Adaptive Unitarity and Magnus Exponential for Scattering Amplitudes

Loops & Legs 2016
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based on the collaborations with:
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Motivation

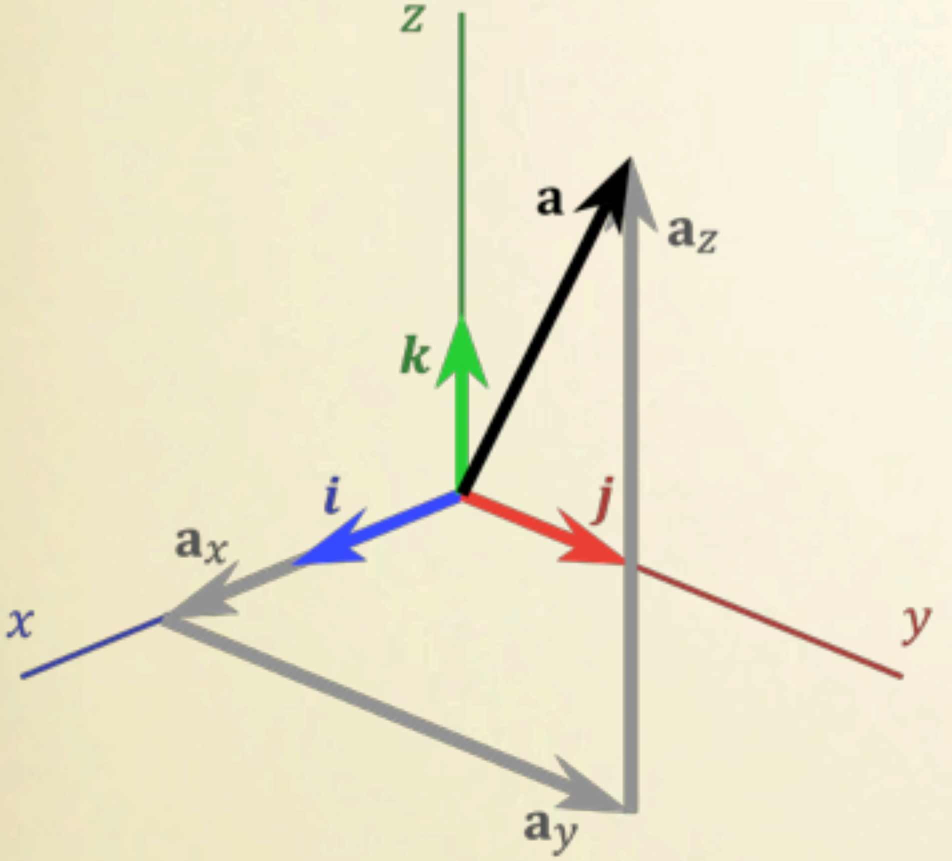
- Amplitudes & Phenomenology
  - Developing an algorithm for multi-loop many-leg processes
  - Extending the NLO automation at high high orders
  - From the beauty of simple formulas (in special kinematics) to the beauty of the structures (in arbitrary kinematics)

Path

- Multi-loop Integrand Reduction: exploiting dimensional regularization
  - Revisiting the 1-loop decomposition
  - Novel decomposition @ any loop: the 2-loop case to begin with
- Magnus Series for Master Integrals
  - Two-loop Master Integrals for the QCD-EW corrections to Drell-Yan scattering

Amplitudes Decomposition:

the algebraic way



$\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$

- **Basis**: {$\mathbf{i}$ $\mathbf{j}$ $\mathbf{k}$}

- **Scalar product**/**Projection**: to extract the components

$a_x = \mathbf{a}.\mathbf{i}$

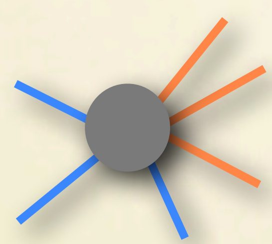
$a_y = \mathbf{a}.\mathbf{j}$

$a_z = \mathbf{a}.\mathbf{k}$

Amplitudes Decomposition:

the algebraic way

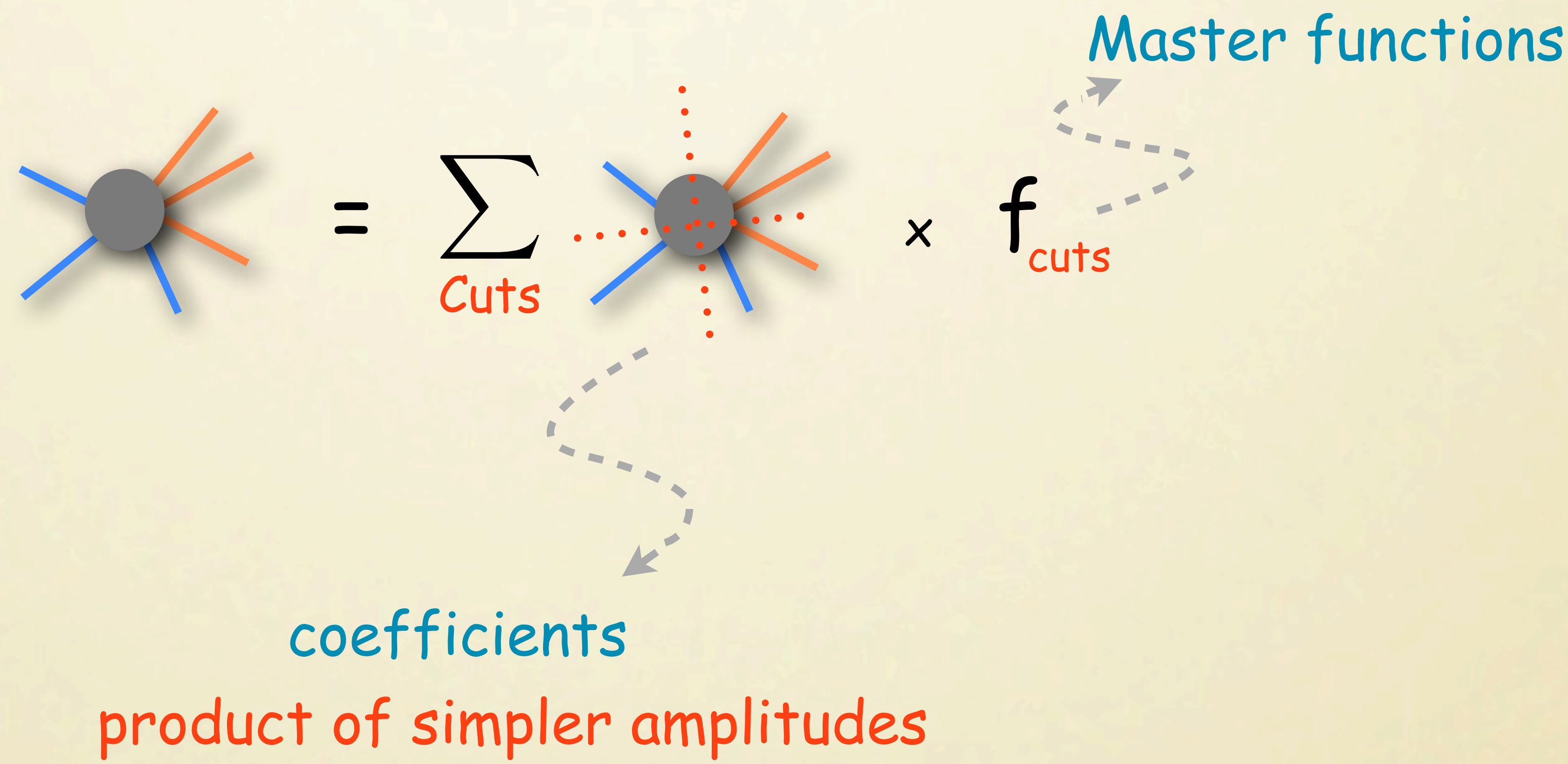
Master functions

 $= \sum_i c_i \times f_i$

coefficients

Amplitudes Decomposition:

the algebraic way



Integrand-Reduction

unitarity at integrand level

Ossola Papadopoulos Pittau (2006)

Ellis Giele Kunszt Melnikov (2007)

Ossola & ***P.M.*** (2011)

Badger, Frellesvig, Zhang (2011)

Zhang (2012)

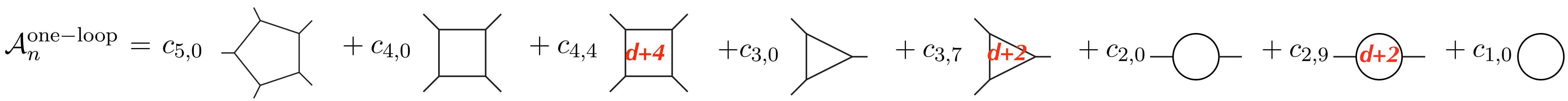
Mirabella, Ossola, Peraro, & ***P.M.*** (2012)

>> Ossola

One-Loop Integrand Decomposition

$$\mathcal{A}_n^{\text{one-loop}} = \int d^{-2\epsilon}\mu \int d^4 q \; A_n(q,\mu^2) \,, \qquad A_n(q,\mu^2) \equiv \frac{\mathcal{N}_n(q,\mu^2)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{n-1}} \qquad \bar{D}_i = (\bar{q}+p_i)^2 - m_i^2 = (q+p_i)^2 - m_i^2 - \mu^2$$

We use a bar to denote objects living in $d = 4 - 2\epsilon$ dimensions $\quad \bar{\not{q}} = \not{q} + \not{\mu}\,, \quad$ with $\quad \bar{q}^2 = q^2 - \mu^2 \,.$



☑ @ the integrand-level

Ossola, Papadopoulos, Pittau

$$A_n(q,\mu^2) \neq \frac{c_{5,0}}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3\bar{D}_4} + \frac{c_{4,0}+c_{4,4}\mu^4}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3} + \frac{c_{3,0}+c_{3,7}\mu^2}{\bar{D}_0\bar{D}_1\bar{D}_2} + \frac{c_{2,0}+c_{2,9}\mu^2}{\bar{D}_0\bar{D}_1} + \frac{c_{1,0}}{\bar{D}_0}$$

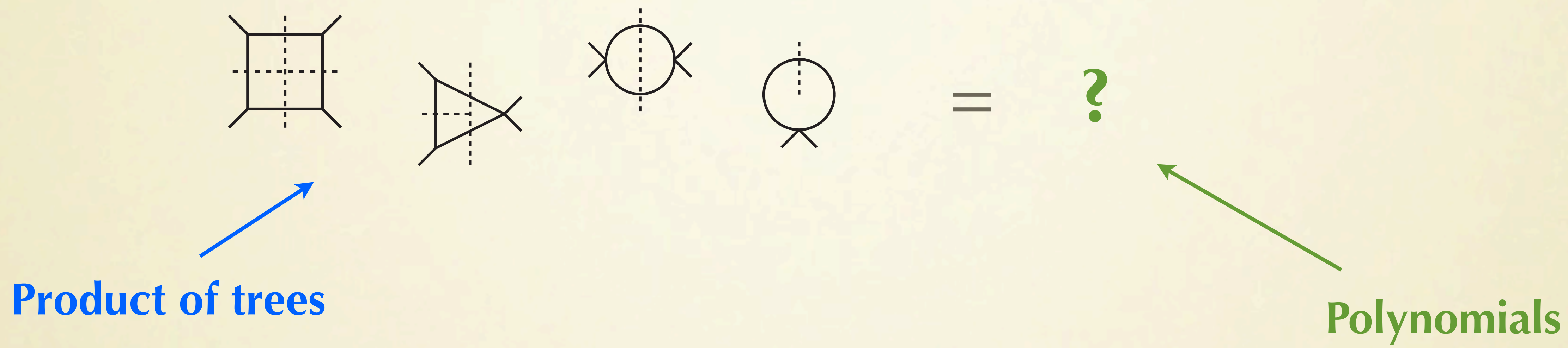
$$= \frac{c_{5,0}+f_{01234}(q,\mu^2)}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3\bar{D}_4} + \frac{c_{4,0}+c_{4,4}\mu^4+f_{0123}(q,\mu^2)}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3} + \frac{c_{3,0}+c_{3,7}\mu^2+f_{012}(q,\mu^2)}{\bar{D}_0\bar{D}_1\bar{D}_2} + \frac{c_{2,0}+c_{2,9}\mu^2+f_{01}(q,\mu^2)}{\bar{D}_0\bar{D}_1} + \frac{c_{1,0}+f_0(q,\mu^2)}{\bar{D}_0}$$

☑ f's are "spurious" ==> integrate to 0 !!!

