Black hole entropy in string theory

Literature:
- Strominger, Vafa hep-th/9601029 (original work)
- Horowitz gr-qc/9604051 (Conference proceedings)
- Peet hep-th/0008241 (TASI lecture)
- Becker, Becker, Schwarz - String theory and M-theory

Idea: Construct (protected) states in string theory that end up as black hole if the coupling is made strong.

Relate : number of these states Ω
to entropy of the black hole S

$$\boxed{S = \ln \Omega}$$

i.e. a microscopical description of black hole entropy

I. Basics

1. Black hole in D dimensions

$$\boxed{ds^2 = -h(r)\,dt^2 + h^{-1}(r)\,dr^2 + r^2\,d\Omega^2_{D-2}}$$

e.g. Schwarzschild: $h(r) = 1 - \left(\frac{r_H}{r}\right)^{D-3}$

with $r_H^{D-3} = \dfrac{16\pi M G_D}{(D-2)\,\Omega_{D-2}}$

(Ω_{D-2} volume of $(D-2)$ sphere, G_D: D-dim Newton constant
i.e. $S_{EH} = \frac{1}{16\pi G_D}\int d^Dx \sqrt{-g}\,R$)

for string theory $D=10$ and $G_{10} = 8\pi^2 g_s^2\, l_s^8$

g_s: string coupling; l_s: string length for 10-D compactified dimensions (on manifold with volume V):

$$G_D = \frac{G_{10}}{V}$$

Area of D dimensional black hole : $A_D = \Omega_{D-2}\, r_H^{D-2}$

$\Rightarrow$ entropy formula valid in arbitrary D:

$$S = \frac{A_D}{4 G_D}$$

Reminder: Schwarzschild in 4d

$$dM = T dS$$

$$T = (8\pi M G_4)^{-1}$$

$$\Rightarrow dS = 8\pi G_4 M dM$$

with $S \to 0$ for $M \to 0$ $\Rightarrow$ $S = 4\pi G_4 M^2 = \frac{A_4}{4G_4}$

2. Reissner-Nordström black hole

black hole with electric charge Q

in 4d:

$$\boxed{ds^2 = -\Delta(r)\,dt^2 + \Delta(r)^{-1} dr^2 + r^2 d\Omega_2^2}$$

with $\Delta(r) = 1 - \frac{2MG_4}{r} + \frac{Q^2 G_4}{r^2}$

has two horizons at : $r_\pm = MG_4 \pm \sqrt{(MG_4)^2 - Q^2 G_4}$

(to avoid naked singularities, impose $M\sqrt{G_4} \geq |Q|$)

for $M\sqrt{G_4} = |Q|$: extremal black hole

$\Rightarrow$ $r_+ = r_-$ and

Hawking temperature : $T_H = \frac{\sqrt{(MG_4)^2 - Q^2 G_4}}{2\pi r_+} = 0$!

$\Rightarrow$ no Hawking radiation $\to$ static, stable solution

In supersymmetric theories such bounds often appear and signal the preservation of part of the supersymmetry in the extremal background $\rightsquigarrow$ BPS-states

These BPS-states are protected!

extremal 5d Reissner-Nordström:

$$ds^2 = -\left(1+\left(\frac{r_0}{r}\right)^2\right)^2 dt^2 + \left(1-\left(\frac{r_0}{r}\right)^2\right)\left(dr^2 + r^2 d\Omega_3^2\right)$$

horizon at $r=0$

$$\Rightarrow \quad A = \Omega_3 r_0^3 = 2\pi^2 r_0^3$$

$$M = \frac{Q}{\sqrt{G_5}} = \frac{3\pi r_0^2}{4 G_5}$$

3. Objects in string theory (type II)

- fundamental strings F couple electrically to B
- NS5-branes couple magnetically to B
- D-branes couple electrically / magnetically to RR-fields C_n

II. Black holes in string theory

"easiest" example: 5d black holes with three charges

in order for supergravity in 5d to be valid:

- small compact dimensions ($10d \to 5d$)
- large black holes $\to$ sufficiently low curvature at horizon

model in type IIB string theory compactified on $T^5 = T^4 \times S^1$

$\rightsquigarrow$ 32 real supercharges $\sim$ $\mathcal{N}=8$ SUGRA

add "matter" to form a black hole:

- Q_5 D5-branes wrapped around T^5
- Q_1 D1-branes wrapped on S^1

- n units of Kaluza-Klein momentum on S^1

$\Rightarrow$ charges with respect to:

