Attacking one-loop multi-leg Feynman integrals with the Loop-Tree Duality

Grigorios Chachamis, IFT UAM-CSIC Madrid

In collaboration with S. Buchta, P. Draggiotis, G. Rodrigo

arXiv:1510.00187

DIS 2016, 11-15 April, DESY Hamburg, Germany

Outline

- Introduction
- Implementation
- Results
- Summary and Outlook

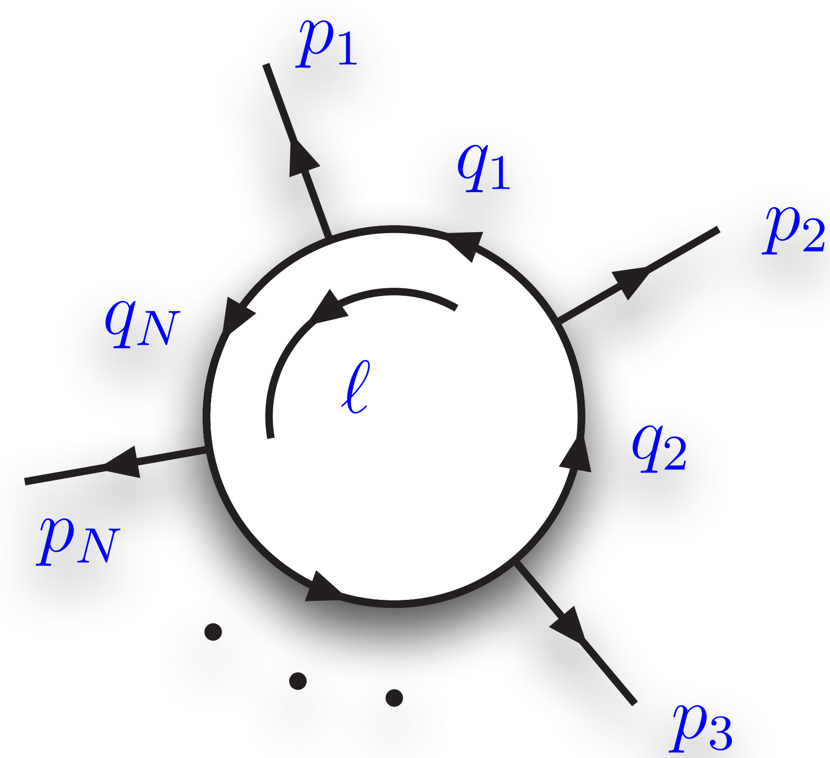
The constant need for higher order radiative corrections

- The LHC is a hadronic collider operating at high energies
    - higher multiplicities
    - proton structure
    - very large soft and collinear corrections
    - logarithms of ratios of very different scales
- Rule of thumb:
    - LO: order of magnitude estimate
    - NLO: first reliable estimate of the central value
    - NNLO: first reliable estimate of the uncertainty
- The Loop-Tree Duality promises to deal with virtual and real corrections on equal footing. In this talk we will see how the method copes with the virtual corrections

A generic one-loop integral

Number of legs N, number of spacetime dimensions is D.

Assume that it is UV and IR finite.



$$L^{(1)}(p_1, p_2, \ldots, p_N) = -i \int \frac{d^d\ell}{(2\pi)^d} \prod_{i=1}^{N} \frac{1}{q_i^2 + i0}$$

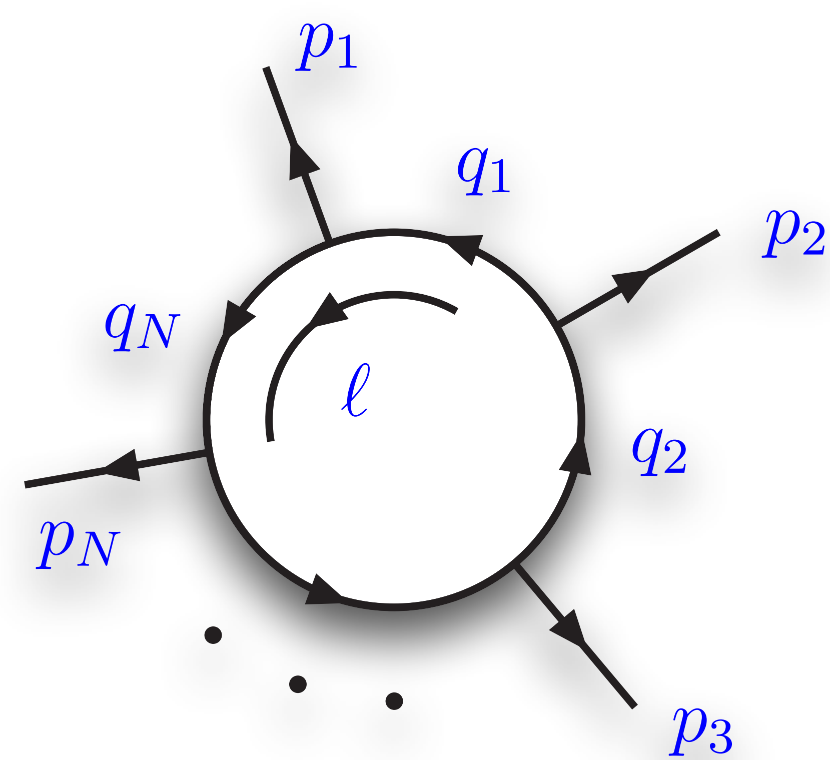
ℓ^μ is the loop momentum and $q_i = \ell + \sum_{k=1}^{i} p_k$ are the momenta of the propagators.