Towards Higher Loop

Ossola & **P.M.** (2011)

Problem: what is the form of the residues?



"find the right variables encoding the cut-structure"

variables

- ISP's = Irreducible Scalar Products:
  - q-components which can variate under cut-conditions
  - spurious: vanishing upon integration
  - non-spurious: non-vanishing upon integration $\Rightarrow$ MI's

Zhang (2012); Badger Frellesvig Zhang (2012)
Mirabella, Ossola, Peraro, & **P.M.** (2012)

- Quantum Field Theory
  - Unitarity-Cuts, Vanishing denominators
  - Cut-residue
  - Amplitudes factorization in tree-amplitudes

Amplitude decomposition

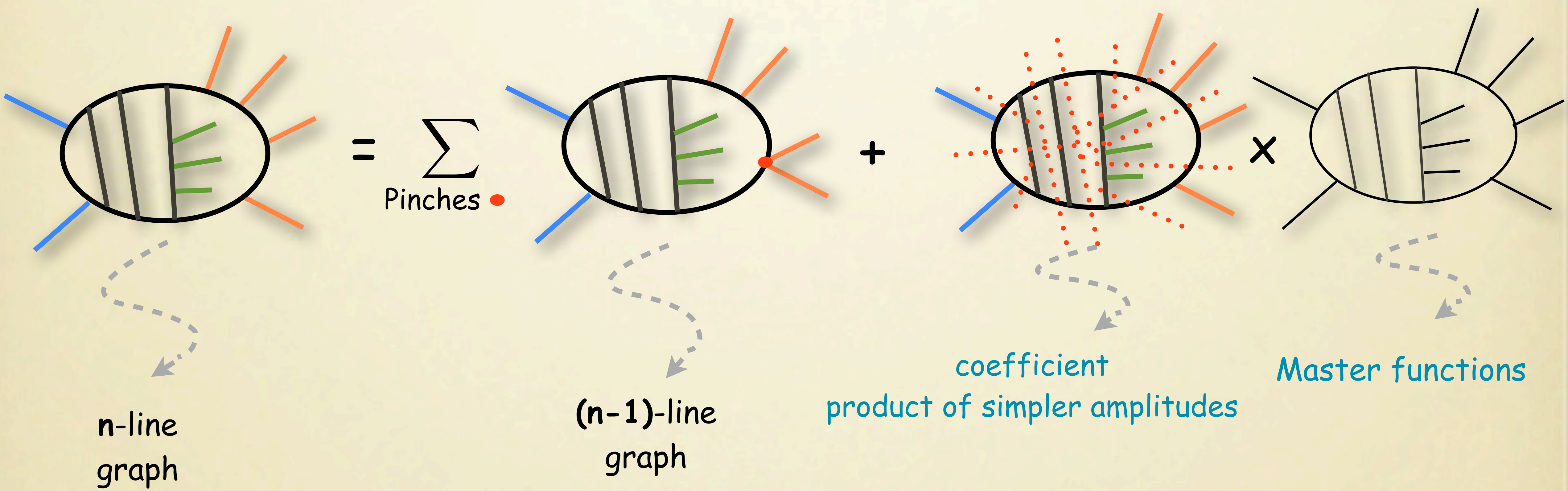
- Algebraic Geometry
  - Polynomial equations, ideals
  - Remainder of polynomial division
  - Polynomials in quotient rings

Multivariate Polynomial division

Multi-Loop Integrand Recurrence

Mirabella, Ossola, Peraro, & ***P.M.*** (2012)

- **ℓ-Loop Recurrence Relation**



- ☑ **all orders** (any number of loops and legs)
- ☑ **any topology** (planar and non-planar)
- ☑ **all kinematics** (massless and massive)
- ☑ **high-power of denominators**

The Maximum-Cut Theorem

Mirabella, Ossola, Peraro, & **P.M.** (2012)

At any loop ℓ, loops we define *maximum cut* as the set of vanishing denominators

$$D_0 = D_1 = \ldots = 0$$

which constrains completely the components of the loop momenta. 0-dimensional
We assume that, in non-exceptional phase-space points, a maximum-cut has a finite number n_s of solutions, each with multiplicity one.
Then,

Theorem 4.1 (Maximum cut). *The residue at the maximum-cut is a polynomial paramatrised by n_s coefficients, which admits a univariate representation of degree $(n_s - 1)$.*

One-Loop Integrand Decomposition $\quad d = 4 - 2\epsilon$

- Choice of 4-dimensional basis for an m-point residue

$$e_1^2 = e_2^2 = 0\,, \qquad e_1 \cdot e_2 = 1\,, \qquad e_3^2 = e_4^2 = \delta_{m4}\,, \qquad e_3 \cdot e_4 = -(1-\delta_{m4})$$

- Coordinates: $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5) \equiv (x_1, x_2, x_3, x_4, \mu^2)$

$$q^\mu_{\text{4-dim}} = -p^\mu_{i_1} + x_1\, e_1^\mu + x_2\, e_2^\mu + x_3\, e_3^\mu + x_4\, e_4^\mu\,, \qquad q^2 = q^2_{\text{4-dim}} - \mu^2$$

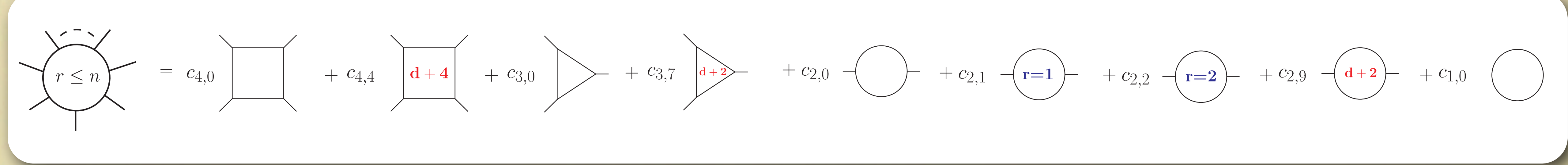
- Generic numerator

$$\mathcal{N}_{i_1\cdots i_m} = \sum_{j_1,\ldots,j_5} \alpha_{\vec{j}}\, z_1^{j_1}\, z_2^{j_2}\, z_3^{j_3}\, z_4^{j_4}\, z_5^{j_5}\,, \qquad (j_1 \ldots j_5) \quad \text{such that} \quad \text{rank}(\mathcal{N}_{i_1\cdots i_m}) \leq m$$

- Residues

reproducing:
Ossola Papadopoulos Pittau
Ellis Giele Kunszt Melnikov

$$\begin{aligned}
\Delta_{i_1 i_2 i_3 i_4 i_5} &= c_0 \\
\Delta_{i_1 i_2 i_3 i_4} &= c_0 + c_1 x_4 + \mu^2 (c_2 + c_3 x_4 + \mu^2 c_4) \\
\Delta_{i_1 i_2 i_3} &= c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2 (c_7 + c_8 x_3 + c_9 x_4) \\
\Delta_{i_1 i_2} &= c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2 \\
\Delta_{i_1} &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{aligned}$$



Longitudinal and Transverse Space

Peraro Primo ***P.M.*** (to appear)

- **Dimensional Regularization**

$$d = 4 - 2\epsilon$$

- **if n-legs < 5**

$$d = d_{//} + d_{\perp}$$

Longitudinal space
spanned by the
(independent) legs

Transverse Space

- ☑ Denominators **do not depend on** "the angular variables" of the Transverse Space $\Omega_{\perp}$
- ☑ Numerators **depend on** "all" loop variables

idea n.1

$$d = d_{//} + d_{\perp}$$

Integrating over Transverse Space

One-Loop Integrals

$$\boxed{d = 4 - 2\epsilon}$$

$$I_n^d[\mathcal{N}] = \int \frac{d^d q}{\pi^{d/2}} \frac{\mathcal{N}(q)}{\prod_{i=0}^{n-1} \mathcal{D}_i}, \qquad \mathcal{D}_i = \Big(q + \sum_{j=0}^{i} p_j\Big)^2 + m_i^2, \qquad p_0 = 0,$$

- loop momentum parametrization

$$q^\alpha = q_{[4]}^\alpha + \mu^\alpha, \qquad q_{[4]}^\alpha = \sum_{i=1}^{4} x_i e_i^\alpha, \qquad q^2 = q_{[4]}^2 + \mu^2.$$

- Integration variables

$$I_n^d[\mathcal{N}] = \frac{\mathcal{K}}{\pi^2 \Gamma\left(\frac{d-4}{2}\right)} \int_{-\infty}^{\infty} \prod_{i=1}^{4} dx_i \int_0^\infty d\mu^2 (\mu^2)^{\frac{d-6}{2}} \frac{\mathcal{N}(x_i, \mu^2)}{\prod_{i=0}^{n-1} \mathcal{D}_i},$$

$$\mathcal{D}_i = \Big(q_{[4]} + \sum_{j=0}^{i} p_j\Big)^2 + \mu^2 + m_i^2, \qquad \mathcal{K} = \sqrt{\det\left(\frac{\partial q_{[4]}^\mu}{\partial x_i} \frac{\partial q_{[4]\,\mu}}{\partial x_j}\right)}.$$

One-Loop Integrals

$$\boxed{d = d_{//} + d_{\perp}}$$

- loop momentum parametrization

$$q^\alpha = q^\alpha_{[k]} + \lambda^\alpha, \qquad q^\alpha_{[k]} = \sum_{j=1}^{k} x_j e^\alpha_j, \qquad q^2 = q^2_{[k]} + \lambda^2,$$

k-dimensional the space spanned by the external momenta

$$\lambda^\alpha = \sum_{j=k+1}^{4} x_j e^\alpha_j + \mu^\alpha, \qquad \lambda^2 = \sum_{j=k+1}^{4} x_j^2 + \mu^2,$$

$(d-k)$-dimensional orthogonal subspace.

