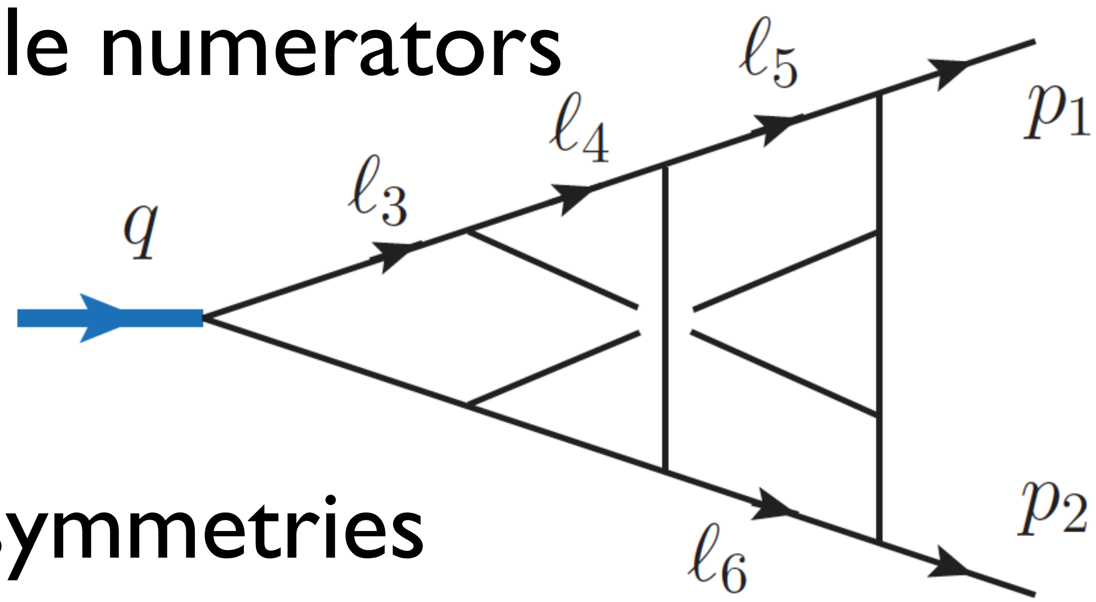

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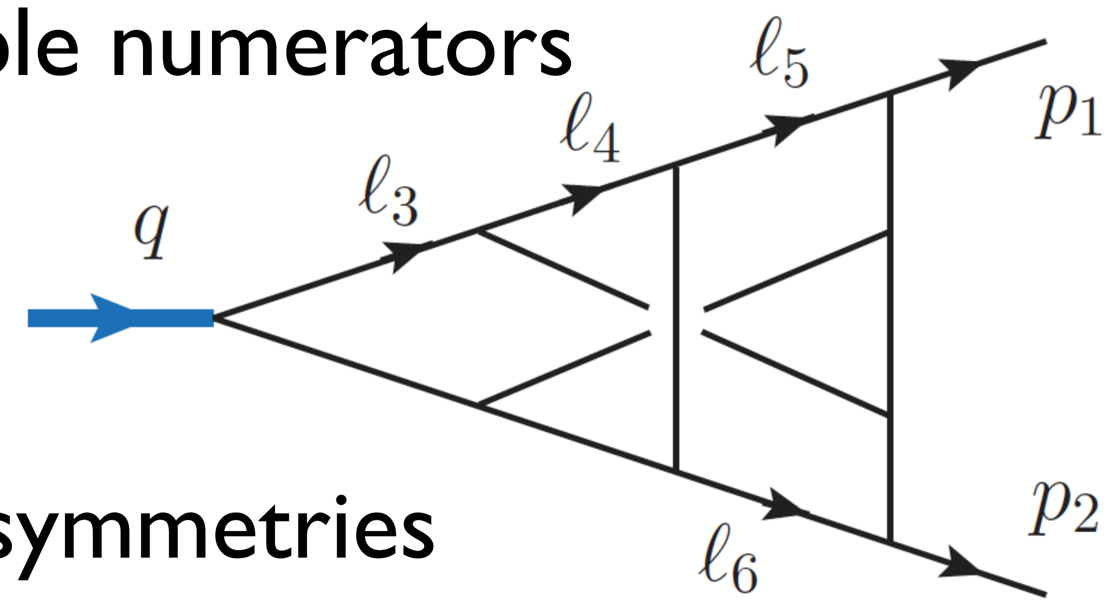