- C_2 (electric: D1-branes)
- B (electric F)
- C_2 (magnetic: D5-branes)

Each of this ingredients breaks half of supersymmetry

$\rightsquigarrow$ left with $\mathcal{N}=1$ (4 real supercharges)

$\Rightarrow$ black holes are $\frac{1}{8}$-BPS states $\rightsquigarrow$ protected

the 5d metric is: (extremal, charged black hole with $T_H=0$)

$$\boxed{ds^2 = -\lambda^{2/3} dt^2 + \lambda^{1/3}\left(dr^2 + r^2 d\Omega_3^2\right)} \quad \text{with } \lambda = \prod_{i=1}^{3}\left(1+\frac{r_i^2}{r^2}\right)$$

- horizon at $r=0$ $\Rightarrow$ $A = 2\pi^2 r_1 r_2 r_3$
- dilaton constant $\Rightarrow$ well-defined string coupling
- r_i related to charges $\Rightarrow$ need all three for $A \neq 0$
- mass: $M = \sum_{i=1}^{3} M_i$ with $M_i = \frac{\pi r_i^2}{4 G_5}$

on T^4 with volume $(2\pi)^4 V$, S with size $2\pi R$

$$\Rightarrow r_i^2 = \frac{g_s^2 \ell_s^8}{RV} M_i \gg \ell_s$$

from string theory:

$$M_1 = 2\pi R \overset{\text{tension}}{T_{D1}} Q_1 = \frac{Q_1 R}{g_s \ell_s^2}$$

$$M_2 = (2\pi)^5 R V T_{D5} Q_5 = \frac{Q_5 R V}{g_s \ell_s^6}$$

$$M_3 = \frac{n}{R}$$

$\Rightarrow \quad g_s Q_1 \gg \frac{V}{\ell_s^4} \;,\quad g_s Q_5 \gg 1 \;,\quad g_s^2 n \gg \frac{R^2 V}{\ell_s^6}$

i.e. all large

$\Rightarrow$ BPS-nature allows us to extrapolate to strong coupling !

resulting entropy:

$$\boxed{S = \frac{A}{4G_5} = \frac{2\pi g_s \ell_s^4}{\sqrt{RV}} \sqrt{M_1 M_2 M_3} = 2\pi \sqrt{Q_1 Q_5 n}}$$

III. Counting of microstates

We know the constituents that build the black hole

$\rightsquigarrow$ they have to form a bound state (single black hole)

$\rightsquigarrow$ count all possible bound states

Go to weak coupling and count the possible string states in the D-brane background with n units of KK momentum that preserve half of the supersymmetry.

The relevant states are: (dominant contribution)

- strings between D1s and D5s

  $\rightsquigarrow$ $Q_1 Q_5$ different ones

- excited oscillators in 4 compact directions $\perp$ to D1

  $\Rightarrow$ $4 Q_1 Q_5$ bosonic states + corresponding fermions

- only left-movers excited $\rightsquigarrow$ part of SUSY preserved

- energy has to be $\frac{n}{R}$ (KK-momentum)

the degeneracy is given by the coefficient of $w^{nQ_1Q_2}$ in the CFT partition function:

$$Z(w) \sim \prod_{m=1}^{\infty} \left(\frac{1+w^m}{1-w^m} \right)^4 = \sum \Omega_m w^m$$

can be approximated for large nQ_1Q_2 by Cardy's formula

$$\Omega \sim \exp\left(2\pi \sqrt{\frac{c}{6} ER}\right) = \exp\left(2\pi \sqrt{\frac{c}{6} n}\right)$$

where c is the central charge $c = n_b + \frac{1}{2} n_f$

(note that we assume the lowest state to have $E=0$, otherwise 24Δ-piece in c)

$$\Rightarrow \quad c = n_b + \frac{1}{2} n_f = 4Q_1Q_5 + \frac{1}{2}(4Q_1Q_5) = 6Q_1Q_5$$

$$\Rightarrow \quad \Omega \sim \exp\left(2\pi \sqrt{nQ_1Q_5}\right)$$

$$\Rightarrow \quad \boxed{S = \ln \Omega \sim 2\pi \sqrt{nQ_1Q_5}}$$

matches the above result !

Also higher corrections match !

For some special cases the microstates constituting a black hole in string theory can be identified !