Larsen Zhang;

$$I_n^d[\mathcal{N}] = \frac{1}{\pi^2 \Gamma\left(\frac{d-4}{2}\right)} \int d^k q_{[k]} \int_0^\infty d\lambda^2 (\lambda^2)^{\frac{d-k-2}{2}} \prod_{i=1}^{4-k} \int_{-1}^{1} d\cos\theta_i (\sin\theta_i)^{d-k-i-2} \frac{\mathcal{N}(q)}{\prod_{i=0}^{n-1} \mathcal{D}_i}.$$

$$\mathcal{N}(q) \equiv \mathcal{N}(q^\alpha_{[k]}, \lambda^2, \{x_{k+1}, \ldots, x_4\}). \qquad \mathcal{D}_i = \left(q_{[k]} + \sum_{j=0}^{i} p_j\right)^2 + \lambda^2 + m_i^2.$$

- ☑ **Denominators do not depend on** "the angular variables" of the **Transverse Space** $\Omega_\perp$

- ☑ **Integration over** $\Omega_\perp$ **: ?**

Integrating over Transverse Angles

Peraro Primo ***P.M.*** (to appear)

Gegenbauer polynomials

orthogonal polynomials over the interval $[-1,1]$

weight function $\omega_\alpha(x) = (1-x^2)^{\alpha-\frac{1}{2}}$

generating function $\frac{1}{(1-2xt+t^2)^\alpha} = \sum_{n=1}^{\infty} C_n^{(\alpha)}(x) t^n.$

$$
\begin{aligned}
C_0^{(\alpha)}(x) &= 1,\\
C_1^{(\alpha)}(x) &= 2\alpha x,\\
C_2^{(\alpha)}(x) &= -\alpha + 2\alpha(1+\alpha)x^2,
\end{aligned}
$$

$$
\begin{aligned}
x &= \frac{1}{2\alpha} C_0^{(\alpha)}(x) C_1^{(\alpha)}(x),\\
x^2 &= \frac{1}{4\alpha^2}[C_1^{(\alpha)}(x)]^2,\\
x^3 &= \frac{1}{4\alpha^2(1+\alpha)} C_1^{(\alpha)}(x)[\alpha C_0^{(\alpha)}(x) + C_2^{(\alpha)}(x)],\\
x^4 &= \frac{1}{4\alpha^2(1+\alpha)^2}[\alpha C_0^{(\alpha)}(x) + C_2^{(\alpha)}(x)]^2,\\
&\dots
\end{aligned}
$$

Orthogonality condition

$$
\int_{-1}^{1} d\cos\theta (\sin\theta)^{2\alpha-1} C_n^{(\alpha)}(\cos\theta) C_m^{(\alpha)}(\cos\theta) = \delta_{mn} \frac{2^{1-2\alpha}\pi\Gamma(n+2\alpha)}{n!(n+\alpha)\Gamma^2(\alpha)}.
$$

- ☑ **Integration over Transverse Angles: trivialized @ all-loop!**
- ☑ Alternative to PV-tensor reduction in the transverse-space

One-Loop Integrals

$$\boxed{d = d_{//} + d_\perp}$$

- loop momentum parametrization

$$q^\alpha = q^\alpha_{[k]} + \lambda^\alpha, \qquad q^\alpha_{[k]} = \sum_{j=1}^{k} x_j e^\alpha_j, \qquad q^2 = q^2_{[k]} + \lambda^2,$$

k-dimensional the space spanned by the external momenta

$$\lambda^\alpha = \sum_{j=k+1}^{4} x_j e^\alpha_j + \mu^\alpha, \qquad \lambda^2 = \sum_{j=k+1}^{4} x_j^2 + \mu^2, \qquad (d-k)\text{-dimensional orthogonal subspace.}$$

$$I_n^d[\mathcal{N}] = \frac{1}{\pi^2 \Gamma\left(\frac{d-4}{2}\right)} \int d^k q_{[k]} \int_0^\infty d\lambda^2 (\lambda^2)^{\frac{d-k-2}{2}} \prod_{i=1}^{4-k} \int_{-1}^{1} d\cos\theta_i (\sin\theta_i)^{d-k-i-2} \frac{\mathcal{N}(q)}{\prod_{i=0}^{n-1} \mathcal{D}_i}.$$

$$\mathcal{N}(q) \equiv \mathcal{N}(q^\alpha_{[k]}, \lambda^2, \{x_{k+1}, \ldots, x_4\}). \qquad \mathcal{D}_i = \Big(q_{[k]} + \sum_{j=0}^{i} p_j\Big)^2 + \lambda^2 + m_i^2.$$

☑ **Denominators do not depend on** "the angular variables" of the **Transverse Space** $\Omega_\perp$

☑ **Integration over** $\Omega_\perp$ **: *Gegenbauer orthogonality condition***

Spurious integrals vanish automatically!

Four-point integrals

$$I_4^d[\mathcal{N}\,] = \int \frac{d^3 q_{[3]}}{\pi^{d/2}} \int d^{d-3}\lambda \frac{\mathcal{N}(q_{[3]}, \lambda^2, x_4)}{\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3}.$$

$$x_4 = \lambda \cos\theta_1$$

$$I_4^d[\mathcal{N}\,] = \frac{1}{\pi^2\Gamma\left(\frac{d-4}{2}\right)} \int d^3 q_{[3]} \int_0^\infty d\lambda^2 (\lambda^2)^{\frac{d-5}{2}} \int_{-1}^1 d\cos\theta_1 (\sin\theta_1)^{d-6} \frac{\mathcal{N}(q_{[3]}, \lambda^2, \cos\theta_1)}{\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3}.$$

Examples

$$\cos^2\theta_1 = \frac{1}{(d-5)^2}\left[C_1^{(\frac{d-5}{2})}(\cos\theta_1)\right]^2,$$

$$\cos^4\theta_1 = \frac{1}{(d-3)^2}\left[C_0^{(\frac{d-5}{2})}(\cos\theta_1) + \frac{4}{(d-5)^2} C_2^{\frac{d-5}{2}}(\cos\theta_1)\right]^2$$

$$I_4^d[\,x_4^2\,] = \frac{1}{d-3} I_4^d[\,\lambda^2\,] = \frac{1}{2} I_4^{d+2}[1],$$

$$I_4^d[\,x_4^4\,] = \frac{3}{(d-3)(d-1)} I_4^d[\,\lambda^4\,] = \frac{3}{4} I_4^{d+4}[1].$$

Gegenbauer integration produces powers of $\lambda_{ij} = \boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j$

Three-point integrals

$$\begin{cases} x_3 = \lambda \cos\theta_1 \\ x_4 = \lambda \sin\theta_1 \cos\theta_2 \end{cases}$$

$$I_3^d[\mathcal{N}] = \frac{1}{\pi^2 \Gamma\left(\frac{d-4}{2}\right)} \int d^2 q_{[2]} \int_0^\infty d\lambda^2 (\lambda^2)^{\frac{d-4}{2}} \int_{-1}^{1} d\cos\theta_1 (\sin\theta_1)^{d-5} \times \int_{-1}^{1} d\cos\theta_2 (\sin\theta_2)^{d-6} \frac{\mathcal{N}(q_{[2]}, \lambda^2, \{\cos\theta_1, \sin\theta_1, \cos\theta_2\})}{\mathcal{D}_0 \mathcal{D}_1 \mathcal{D}_2}.$$

Two-point integrals

$$\begin{cases} x_2 = \lambda \cos\theta_1 \\ x_3 = \lambda \sin\theta_1 \cos\theta_2, \\ x_4 = \lambda \sin\theta_1 \sin\theta_2 \cos\theta_3 \end{cases}$$

$$I_2^d[\mathcal{N}] = \frac{1}{\pi^2 \Gamma\left(\frac{d-4}{2}\right)} \int dq_{[1]} \int_0^\infty d\lambda^2 (\lambda^2)^{\frac{d-3}{2}} \int_{-1}^{1} d\cos\theta_1 (\sin\theta_1)^{d-4} \times \int_{-1}^{1} d\cos\theta_2 (\sin\theta_2)^{d-5} \int_{-1}^{1} d\cos\theta_3 (\sin\theta_3)^{d-6} \times \frac{\mathcal{N}(q_{[1]}, \lambda^2, \cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2, \cos\theta_3)}{\mathcal{D}_0 \mathcal{D}_1},$$

$$I_2^d[\mathcal{N}]\big|_{p^2=0} = \frac{1}{\pi^2 \Gamma\left(\frac{d-4}{2}\right)} \int d^2 q_{[2]} \int_0^\infty d\lambda^2 (\lambda^2)^{\frac{d-4}{2}} \int_{-1}^{1} d\cos\theta_1 (\sin\theta_1)^{d-5} \times \int_{-1}^{1} d\cos\theta_2 (\sin\theta_2)^{d-6} \frac{\mathcal{N}(q_{[2]}, \lambda^2, \cos\theta_1, \sin\theta_1, \cos\theta_2)}{\mathcal{D}_0 \mathcal{D}_1};$$

$$\begin{cases} x_3 = \lambda \cos\theta_1 \\ x_4 = \lambda \sin\theta_1 \cos\theta_2 \end{cases}$$

One-point integrals

$$\begin{cases} x_1 = \lambda \cos\theta_1, \\ x_2 = \lambda \sin\theta_1 \cos\theta_2, \\ x_3 = \lambda \sin\theta_1 \sin\theta_2 \cos\theta_3 \\ x_4 = \lambda \sin\theta_1 \sin\theta_2 \sin\theta_3 \cos\theta_4 \end{cases}$$

$$I_1^d[\mathcal{N}] = \frac{1}{\pi^2 \Gamma\left(\frac{d-4}{2}\right)} \int_0^\infty d\lambda^2 (\lambda^2)^{\frac{d-2}{2}} \int_{-1}^{1} d\cos\theta_1 (\sin\theta_1)^{d-3} \int_{-1}^{1} d\cos\theta_1 (\sin\theta_1)^{d-4} \times \int_{-1}^{1} d\cos\theta_2 (\sin\theta_2)^{d-5} \times \int_{-1}^{1} d\cos\theta_3 (\sin\theta_3)^{d-6} \times \frac{\mathcal{N}(q_{[1]}, \lambda^2, \cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2, \cos\theta_3, \sin\theta_3, \cos\theta_4)}{\mathcal{D}_0}$$

idea n.2

$$d = d_{//} + d_{\perp}$$

Cutting in the Longitudinal Space

Adaptive Unitarity @ 1-loop

$$d = d_{//} + d_{\perp}$$

- **Integrand red'n**

$$q^\alpha = q^\alpha_{[k]} + \lambda^\alpha, \qquad \mathcal{D}_i = \Big(q_{[k]} + \sum_{j=0}^{i} p_j\Big)^2 + \lambda^2 + m_i^2.$$

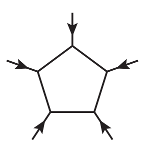
$$q^\alpha_{[k]} = \sum_{j=1}^{k} x_j e^\alpha_j, \qquad \lambda^2$$

reducible

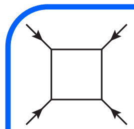
Cutting in different dimensions
according to the # of legs

1-loop :: always MAXIMUM CUTS

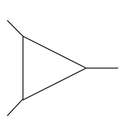
New residue parametrization



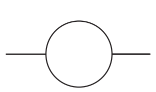
$$\Delta_{i_0 \cdots i_4} = c_0.$$



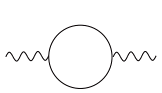
$$\Delta_{i_0 \cdots i_3} = c_0 + c_1 x_4 + c_2 x_4^2 + c_3 x_4^3 + c_4 x_4^4,$$



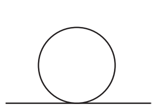
$$\Delta_{i_0 i_1 i_2} = c_0 + c_1 x_3 + c_2 x_4 + c_3 x_3^2 + c_4 x_3 x_4 + c_5 x_4^2 + c_6 x_3^3 + c_7 x_3^2 x_4 + c_8 x_3 x_4^2 + c_9 x_4^3.$$



$$\Delta_{i_0 i_1} = c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2 x_3 + c_5 x_2 x_4 + c_6 x_3 x_4 + c_7 x_2^2 + c_8 x_3^2 + c_9 x_4^2.$$



$$\Delta_{i_0 i_1}|_{p^2=0} = c_0 + c_1 x_1 + c_2 x_3 + c_3 x_4 + c_4 x_1 x_3 + c_5 x_1 x_4 + c_6 x_3 x_4 + c_7 x_1^2 + c_8 x_3^2 + c_9 x_4^2.$$



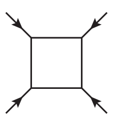
$$\Delta_{i_0} = c_0 + \sum_{i=1}^{4} c_i x_i$$

Adaptive Unitarity @ 1-loop

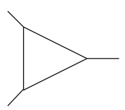
$$\boxed{d = d_{//} + d_{\perp}}$$

- **Integrand red'n**

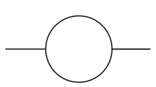
Integration of the Residues over Transverse Angles



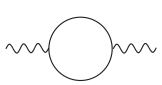
$$\int \frac{d^dq}{\pi^{d/2}} \frac{\Delta_{i_0 i_1 i_2 i_3}}{\mathcal{D}_{i_0}\mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3}} \boxed{= c_0 I_4^d[1] + \frac{1}{(d-3)} c_2 I_4^d[\lambda^2] + \frac{3}{(d-3)(d-1)} c_4 I_4^d[\lambda^4]} = c_0 I_4^d[1] + \frac{1}{2} c_2 I_4^{d+2}[1] + \frac{3}{4} c_4 I_4^{d+4}[1].$$