$G_F(q) \equiv \frac{1}{q^2 + i0}$ is the Feynman propagator.

Introduce the shorthand notation $-i \int \frac{d^d\ell}{(2\pi)^d} \bullet \equiv \int_\ell \bullet$, then

$$L^{(1)}(p_1, p_2, \ldots, p_N) = \int_\ell \prod_{i=1}^{N} G_F(q_i)$$

"Feynman" and "advanced" propagators



$$L^{(1)}(p_1, p_2, \dots, p_N) = \int_\ell \prod_{i=1}^{N} G_F(q_i)$$

$$G_F(q) \equiv \frac{1}{q^2 + i0} \text{ and } G_A(q) \equiv \frac{1}{q^2 - i0\, q_0}$$

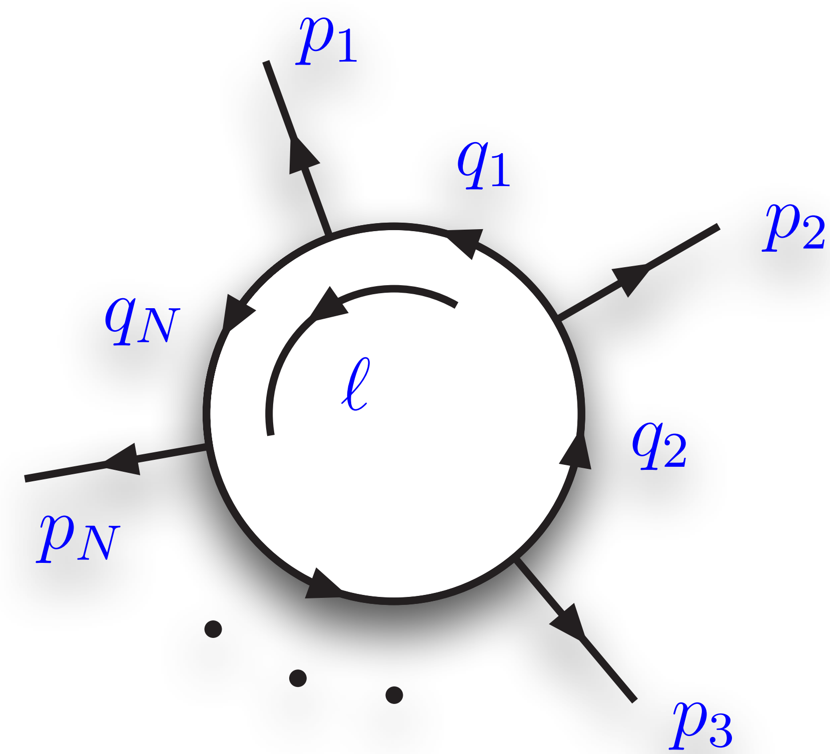
Feynman and advanced propagators are related:

$$G_A(q) = G_F(q) + \widetilde{\delta}(q) \quad \text{with} \quad \widetilde{\delta}(\ell) \equiv 2\pi\, i\, \theta(\ell_0)\, \delta(\ell^2)$$

This also holds when the propagators are

massive but now $\quad \widetilde{\delta}(q_i) \to \widetilde{\delta}(q_i) = 2\pi\, i\, \theta(q_{i,0})\, \delta(q_i^2 - m_i^2)$

“Feynman” and “advanced” propagators



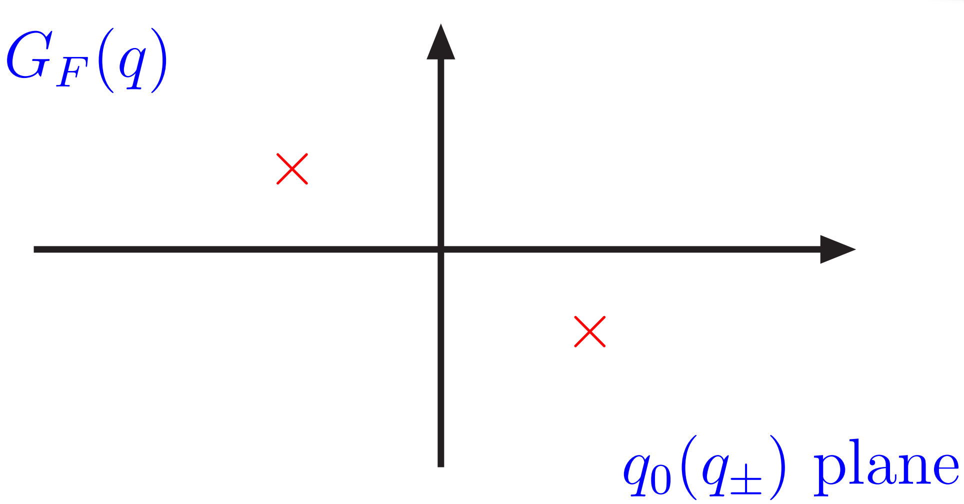
$$L^{(1)}(p_1, p_2, \ldots, p_N) = \int_\ell \prod_{i=1}^{N} G_F(q_i)$$