(26)

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- generically, 18 independent propagators, 6 irreducible numerators / graph topology

IBP reduction [RB-Kniehl-Yang, 12]

non-planar topology integrals with up-to quadratic numerators are still hard to integrate → need simpler integrals

IBPs implemented in many ways. Public:

- AIR [Anastasiou, Lazopoulos, 04],
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Reduze doesn’t scale well for beyond N=4, e.g. for QCD

IBP reduction

Large memory use on single machine:

11031	boels	20	0	18.9g	18g	16m	R	64.2	1.9	6147:50	reduze
11024	boels	20	0	16.6g	16g	16m	R	53.4	1.6	5864:25	reduze
11032	boels	20	0	15.7g	15g	20m	R	63.2	1.6	6058:05	reduze
11093	boels	20	0	15.3g	15g	25m	R	99.3	1.5	5997:52	reduze
11038	boels	20	0	14.5g	14g	24m	R	100.0	1.4	6444:40	reduze
11044	boels	20	0	14.5g	14g	12m	R	100.0	1.4	5258:38	reduze
11087	boels	20	0	14.5g	14g	16m	R	98.3	1.4	5519:41	reduze
11035	boels	20	0	14.4g	14g	13m	R	100.0	1.4	5660:43	reduze
11091	boels	20	0	14.4g	14g	22m	R	75.7	1.4	6635:26	reduze
11036	boels	20	0	14.3g	14g	14m	R	62.3	1.4	6230:30	reduze
11085	boels	20	0	14.3g	14g	13m	R	69.5	1.4	6133:47	reduze
11030	boels	20	0	13.7g	13g	21m	R	93.7	1.4	5396:21	reduze
11041	boels	20	0	13.6g	13g	12m	R	100.0	1.3	6550:45	reduze
11027	boels	20	0	13.5g	13g	15m	R	60.3	1.3	5835:31	reduze
11088	boels	20	0	13.2g	13g	10m	R	100.0	1.3	5135:12	reduze
11025	boels	20	0	12.7g	12g	23m	R	67.8	1.3	5784:08	reduze
11028	boels	20	0	12.6g	12g	12m	R	51.4	1.2	6609:43	reduze
11042	boels	20	0	12.3g	12g	14m	R	51.4	1.2	5718:09	reduze
11090	boels	20	0	12.3g	12g	12m	R	100.0	1.2	6565:02	reduze
11095	boels	20	0	11.7g	11g	25m	R	91.4	1.2	5593:43	reduze
11043	boels	20	0	11.7g	11g	19m	R	65.5	1.2	5523:44	reduze
11039	boels	20	0	11.5g	11g	12m	R	100.0	1.1	6213:14	reduze
11033	boels	20	0	11.3g	11g	11m	R	65.2	1.1	5780:59	reduze
11029	boels	20	0	11.2g	10g	14m	R	99.6	1.1	5996:48	reduze
11086	boels	20	0	10.6g	10g	11m	R	94.7	1.0	5672:34	reduze
11023	boels	20	0	9.9g	9.7g	14m	R	79.0	1.0	6489:06	reduze
11136	boels	20	0	9798m	9.4g	27m	R	91.8	0.9	6417:35	reduze
11026	boels	20	0	9308m	9.0g	27m	R	90.4	0.9	6421:29	reduze
11037	boels	20	0	9022m	8.7g	19m	R	59.0	0.9	5713:30	reduze
11040	boels	20	0	8876m	8.5g	14m	R	57.3	0.9	6139:33	reduze
11092	boels	20	0	8431m	8.1g	20m	S	100.0	0.8	6483:24	reduze
11089	boels	20	0	7900m	7.6g	21m	R	69.5	0.8	6322:00	reduze
11116	boels	20	0	7044m	6.7g	26m	R	100.0	0.7	6417:08	reduze
11034	boels	20	0	7042m	6.7g	20m	R	100.0	0.7	6022:42	reduze