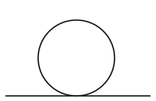
$$\int \frac{d^dq}{\pi^{d/2}} \frac{\Delta_{i_0 i_1 i_2}}{\mathcal{D}_{i_0}\mathcal{D}_{i_1}\mathcal{D}_{i_2}} = \boxed{c_0 I_3^d[1] + \frac{1}{(d-3)} (c_3 + c_5) I_3^d[\lambda^2]} = c_0 I_3^d[1] + \frac{1}{2} (c_3 + c_5) I_3^{d+2}[1].$$



$$\int \frac{d^dq}{\pi^{d/2}} \frac{\Delta_{i_0 i_1}}{\mathcal{D}_{i_0}\mathcal{D}_{i_1}} = \boxed{c_0 I_2^d[1] + \frac{1}{(d-3)} (c_7 + c_8 + c_9) I_2^d[\lambda^2]} = c_0 I_2^d[1] + \frac{1}{2} (c_7 + c_8 + c_9) I_2^{d+2}[1].$$



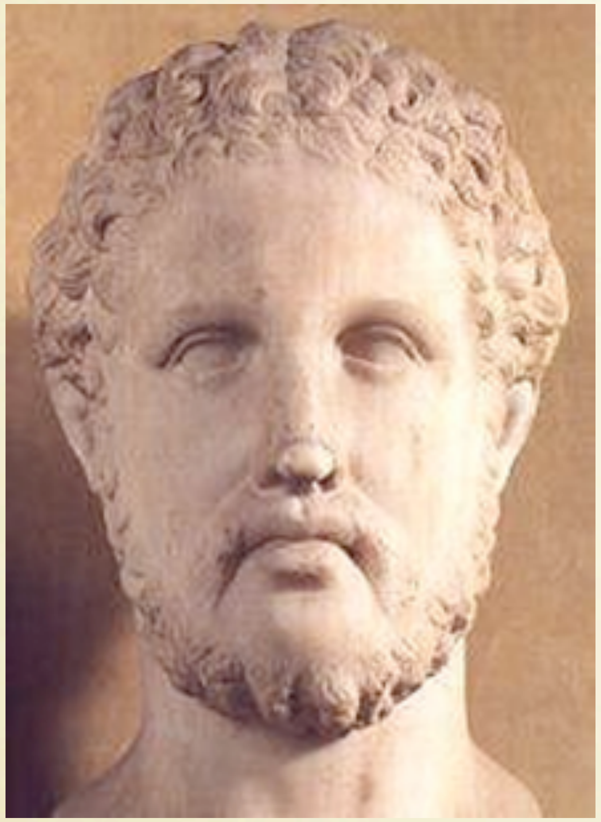
$$\int \frac{d^dq}{\pi^{d/2}} \frac{\Delta_{i_0 i_1}}{\mathcal{D}_{i_0}\mathcal{D}_{i_1}}\Bigg|_{p^2=0} = \boxed{c_0 I_2^d[1] + c_1 I_2^d[x_1] + c_7 I_2^d[x_1^2] + \frac{1}{(d-3)} (c_8 + c_9) I_3^d[\lambda^2]} = c_0 I_2^d[1] + c_1 I_2^d[x_1] + c_7 I_2^d[x_1^2] + \frac{1}{2} (c_8 + c_9) I_2^{d+2}[1].$$



$$\int \frac{d^dq}{\pi^{d/2}} \frac{\Delta_{i_0}}{\mathcal{D}_{i_0}} = c_0 I_1^d[1].$$

λ^2

reducible



Divide et Impera

Philip II of Macedon

i) Divide...
ii) et Integra...
iii) et Divide

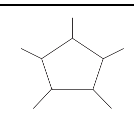
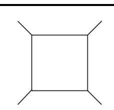
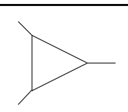
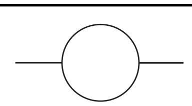
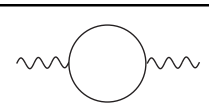
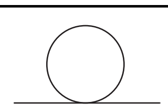
Divide-et-Integra-et-Divide

Additional Polynomial Division

λ^2

reducible

divide integra divide

Topology		$\Delta_{i_0 \cdots i_n}$	$\Delta^{\rm int}_{i_0 \cdots i_n}$	$\Delta'_{i_0 \cdots i_n}$
$\mathcal{I}_{01234}$		1 $\{1\}$	– –	– –
$\mathcal{I}_{0123}$		5 $\{1, x_4, x_4^2, x_4^3, x_4^4\}$	3 $\{1, \lambda^2, \lambda^4\}$	1 $\{1\}$
$\mathcal{I}_{012}$		10 $\{1, x_3, x_4, x_3^2, x_3 x_4, x_4^2, x_3^3, x_3^2 x_4, x_3 x_4^2, x_4^3\}$	2 $\{1, \lambda^2\}$	1 $\{1\}$
$\mathcal{I}_{02}$		10 $\{1, x_2, x_3, x_4, x_2^2, x_2 x_3, x_2 x_4, x_3^2, x_3 x_4, x_4^2\}$	2 $\{1, \lambda^2\}$	1 $\{1\}$
$\mathcal{I}_{01}$		10 $\{1, x_1, x_3, x_4, x_1^2, x_1 x_3, x_1 x_4, x_3^2, x_3 x_4, x_4^2\}$	4 $\{1, x_1, x_1^2, \lambda^2\}$	3 $\{1, x_1, x_1^2\}$
$\mathcal{I}_{0}$		5 $\{1, x_1, x_2, x_3, x_4\}$	1 $\{1\}$	– –

- ☑ minimal number of irreducible non-spurious monomials (irr. scal. prod.s)!
- ☑ Second polynomial division <==> Dimensional Recurrence @ integrand level

$$d = d_{//} + d_{\perp}$$

idea n.1 Integrating over Transverse Space

idea n.2 Cutting in the Longitudinal Space

☑ both ideas can be applied @ all-loops

Two-Loop Integrals

$$\boxed{d = 4 - 2\epsilon}$$

$$I_n^d[\mathcal{N}] = \int \frac{d^d q_1 d^d q_2}{\pi^d} \frac{\mathcal{N}(q_1, q_2)}{\prod_i \mathcal{D}_i}, \qquad q_1^\alpha = q_{1[4]}^\alpha + \mu_1^\alpha, \quad q_2^\alpha = q_{2[4]}^\alpha + \mu_2^\alpha, \qquad \mu_i \cdot \mu_j = \mu_{ij}, \quad q_i \cdot q_j = q_{i[4]} \cdot q_{j[4]} + \mu_{ij},$$

- loop momentum parametrization

$$q_{1[4]}^\alpha = \sum_{i=1}^4 x_i e_i^\alpha, \qquad q_{2[4]}^\alpha = \sum_{i=1}^4 y_i f_i^\alpha,$$

$$I_n^d[\mathcal{N}] = \frac{2^{d-6}\mathcal{K}_1\mathcal{K}_2}{\pi^5 \Gamma(d-5)} \int \prod_{i=1}^4 dx_i dy_i \int_0^\infty d\mu_{11} \int_0^\infty d\mu_{22} \int_{-\sqrt{\mu_{11}\mu_{22}}}^{\sqrt{\mu_{11}\mu_{22}}} d\mu_{12} (\mu_{11}\mu_{22} - \mu_{12}^2)^{\frac{d-6}{2}} \times \frac{\mathcal{N}(x_j, y_i, \mu_{ij})}{\prod_i \mathcal{D}_i},$$

Two-Loop Integrals

$$\boxed{d = d_{//} + d_\perp}$$

- loop momentum parametrization

$$q_1^\alpha = q_{1[k]}^\alpha + \lambda_1^\alpha, \qquad q_2^\alpha = q_{2[k]}^\alpha + \lambda_2^\alpha, \qquad k \leq 3,$$

$$q_{1[k]}^\alpha = \sum_{j=1}^{k} x_j e_j^\alpha, \qquad q_{2[k]}^\alpha = \sum_{j=1}^{k} y_j e_j^\alpha,$$

k-dimensional space spanned by the external kinematics

$$\lambda_1^\alpha = \sum_{j=k+1}^{4} x_j e_j^\alpha + \mu_1^\alpha, \qquad \lambda_2^\alpha = \sum_{j=k+1}^{4} y_j e_j^\alpha + \mu_2^\alpha$$

$(d-k)$-dimensional orthogonal subspaces,

$$\cos\theta_{12} = \frac{\lambda_{12}}{\sqrt{\lambda_{11}\lambda_{22}}},$$

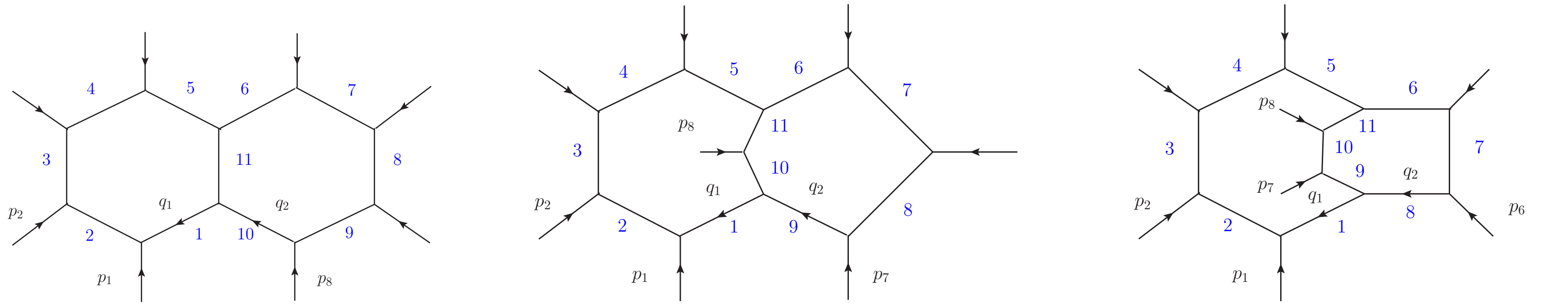
$$I_n^d[\mathcal{N}] = \frac{2^{d-6}}{\pi^5\Gamma(n-k-1)} \int d^k q_{1[k]} d^k q_{2[k]} \int_0^\infty d\lambda_{11} (\lambda_{11})^{\frac{d-k-2}{2}} \int_0^\infty d\lambda_{22} (\lambda_{22})^{\frac{d-k-2}{2}} \times$$
$$\int_{-1}^{1} d\cos\theta_{12} (\sin\theta_{12})^{d-k-3} \int_{-1}^{1} \prod_{i=1}^{4-k} d\cos\theta_{i1} d\cos\theta_{i+1\,2} (\sin\theta_{i1})^{d-k-i-2} (\sin\theta_{i+1\,2})^{d-k-i-3} \times \frac{\mathcal{N}(q_1,q_2)}{\prod_i \mathcal{D}_i}.$$

- ☑ **Denominators do not depend on** "the angular variables" of the **Transverse Space** $\Omega_\perp$
- ☑ **Numerators depend on** "all" loop variables
- ☑ **Integration over** $\Omega_\perp$ **:** ***Gegenbauer orthogonality condition***

  Spurious integrals vanish automatically @ all-loop!