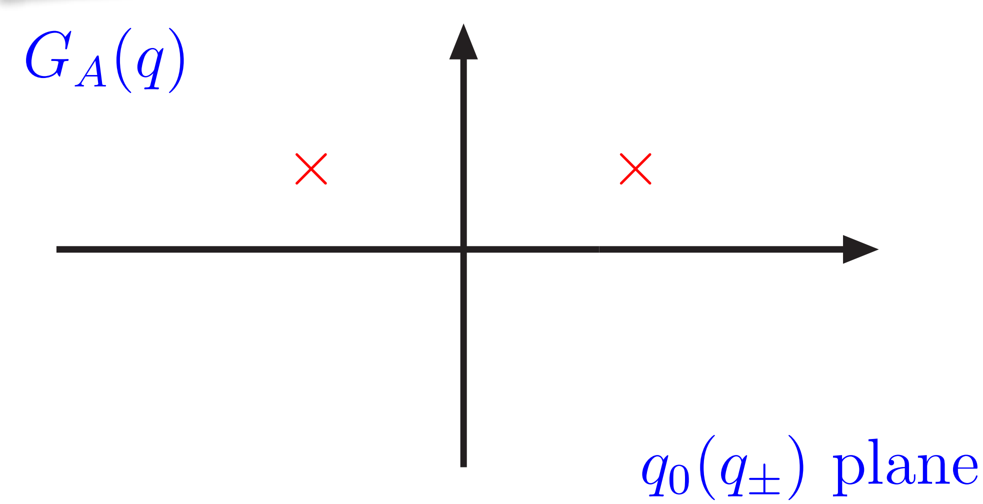
$$G_F(q) \equiv \frac{1}{q^2 + i0} \text{ and } G_A(q) \equiv \frac{1}{q^2 - i0\, q_0}$$

Feynman and advanced propagators differ in the position of the poles in the complex plane

$$[G_F(q)]^{-1} = 0 \quad \Longrightarrow \quad q_0 = \pm\sqrt{\mathbf{q}^2 - i0}$$

$$[G_A(q)]^{-1} = 0 \quad \Longrightarrow \quad q_0 \simeq \pm\sqrt{\mathbf{q}^2} + i0$$