IBP reduction: output

Reduze solves finite ranges of identities: choice up to 2 numerator powers, up to 12 denominator powers
(extension to 13, beyond unrealistic)

one unreduced master detected (file size) → obtained from symmetry

Table 1: Master integral statistics of obtained IBP reduction

(a) planar form factor

# props	$s=0$	s=1	$s=2$
12	8	6	0
11	18	2	0
10	43	9	0
9	49	1	0
8	51	4	1
7	25	0	0
6	8	0	0
5	0	0	0
sum	203	22	1

(b) non-planar form factor

# props	$s=0$	s=1	$s=2$
12	10	10	1
11	13	3	0
10	34	10	0
9	29	1	0
8	32	3	1
7	13	0	0
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Problem: integrals are still very hard and there's very many

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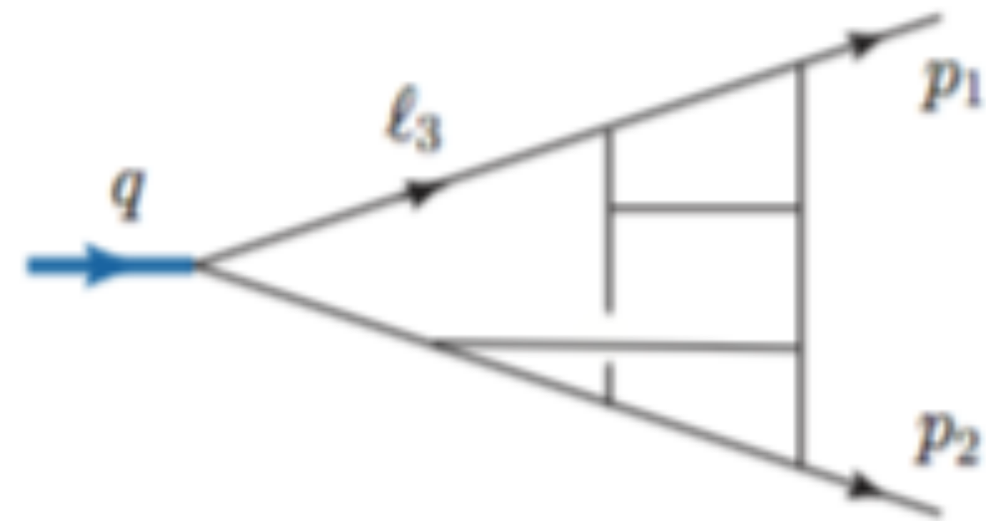
e.g. integral topology 24 using FIESTA 4 [Smirnov, 15]

choose numerator $(l3 \cdot (p_1 - p_2))^2$



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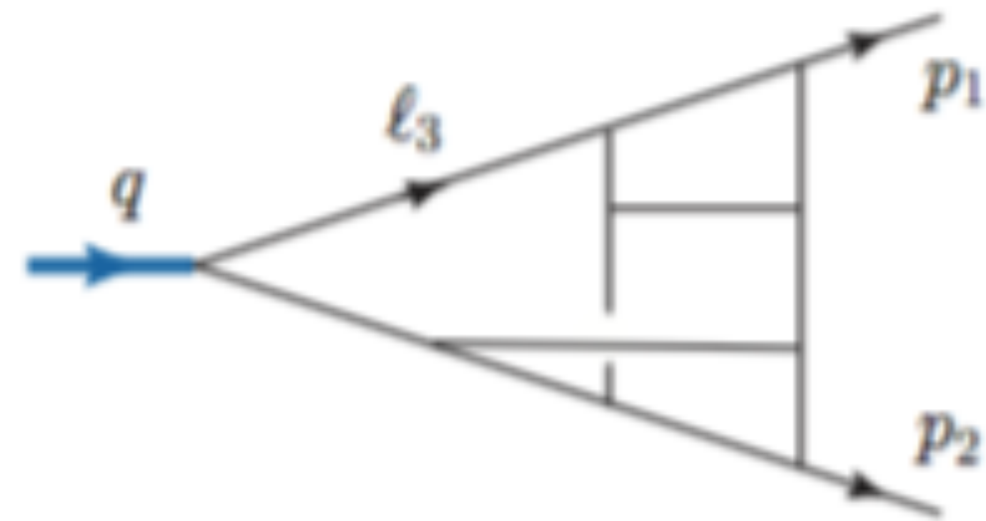