Adaptive Unitarity @ 2-loop

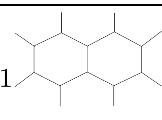
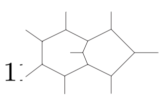
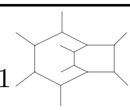
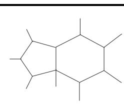
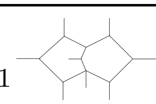
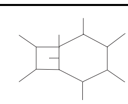
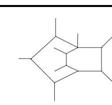
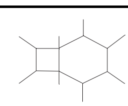
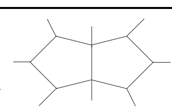
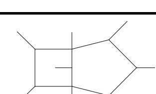
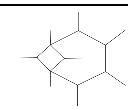
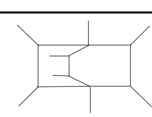
- **Novel Integrand red'n**

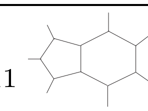
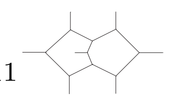
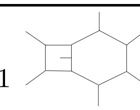
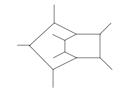
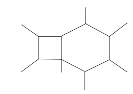
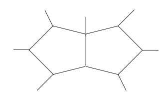
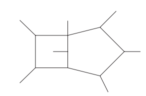
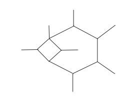
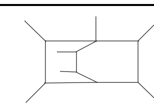
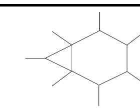
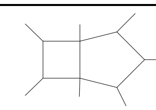
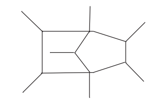
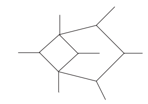


(a) $\mathcal{I}^{\mathrm{P}}_{12345678910\,11}$ (b) $\mathcal{I}^{\mathrm{NP1}}_{12345678910\,11}$ (c) $\mathcal{I}^{\mathrm{NP2}}_{12345678910\,11}$

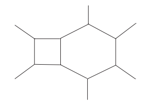
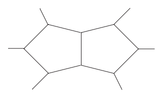
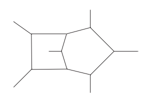
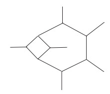
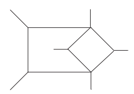
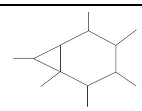
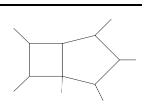
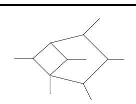
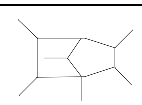
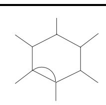
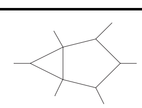
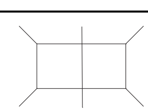
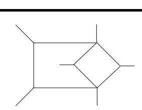
☑ Arbitrary (external and internal) kinematics!

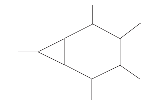
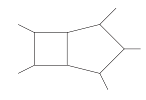
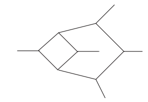
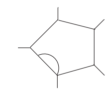
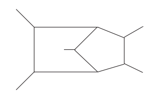
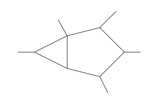
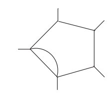
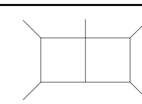
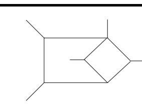
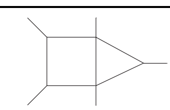
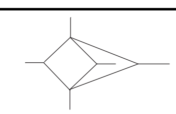
8 and 7 legs

$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$
$\mathcal{I}^{\mathrm{P}}_{12345678910\,11}$ 	1 $\{1\}$
$\mathcal{I}^{\mathrm{NP1}}_{12345678910\,11}$ 	1 $\{1\}$
$\mathcal{I}^{\mathrm{NP2}}_{12345678910\,11}$ 	1 $\{1\}$
$\mathcal{I}^{\mathrm{P}}_{2345678910\,11}$ 	6 $\{1, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{2345678910\,11}$ 	10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{1234578910\,11}$ 	6 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{1234678910\,11}$ 	10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{234678910\,11}$ 	15 $\{1, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{234578910\,11}$ 	33 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{234578910\,11}$ 	39 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{123456910\,11}$ 	15 $\{1, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{234678910\,11}$ 	45 $\{1, x_{41}, x_{42}\}$

$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$
$\mathcal{I}^{\mathrm{P}}_{1245678910\,11}$ 	6 $\{1, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{1245678910\,11}$ 	10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{1234568910\,11}$ 	6 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{1245678910\,11}$ 	10 $\{1, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{245678910\,11}$ 	15 $\{1, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{NP2}}_{234567910\,11}$ 	33 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{124568910\,11}$ 	39 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{123456810\,11}$ 	15 $\{1, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{124678910\,11}$ 	45 $\{1, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{2478910\,11}$ 	20 $\{1, x_{21}, x_{31}, x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{23478910\,11}$ 	76 $\{1, x_{31}, x_{41}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{24578910\,11}$ 	116 $\{1, x_{41}, x_{32}, x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{12457810\,11}$ 	80 $\{1, x_{31}, x_{41}, x_{42}\}$

6 and 5 legs

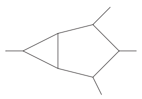
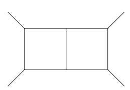
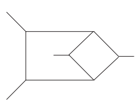
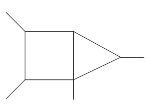
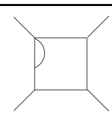
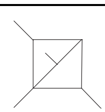
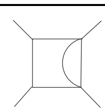
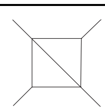
$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$
$\mathcal{I}^{\mathrm{P}}_{135678910\,11}$ 	15 $\{1,x_{31},x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{124567910\,11}$ 	62 $\{1,x_{41},x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{23568910\,11}$ 	39 $\{1,x_{41},x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{123456910\,11}$ 	15 $\{1,x_{32},x_{42}\}$
$\mathcal{I}^{\mathrm{NP2}}_{135678910\,11}$ 	45 $\{1,x_{41},x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{25678910\,11}$ 	20 $\{1,x_{21},x_{31},x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{23568910\,11}$ 	76 $\{1,x_{31},x_{41},x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{25678910\,11}$ 	80 $\{1,x_{31},x_{41},x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{24568910\,11}$ 	116 $\{1,x_{41},x_{32},x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{3678910\,11}$ 	15 $\{1,x_{11},x_{21},x_{31},x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{2578910\,11}$ 	94 $\{1,x_{21},x_{31},x_{41},x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{2357910\,11}$ 	160 $\{1,x_{31},x_{41},x_{32},x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{2457910\,11}$ 	185 $\{1,x_{31},x_{41},x_{32},x_{42}\}$

$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$
$\mathcal{I}^{\mathrm{P}}_{15678910\,11}$ 	20 $\{1,x_{21},x_{31},x_{41}\}$
$\mathcal{I}^{\mathrm{P}}_{13567910\,11}$ 	76 $\{1,x_{31},x_{41},x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{15678910\,11}$ 	80 $\{1,x_{31},x_{41},x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1678910\,11}$ 	15 $\{1,x_{11},x_{21},x_{31},x_{41}\}$
$\mathcal{I}^{\mathrm{NP1}}_{13568910\,11}$ 	116 $\{1,x_{31},x_{32},x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1467910\,11}$ 	94 $\{1,x_{21},x_{31},x_{41},x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1678911}$ 	66 $\{1,x_{11},x_{21},x_{31},x_{41},x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1256910\,11}$ 	160 $\{1,x_{31},x_{41},y_{32},x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{1357910\,11}$ 	185 $\{1,x_{31},x_{41},x_{32},x_{42}\}$
$\mathcal{I}^{\mathrm{P}}_{1256911}$ 	180 $\{1,x_{11},x_{31},x_{41},x_{32},x_{42}\}$
$\mathcal{I}^{\mathrm{NP1}}_{246910\,11}$ 	246 $\{1,x_{31},x_{41},x_{22},x_{32},x_{42}\}$

4 legs: *divide-integra-divide*

λ_{ij}

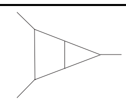
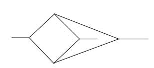
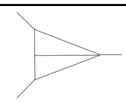
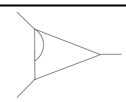
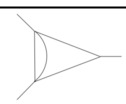
reducible

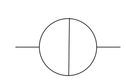
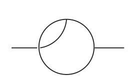
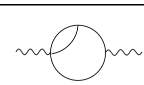
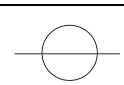
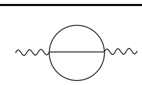
$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$	$\Delta^{\mathrm{int}}_{i_1\cdots i_n}$	$\Delta'_{i_1\cdots i_n}$
$\mathcal{I}^{\mathrm{P}}_{1567910\,11}$ 	94 $\{1,x_{21},x_{31},x_{41},x_{42}\}$	53 $\{1,x_{21},x_{31},\lambda_{11},\lambda_{22},\lambda_{12}\}$	10 $\{1,x_{21},x_{31}\}$
$\mathcal{I}^{\mathrm{P}}_{12256910\,11}$ 	160 $\{1,x_{31},x_{41},x_{32},x_{42}\}$	93 $\{1,x_{31},x_{32},\lambda_{11},\lambda_{22},\lambda_{12}\}$	22 $\{1,x_{31},x_{32}\}$
$\mathcal{I}^{\mathrm{NP1}}_{1356910\,11}$ 	184 $\{1,x_{31},x_{42},x_{32},x_{42}\}$	105 $\{1,x_{31},x_{32},\lambda_{11},\lambda_{22},\lambda_{12}\}$	25 $\{1,x_{31},x_{32}\}$
$\mathcal{I}^{\mathrm{P}}_{1356811}$ 	180 $\{1,x_{31},x_{41},x_{22},x_{32},x_{42}\}$	101 $\{1,x_{31},x_{22},x_{32},\lambda_{11},\lambda_{22},\lambda_{12}\}$	39 $\{1,x_{31},x_{22},y_{32}\}$
$\mathcal{I}^{\mathrm{P}}_{168910\,11}$ 	66 $\{1,x_{11},x_{21},x_{31},x_{41},x_{42}\}$	35 $\{1,x_{11},x_{21},x_{31},\lambda_{11},\lambda_{22},\lambda_{12}\}$	10 $\{1,x_{11},x_{21},x_{31}\}$
$\mathcal{I}^{\mathrm{NP1}}_{246910\,11}$ 	245 $\{1,x_{31},x_{41},x_{21},x_{32},x_{42}\}$	137 $\{1,x_{31},x_{22},x_{32},\lambda_{11},\lambda_{22},\lambda_{12}\}$	55 $\{1,x_{31},x_{22},y_{32}\}$
$\mathcal{I}^{\mathrm{P}}_{36810\,11}$ 	115 $\{1,x_{31},x_{41},x_{12},x_{22},x_{32},x_{42}\}$	66 $\{1,x_{31},x_{12},x_{22},x_{32},\lambda_{11},\lambda_{22},\lambda_{12}\}$	35 $\{1,x_{31},x_{12},x_{22},x_{32}\}$
$\mathcal{I}^{\mathrm{P}}_{136811}$ 	180 $\{1,x_{11},x_{31},x_{41},x_{22},x_{32},x_{42}\}$	103 $\{1,x_{11},x_{31},x_{22},x_{32},\lambda_{11},\lambda_{22},\lambda_{12}\}$	60 $\{1,x_{11},x_{31},x_{22},x_{32}\}$

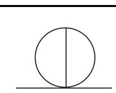
3, 2 , 1 legs: *divide-integra-divide*

λ_{ij}

reducible

$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$	$\Delta^{\rm int}_{i_1\cdots i_k}$	$\Delta'_{i_1\cdots i_k}$
$\mathcal{I}^{\rm P}_{1356911}$ 	180 $\{1, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	22 $\{1, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	4 $\{1, x_{22}\}$
$\mathcal{I}^{\rm NP1}_{156910\,11}$ 	240 $\{1, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	30 $\{1, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	6 $\{1, x_{22}\}$
$\mathcal{I}^{\rm P}_{15710\,11}$ 	180 $\{1, x_{21}, x_{31}, x_{41}, x_{12}, x_{32}, x_{42}\}$	33 $\{1, x_{21}, x_{12}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	13 $\{1, x_{21}, x_{12}\}$
$\mathcal{I}^{\rm P}_{16910\,11}$ 	115 $\{1, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	20 $\{1, x_{11}, x_{22}\lambda_{11}, \lambda_{22}, \lambda_{12}\}$	6 $\{1, x_{12}, x_{22}\}$
$\mathcal{I}^{\rm P}_{3610\,11}$ 	100 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	26 $\{1, x_{11}, x_{21}, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	16 $\{x_{11}, x_{21}, x_{22}\}$