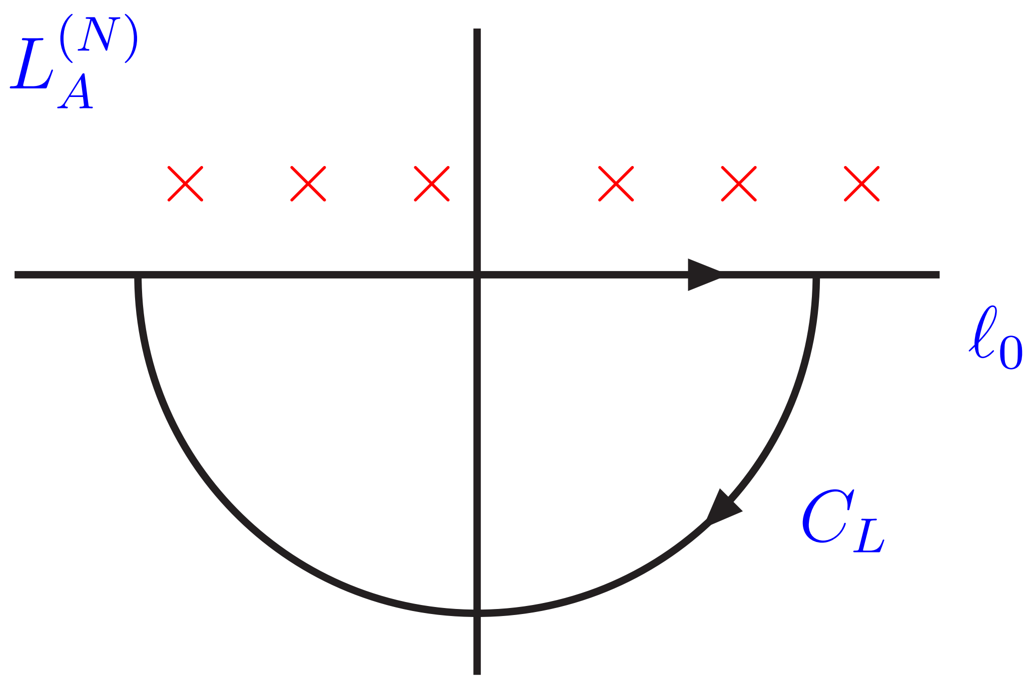


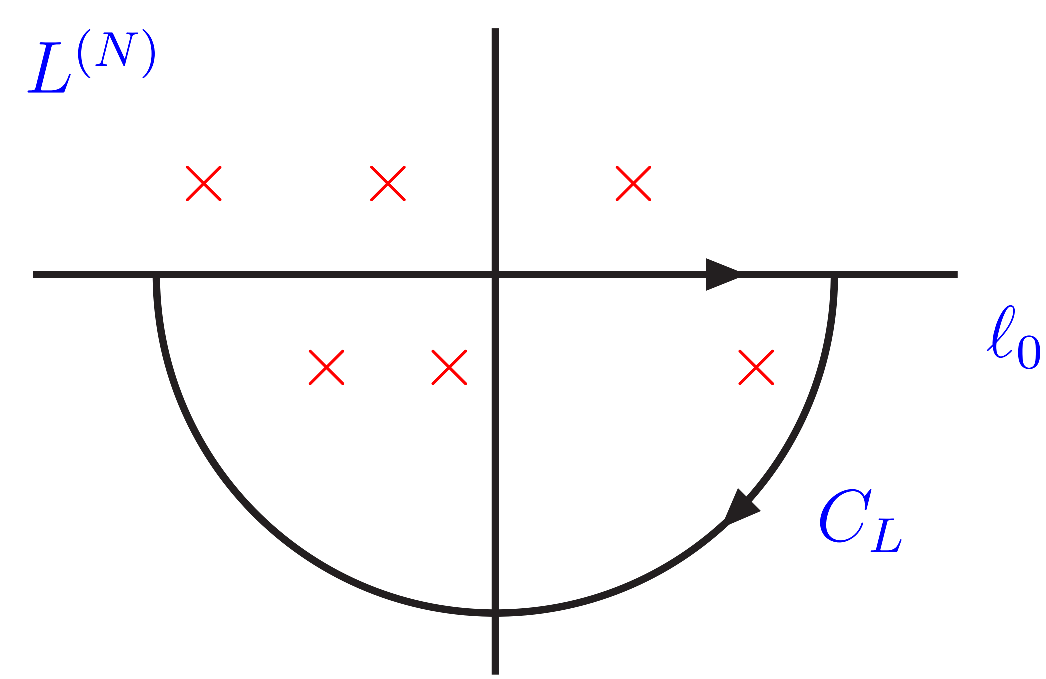


The Feynman Tree Theorem

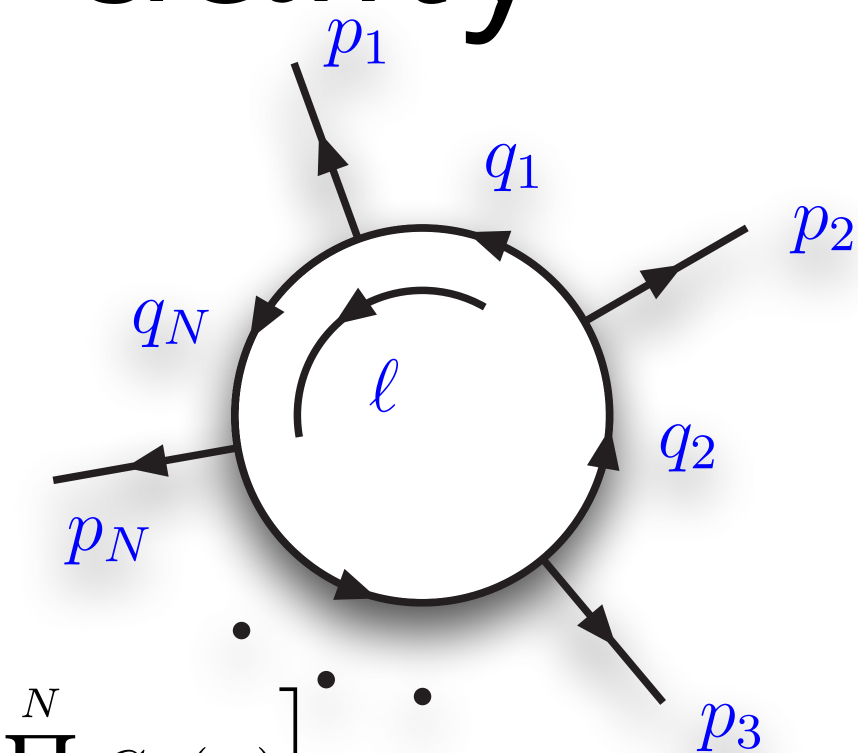
$$L^{(1)}(p_1, p_2, \ldots, p_N) = \int_\ell \prod_{i=1}^{N} G_F(q_i) \quad \Rightarrow \quad L_A^{(1)}(p_1, p_2, \ldots, p_N) = \int_\ell \prod_{i=1}^{N} G_A(q_i)$$