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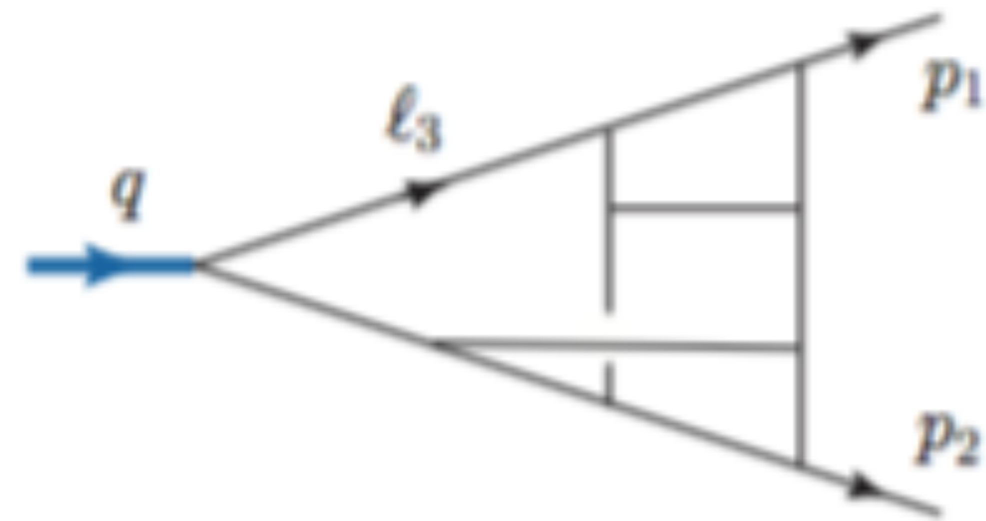

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(Reduze+FIESTA give the three loop cusp in ~ 2 days up to percent level)

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• → find an IBP reduction without \epsilon factors

• likely to be much faster, expect more compact results coefficient-wise, but also more master integrals

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select subset $$V \subset \{int_j\}$$

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- here: implementable by Mathematica, with one embellishment

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problem for form factor?

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- problem of the large rationals

11.allrels.txt

```
{-396193/3456, 314330163103/278691840, 9780866891/3096576, 0, -770143973743/278691840, 0, 333/160,
-118192620031/278691840, -251561659/483840, 237847776493/92897280, -11583815669/46448640, -477151/7680,
-10400892953/34836480, -58187987/145152, -1241830621/3870720, 14630804879/11612160, 2220571/221184,
4403543/103680, 24605521847/34836480, -2772151/552960, 1045180741/2903040, -287/20, 0, 13/24, -539280929/580608,
-2622127141/2903040, 270334399/725760, -270334399/725760, 0, 0, -118192620031/278691840, -29936311/552960,
-1074398251/2903040, -8622627271/8709120, -10463823863/69672960, 2576272241/3317760, -20354335591/23224320,
3485022715/13934592, -2155735847/2903040, -325143261343/278691840, 0, 325143261343/278691840, 0,
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270334399/483840, 270334399/483840, 0, 0, 0, 0, 0, 13/12, 0, 13/24, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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10657/5040, 6809/1512, -7807/10080, 0, 0, 0, 0, 0, 0, -25727/10080, 0, 6109/1680, -6109/3360, 6109/3360, 0, 0, 0,
0, 0, 17677/2520, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -3601/1440, 3601/960, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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-2622127141/2903040, 270334399/725760, -270334399/725760, 0, 0, -118192620031/278691840, -29936311/552960,
-1074398251/2903040, -8622627271/8709120, -10463823863/69672960, 2576272241/3317760, -20354335591/23224320,
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problem = opportunity

Mini-IBP (given IBP generate sub-IBP)

- problem of the large rationals

11.allrels.txt


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```


problem = opportunity

- reduce form factor without introducing ‘large’ rationals
- steer by # of master integrals in full ff (somewhat arbitrary)

Mini-IBP (given IBP generate sub-IBP)

- problem of the large rationals

11.allrels.txt

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```


problem = opportunity

- reduce form factor without introducing ‘large’ rationals

- steer by # of master integrals in full ff (somewhat arbitrary)

  → planar ff in terms of ~80 integrals with integer coefficients

Outlook

progress reported toward four loop form factors (any theory)

- basis of masters
- mini-IBP reduction

Outlook

progress reported toward four loop form factors (any theory)

- basis of masters
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still not quite the right way to compute….

- better basis?
  Von Manteuffel/Smirnov talks
- solve IBPs with 'four dots' to open more possibilities

Your Idea Here?