$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$	$\Delta^{\rm int}_{i_1\cdots i_n}$	$\Delta'_{i_1\cdots i_n}$
$\mathcal{I}^{\rm P}_{15610\,11}$ 	180 $\{1, x_{21}, x_{31}, x_{41}, x_{22}, x_{32}, x_{42}\}$	8 $\{1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	1 $\{1\}$
$\mathcal{I}^{\rm P}_{1610\,11}$ 	100 $\{1, x_{11}, x_{21}, x_{31}, x_{4}, x_{22}, y_{3}, x_{42}\}$	8 $\{1, x_{11}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	3 $\{1, x_{11}\}$
$\mathcal{I}^{\rm P}_{1310\,11}$ 	100 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{32}, x_{42}\}$	26 $\{1, x_{11}, x_{21}, x_{12}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	16 $\{1, x_{11}, x_{21}, x_{12}\}$
$\mathcal{I}^{\rm P}_{210\,11}$ 	45 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	9 $\{1, x_{11}, x_{12}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	6 $\{1, x_{11}, x_{12}\}$
$\mathcal{I}^{\rm P}_{210\,11}$ 	45 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	18 $\{1, x_{11}, x_{21}, x_{12}, x_{22}, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	15 $\{1, x_{11}, x_{22}, x_{21}, x_{22}\}$

$\mathcal{I}_{i_1\cdots i_n}$	$\Delta_{i_1\cdots i_n}$	$\Delta^{\rm int}_{i_1\cdots i_n}$	$\Delta'_{i_1\cdots i_n}$
$\mathcal{I}^{\rm P}_{110\,11}$ 	45 $\{1, x_{11}, x_{21}, x_{31}, x_{41}, x_{12}, x_{22}, x_{32}, x_{42}\}$	4 $\{1, \lambda_{11}, \lambda_{22}, \lambda_{12}\}$	1 $\{1\}$

2-loop Automation

- **Integrand generation + *Integration over Transverse Angles*** **GoSam** >> Jones

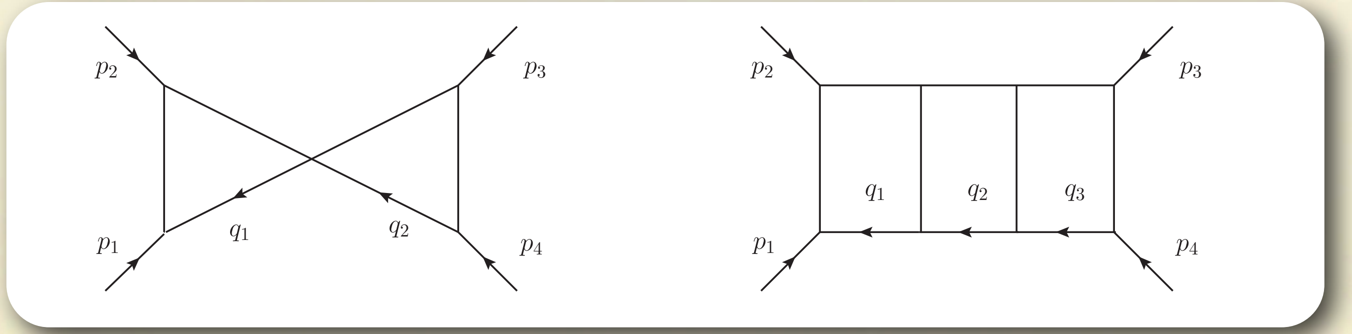
  - ☑ Simplifying the integrands to be reduced (nleg < 5)
    Removing the transverse direction ==> less coefficients to be determined

2-loop Automation

- **Integrand generation + *Integration over Transverse Angles*** GoSam

  - ☑ Simplifying the integrands to be reduced (nleg < 5)
    Removing the transverse direction ==> less coefficients to be determined

  - ☑ Integrals whose denominators depend on a reduced set of variables (@ higher-loops)



Factorized diagrams

Denominators do not depend on

$$\mu_{12}$$

Ladder diagrams

Denominators do not depend on

$$\lambda_{13}$$

integration with
Gegenbauer orthogonality condition

2-loop Automation

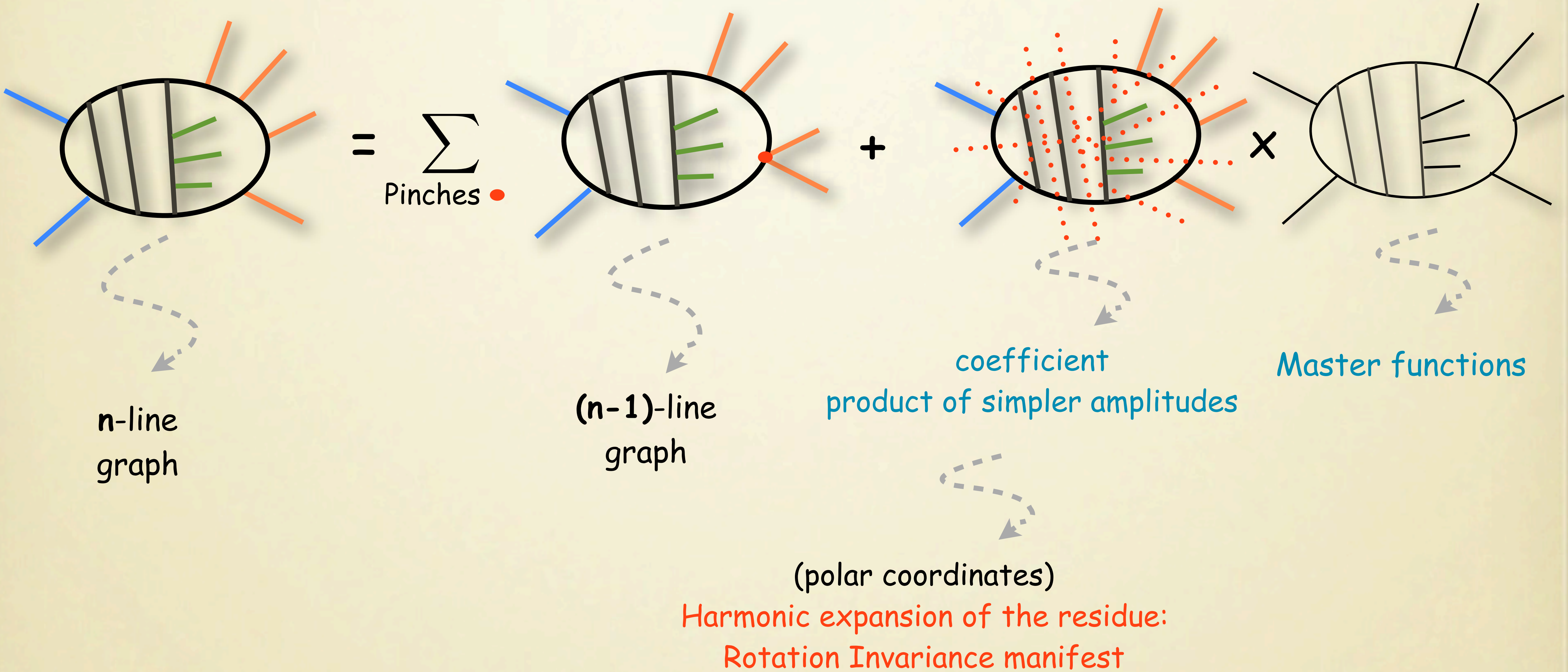
- **Integrand generation + *Integration over Transverse Angles*** **GoSam**
  - ☑ Simplifying the integrands to be reduced (nleg < 5)
    Removing the transverse direction ==> less coefficients to be determined
  - ☑ Integrals whose denominators depend on a reduced set of variables (@ higher-loops)
  - ☑ note :: Generalizing and extending to all-loop the R2-integration

2-loop Automation

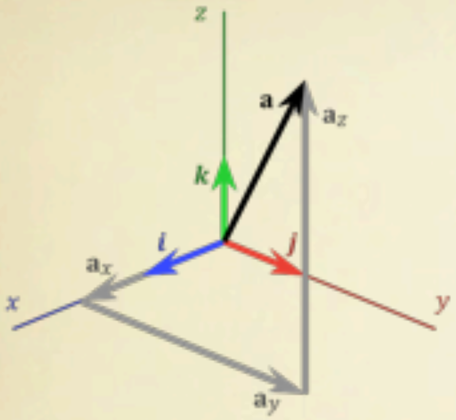
- **Integrand generation + *Integration over Transverse Angles*** **GoSam**
  - ☑ Simplifying the integrands to be reduced (nleg < 5)
    Removing the transverse direction ==> less coefficients to be determined
  - ☑ Integrals whose denominators depend on a reduced set of variables (@ higher-loops)
  - ☑ note :: Generalizing and extending to all-loop the R2-integration
- ***Adaptive Integrand Reduction* + IBP-id's**
  - Improved IBP Solver Fire; Reduze; LiteRed ...
  - Algebraic Geometry Methods (exploiting sygyzy's)
    Kosower Gluza Kaida; Ita; Larsen Zhang;
- ***Adaptive Integrand Reduction* + Numerical MIs**
  - Improved Sector Decomposition **SecDec** >> Jones, Kerner