Then $L_A^{(1)}(p_1, p_2, \ldots, p_N) = 0$





The Loop-Tree Duality

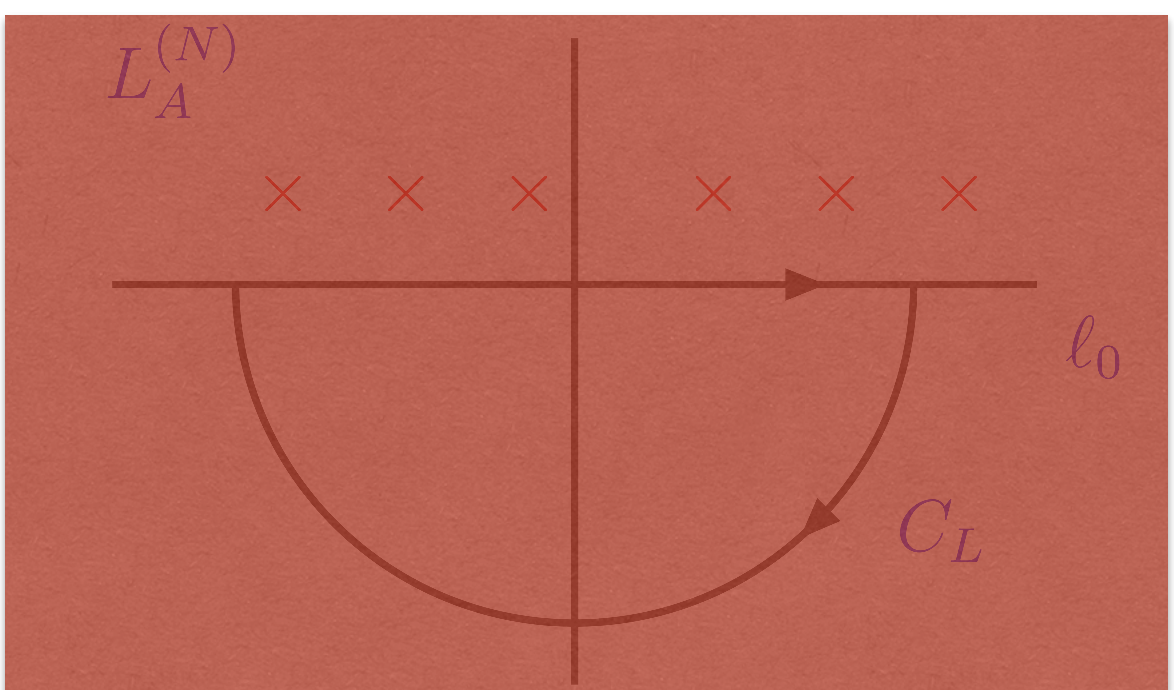


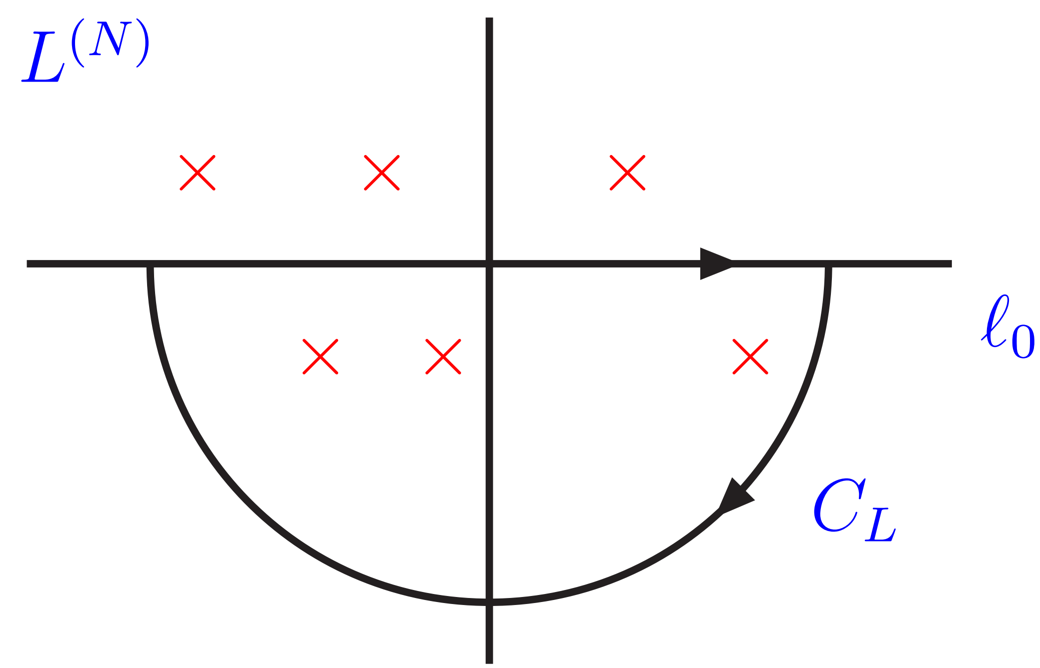
$$L^{(1)}(p_1, p_2, \dots, p_N) = \int_\ell \prod_{i=1}^{N} G_F(q_i)$$

$$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0}$$

$$L^{(1)}(p_1, p_2, \dots, p_N) = \int_{\vec{\ell}} \int d\ell_0 \prod_{i=1}^{N} G_F(q_i)$$

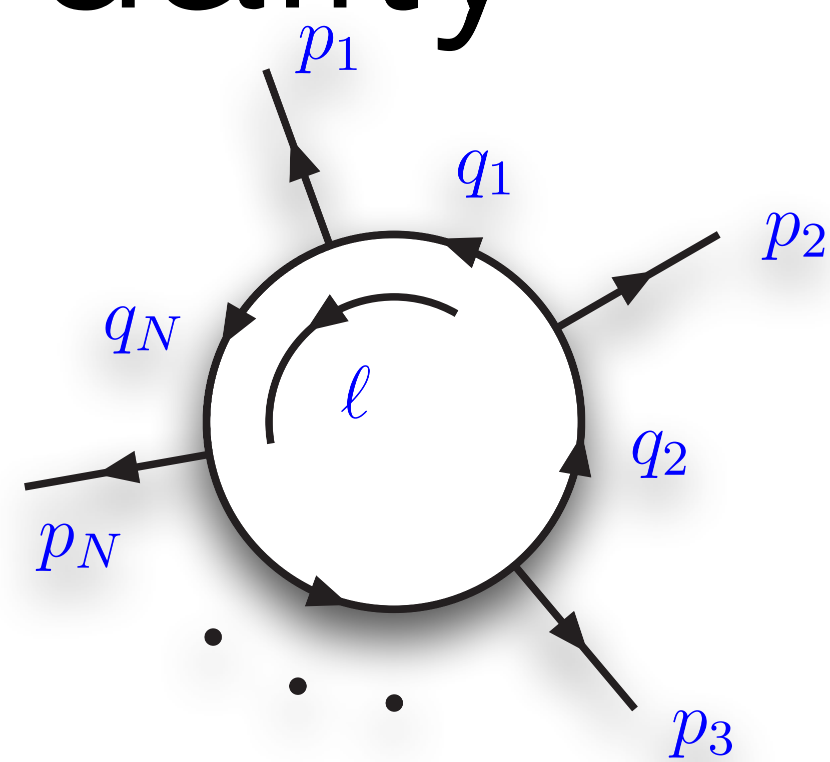
$$= \int_{\vec{\ell}} \int_{C_L} d\ell_0 \prod_{i=1}^{N} G_F(q_i) = -2\pi i \int_{\vec{\ell}} \sum \mathrm{Res}_{\{\mathrm{Im}\ \ell_0 < 0\}} \left[\prod_{i=1}^{N} G_F(q_i) \right]$$