The Geometry of Cut-Residues

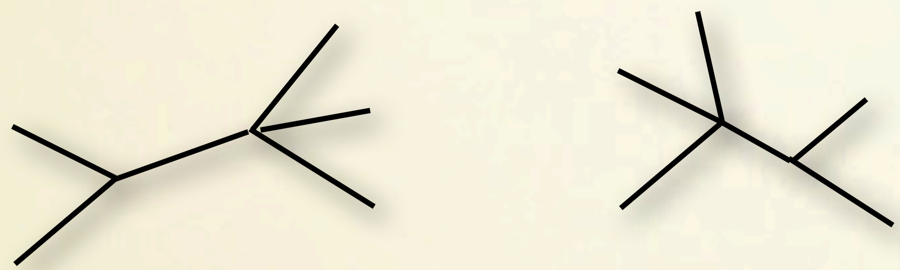
- **ℓ-Loop Recurrence Relation**



Basis :: Master Functions

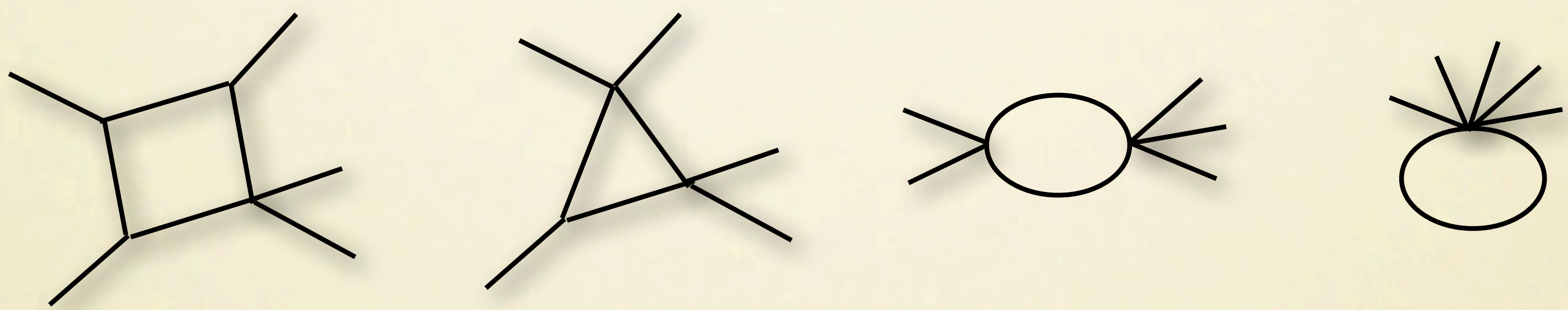


- Tree level



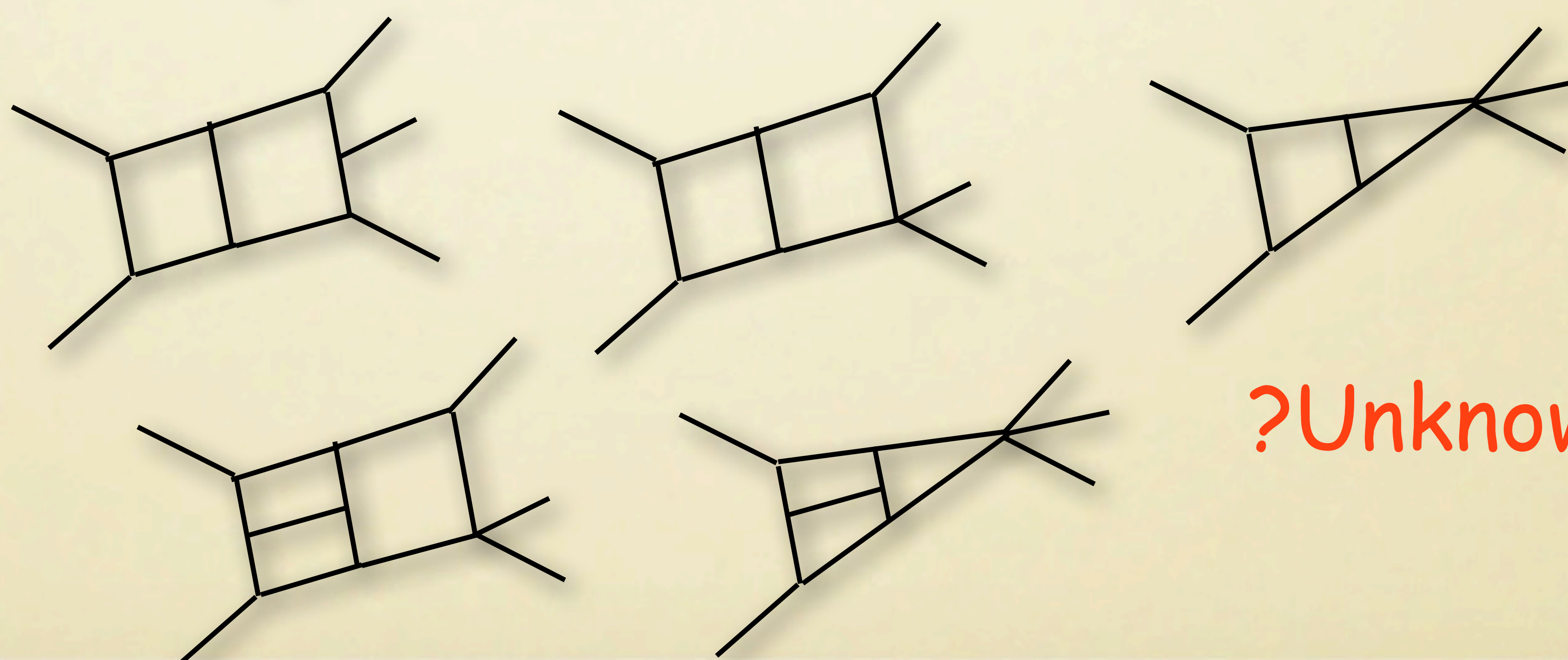
Known!

- One Loop



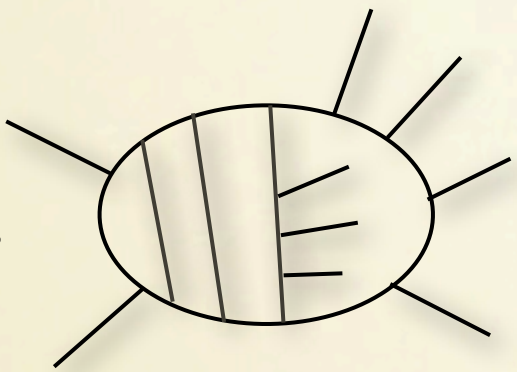
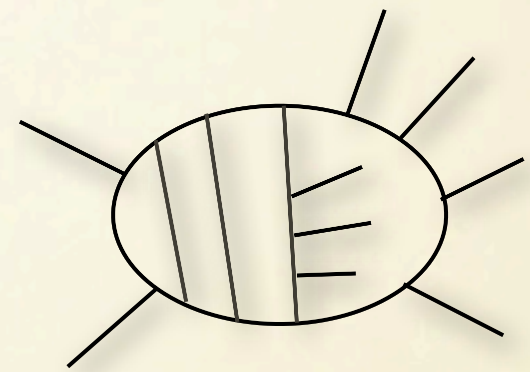
Known!

- Higher Loops



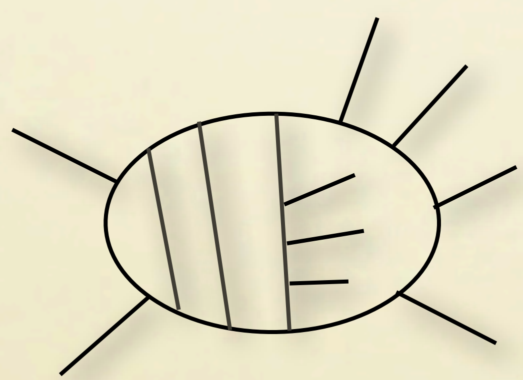
?Unknown?

Differential Equations for Master Integrals

∂_x  $= A(d, x)$ 

kinematic variable
(s,t,u, masses)

space-time
dimensions

 = ?

Kotikov; Remiddi;
Gehrmann Remiddi
Argeri Bonciani Ferroglia Remiddi **P.M.**
Aglietti Bonciani DeGrassi Vicini
Weinzierl
...
Henn;
Henn Smirnov & Smirnov
Henn Melnikov, Smirnov
Caron-Huot Henn
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Tancredi **P.M.**
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Papadopoulos Tommasini Wever
Ablinger Bluemlein DeFreitas Schneider

Quantum Mechanics

- **Schroedinger Eq'n (ε-*linear Hamiltonian*)**

$$i\hbar\, \partial_t |\Psi(t)\rangle = H(\epsilon, t)|\Psi(t)\rangle\ , \qquad H(\epsilon, t) = H_0(t) + \epsilon H_1(t)$$

- **Interaction Picture**

$$H_{i,I}(t) = B^\dagger(t)\ H_i(t)\ B(t)$$

- **Matrix Transform**

$$i\hbar\, \partial_t B(t) = H_0(t) B(t) \qquad B(t) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} d\tau H_0(\tau)}$$

- **Schroedinger Eq'n (*canonical form*)**

$$i\hbar\, \partial_t |\Psi_I(t)\rangle = \epsilon\, H_{1,I}(t)|\Psi_I(t)\rangle\ ,$$

Magnus Expansion

Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

System of 1st ODE

$$\partial_x Y(x) = A(x) Y(x) \ , \quad Y(x_0) = Y_0 \ .$$

$A(x)$ non-commutative

solution: *Matrix Exponential*

$$Y(x) = e^{\Omega(x,x_0)} \, Y(x_0) \equiv e^{\Omega(x)} \, Y_0 \, , \qquad \Omega(x) = \sum_{n=1}^{\infty} \Omega_n(x) \ .$$

BCH-formula

$$\Omega_1(x) = \int_{x_0}^{x} d\tau_1 A(\tau_1) \ ,$$
$$\Omega_2(x) = \frac{1}{2} \int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \ [A(\tau_1), A(\tau_2)] \ ,$$
$$\Omega_3(x) = \frac{1}{6} \int_{x_0}^{t} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3 \ [A(\tau_1), [A(\tau_2), A(\tau_3)]] + [A(\tau_3), [A(\tau_2), A(\tau_1)]] \ .$$

....

Iterated Integrals

$$\mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1} \equiv \int_\gamma d\log \eta_{i_1} \ldots d\log \eta_{i_k} \equiv \int_{0 \le t_1 \le \ldots \le t_k \le 1} g^\gamma_{i_k}(t_k) \ldots g^\gamma_{i_1}(t_1) \, dt_1 \ldots dt_k \qquad g^\gamma_i(t) = \frac{d}{dt} \log \eta_i(\gamma(t))$$

$$\mathcal{C}^{[\gamma]}_{\vec{m}} \, \mathcal{C}^{[\gamma]}_{\vec{n}} = \mathcal{C}^{[\gamma]}_{\vec{m}} \sqcup\!\sqcup \, \mathcal{C}^{[\gamma]}_{\vec{n}} = \sum_{\vec{p} = \vec{m} \sqcup\!\sqcup \vec{n}} \mathcal{C}^{[\gamma]}_{\vec{p}} \qquad \mathcal{C}^{[\alpha\beta]}_{i_k,\ldots,i_1} = \sum_{p=0}^{k} \mathcal{C}^{[\alpha]}_{i_k,\ldots,i_{p+1}} \, \mathcal{C}^{[\beta]}_{i_p,\ldots,i_1}$$

Chen
Goncharov
Remiddi Vermaseren
Gehrmann Remiddi
Bonciani Remiddi **P.M.**
Vollinga Weinzierl
Brown
Duhr Gangl Rhodes
....

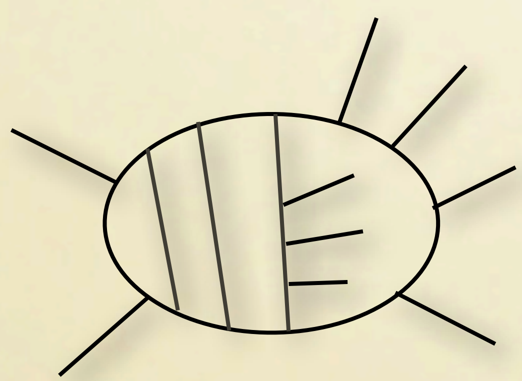
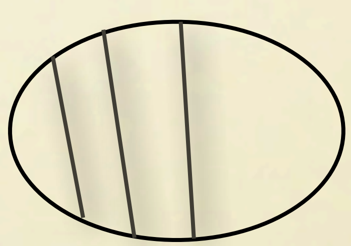
Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

Quantum Mechanics

- Time-evolution in Perturbation Theory
- perturbation parameter: ε
- **Linear** Hamiltonian in ε
- **Unitary transform**
- Schroedinger Equation
  in the interaction picture (ε-factorization)
- solution: Dyson series

Feynman Integrals

- Kinematic-evolution in Dimensional Regularization
- space-time dimensional parameter: ε = (4-d)/2
- **Linear** system in ε
- non-Unitary **Magnus transform**
- System of Differential Equations
  in canonical form (ε-factorization) Henn (2013)
- solution: Dyson/Magnus series


$$= \mathrm{e}^{\Omega(d,x)}$$


boundary term
(simpler integral)

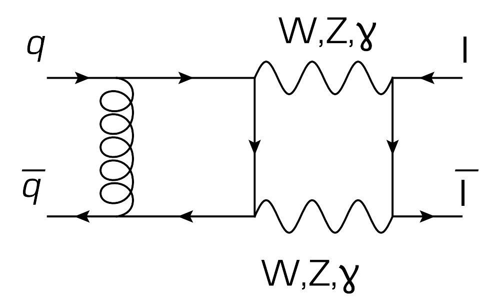
- Feynman integrals can be determined from differential equations that looks like **gauge transformations**

Drell-Yan @ 2loop EW-QCD

Bonciani, Di Vita, Schubert, **P.M.** (to appear)

$$q(p_1) + \bar{q}(p_2) \to l^-(p_3) + l^+(p_4)\,,$$
$$q(p_1) + \bar{q}'(p_2) \to l^-(p_3) + \overline{\nu}(p_4)\,.$$