The Loop-Tree Duality

$$L^{(1)}(p_1, p_2, \dots, p_N) = \int_\ell \prod_{i=1}^{N} G_F(q_i)$$



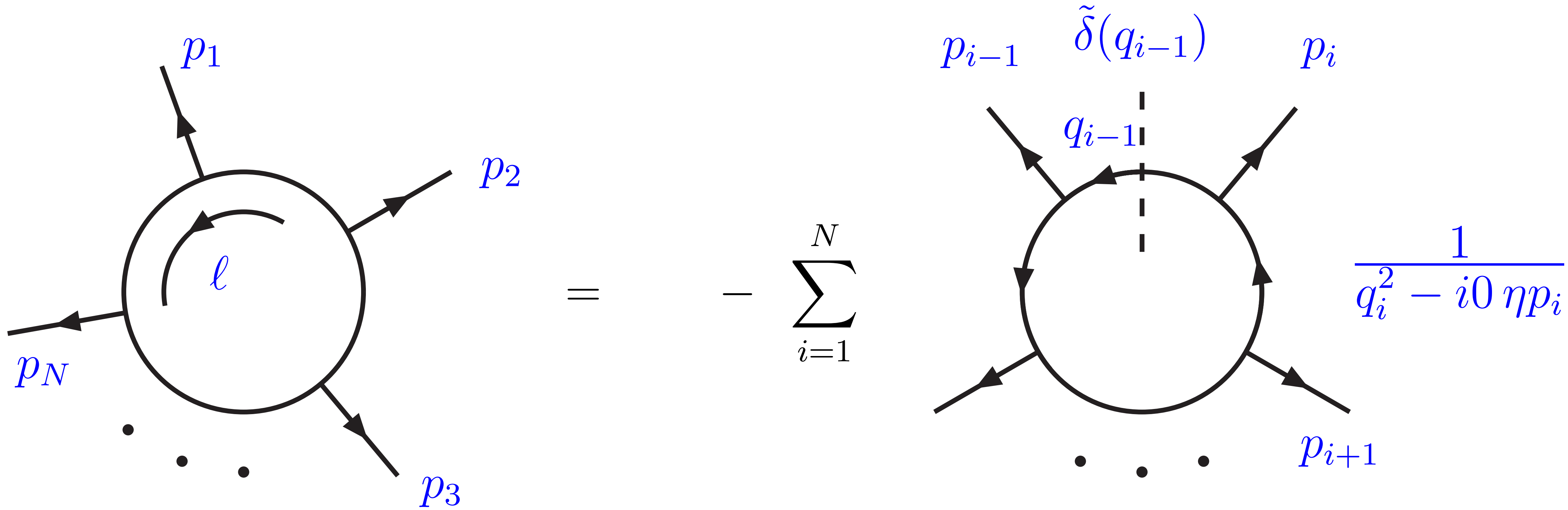
$$\boxed{L^{(1)}(p_1, p_2, \dots, p_N) \;=\; -\sum \int_{\ell_1} \widetilde{\delta}(q_i) \prod_{\substack{j=1 \\ j\neq i}}^{N} G_D(q_i; q_j)}$$

$$G_D(q_i; q_j) = \frac{1}{q_j^2 - m_j^2 - i0\, \eta(q_j - q_i)}$$

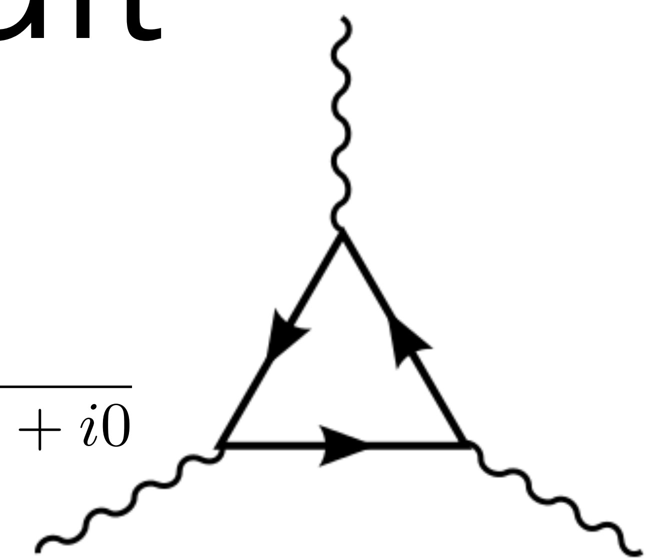
η is a future-like vector such that $\eta_\mu = (\eta_0, \eta)$, with $\eta_0 \geq 0,\ \eta^2 = \eta_\mu \eta^\mu \geq 0$