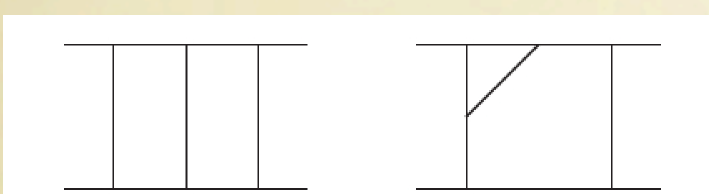
$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0.$$



$$\frac{1}{p^2 - m_Z^2} = \frac{1}{p^2 - m_W^2 - \Delta m^2} \approx \frac{1}{p^2 - m_W^2} + \frac{m_Z^2}{(p^2 - m_W^2)^2}\,\xi + \dots$$

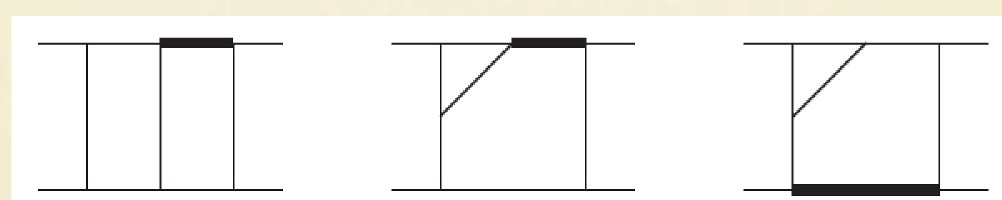
$$\xi = \frac{\Delta m^2}{m_Z^2} = \frac{m_Z^2 - m_W^2}{m_Z^2} \sim \frac{1}{4}$$

no-mass



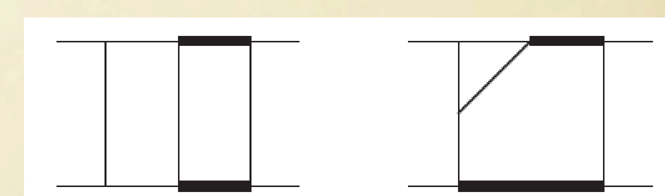
known

1-mass



new

2-mass



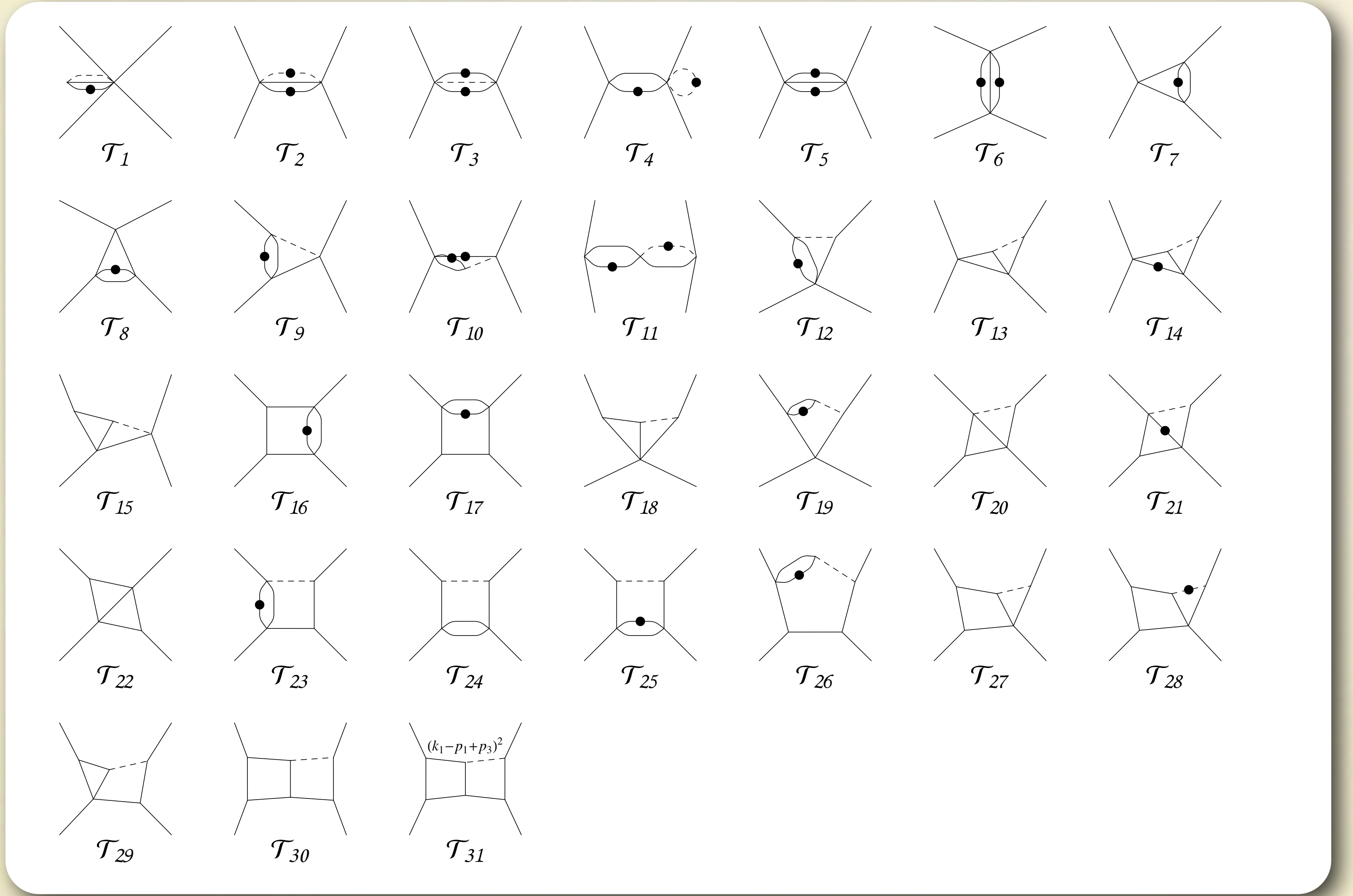
new

- **Systems of 1st ODE for 41 Master Integrals** using **Reduze 2**

$$d\mathcal{I} = \epsilon\, d\hat{A}\, \mathcal{I} \qquad \text{with} \qquad d\hat{A} = \hat{A}_x\, dx + \hat{A}_y\, dy\,.$$

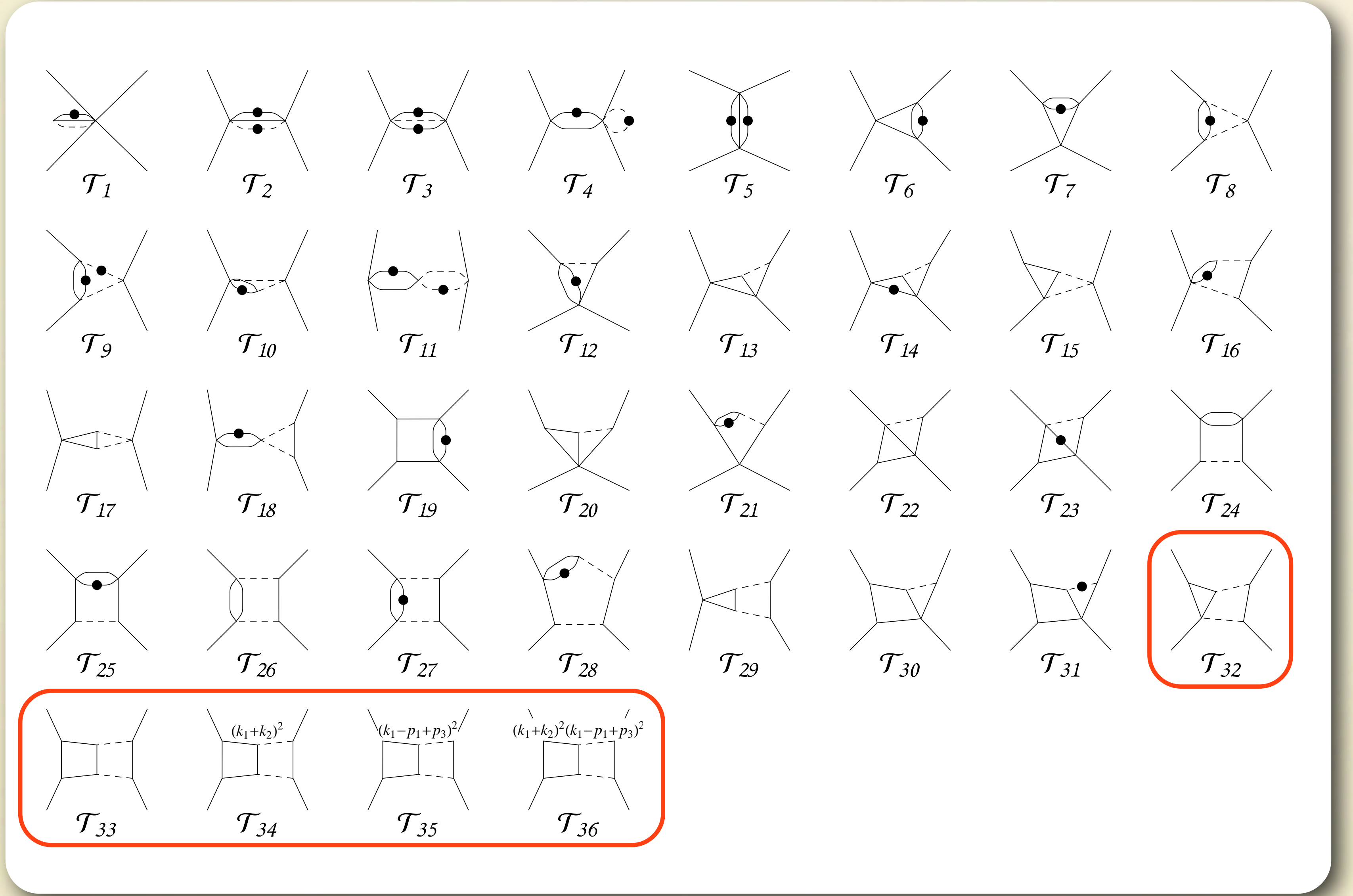
$$dA = \sum_{i=1}^{n} M_i \; d\log \eta_i$$

1-Mass



- **31 MIs**
- **alphabet:** 6 rational letters
- **solution:** GPL's
- **numerical checks** using **GiNac** vs **SecDec**
- **constants:** GPL's @ 1 • $\{-1, 0, \frac{1}{2}, 1\}$

2-Mass



- **36 MIs**
- **alphabet:** 12 rational + 5 irrational letters
- **solution:** Iterated integrals ::
  - mixed Chen-Goncharov representation
- **numerical checks** using **GiNac** vs **SecDec**
  - 1-fold representation over GPLs-kernel by using ibp for Chen-integrals
- **constants:** GPL's @ 1 • $\{-1,0,-i,i,1,(-1)^{\frac{1}{3}},-(-1)^{\frac{2}{3}}\}$, • $\{-1,0,-i,i,1,-(-1)^{\frac{1}{6}},-(-1)^{\frac{5}{6}}\}$,

Summary and Outlook

- IntegrANDS
  - Multi-Loop Integrand Reduction
    - Complete Development :: for generic kinematics
    - Exploiting DimReg :: Adaptive Unitarity and Transverse space integration
      - **any loop** :: we are at the same point as OPP for 1-loop.
    - Applying symmetries to the coefficients w/in the integrand decomposition
      - FDF: simple implementation of FDH scheme for generalized unitarity cuts
        Fazio, Mirabella, Torres, **PM** (2014)
      - BCJ relations @ tree-level in DimReg w/in FDF Primo, Schubert, Torres, **PM** (2015)
      - BCJ relations @ 1-Loop Chester (2016) Primo, Torres (2016)
- IntegrALS
  - Multi-Loop Master Integrals evaluation
    - Differential Equations (analytic as well as numerical) :: Magnus Exponential
      - exploiting Path invariance
      - MI's in different dimensions ==> Adaptive Differential Equations?
    - Numerical methods: the big short