Dual propagator, keeps proper track of the small imaginary parts. Notice that $(q_j - q_i)$ does not depend on the loop momentum. Recall that $\widetilde{\delta}(q_i) \to \widetilde{\delta}(q_i) = 2\pi\, i\, \theta(q_{i,0})\, \delta(q_i^2 - m_i^2)$

A graphical representation of the Loop-Tree Duality



En explicit result



$$L^{(1)}(p_1, p_2, p_3) = \int_\ell G_F(q_1) G_F(q_2) G_F(q_3)$$

$$G_F(q_1) = \frac{1}{q_1^2 - m_1^2 + i0}, \; G_F(q_2) = \frac{1}{q_2^2 - m_2^2 + i0}, \; G_F(q_3) = \frac{1}{q_3^2 - m_3^2 + i0}$$

$$q_1 = \ell + p_1, \; q_2 = \ell + p_1 + p_2 = \ell, \; q_3 = \ell$$

Let us apply the Loop-Tree Duality

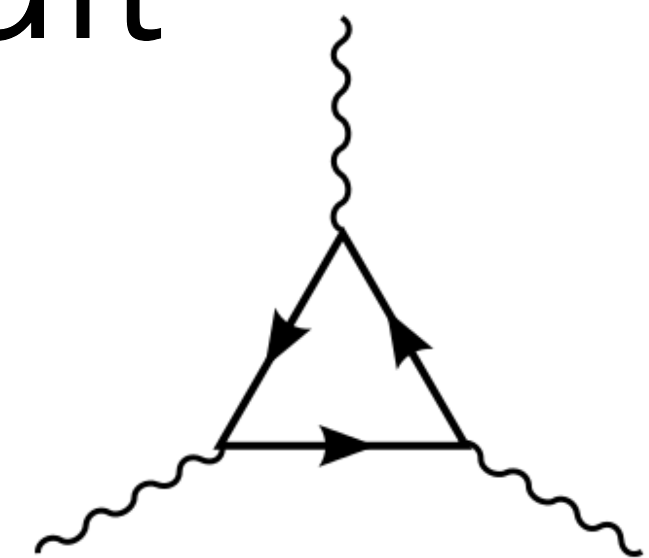
$$L^{(1)}(p_1, p_2, p_3) = \int_\ell \widetilde{\delta}(q_1) G_D(q_1; q_2) G_D(q_1; q_3) \qquad \text{first contribution} \qquad (\mathrm{I}_1)$$

$$+ \int_\ell G_D(q_2; q_1) \widetilde{\delta}(q_2) G_D(q_2; q_3) \qquad \text{second contribution} \qquad (\mathrm{I}_2)$$

$$+ \int_\ell G_D(q_3; q_1) G_D(q_3; q_2) \widetilde{\delta}(q_3) \qquad \text{third contribution} \qquad (\mathrm{I}_3)$$

En explicit result

$$L^{(1)}(p_1,p_2,p_3) = \int_\ell \widetilde{\delta}(q_1) G_D(q_1;q_2) G_D(q_1;q_3) \qquad \text{first contribution} \quad (\mathrm{I}_1)$$
$$+ \int_\ell G_D(q_2;q_1)\widetilde{\delta}(q_2) G_D(q_2;q_3) \qquad \text{second contribution} \quad (\mathrm{I}_2)$$
$$+ \int_\ell G_D(q_3;q_1) G_D(q_3;q_2)\widetilde{\delta}(q_3) \qquad \text{third contribution} \quad (\mathrm{I}_3)$$



$$\widetilde{\delta}(q_1) = \frac{\delta(\ell_0 - (-p_{1,0} + \sqrt{(\boldsymbol{\ell}+\mathbf{p}_1)^2 + m_1^2}))}{2\sqrt{(\boldsymbol{\ell}+\mathbf{p}_1)^2 + m_1^2)}} \,,$$

$$\widetilde{\delta}(q_2) = \frac{\delta(\ell_0 - (-p_{1,0} - p_{2,0} + \sqrt{(\boldsymbol{\ell}+\mathbf{p}_1+\mathbf{p}_2)^2 + m_2^2}))}{2\sqrt{(\boldsymbol{\ell}+\mathbf{p}_1+\mathbf{p}_2)^2 + m_2^2)}} \,,$$

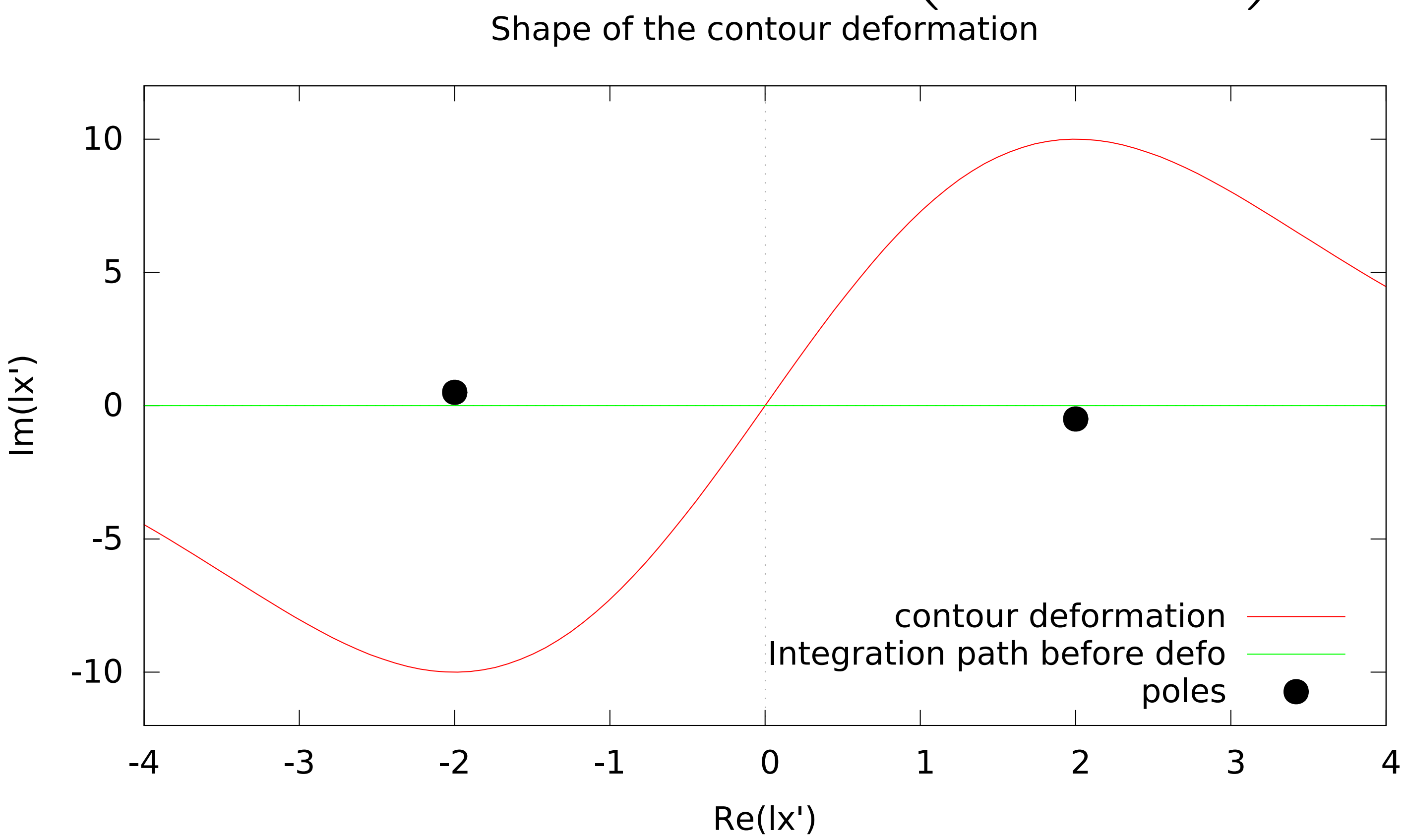
$$\widetilde{\delta}(q_3) = \frac{\delta(\ell_0 - \sqrt{\boldsymbol{\ell}^2 + m_3^2})}{2\sqrt{\boldsymbol{\ell}^2 + m_3^2}}$$

$$\boxed{I_3 = -\int_{\boldsymbol{\ell}} \frac{1}{2p_{1,0}\sqrt{\boldsymbol{\ell}^2+m_3^2} + 2\boldsymbol{\ell}\cdot\mathbf{p}_1 - m_1^2 + m_3^2 + p_1^2 - i0\eta k_{13}} \cdot \frac{1}{2\sqrt{\boldsymbol{\ell}^2+m_3^2}} \cdot \frac{1}{2(p_{1,0}+p_{2,0})\sqrt{\boldsymbol{\ell}^2+m_3^2} + 2\boldsymbol{\ell}\cdot(\mathbf{p}_1+\mathbf{p}_2) + (p_1+p_2)^2 - m_2^2 + m_3^2 - i0\eta k_{23}}}$$

Contour deformation

Assume $f(\ell_x) = \dfrac{1}{\ell_x^2 - E^2 + i0}$ with poles $\ell_{x\pm} = \pm(E - i0)$

$$\ell_x \to \ell'_x = \ell_x + i\lambda\ell_x \exp\left(-\frac{\ell_x^2 - E^2}{2E^2}\right)$$



$$\ell \to \ell' = \ell + i\kappa$$

Implementation

- In C++ (double and extended precision)
- Uses the Cuba library for numerical integration (T. Hahn)
- In particular, Cuhre (G. Berntsen, T. O. Espelid, A. Genz) and Vegas (G. P. Lepage)
- Input: - number of legs
  - external momenta
  - internal masses
- The user is free to choose between Cuhre and Vegas and also to change the parameters of the contour deformation
- MATHEMATICA was used extensively for cross-checking and during the development
- Two other programs were heavily used Looptools (T. Hahn, M. Perez-Victoria) and SecDec v3 (S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, T. Zirke) to get reference values and generally for cross-checks
- **Special thanks** to S. Borowka and to G. Heinrich for advice in running SecDec for some special cases

Results

- All results were obtained on a Desktop machine with an Intel i7 (3.4 GHz) processor, # cores = 4 and # threads = 8

- The SecDec run times in the following are only indicative, no optimisations were used and the important for us was the SecDec result as a reference value. Wherever run times of SecDec and the Loop-Tree Duality are displayed it is only to give a feeling of the increasing complexity of the integrals calculated and not a comparison of the two programs!

Scalar triangles

	Real Part	Real Error	Imaginary Part	Imaginary Error
LoopTools P.3	5.37305E-4	0	-6.68103E-4	0
Loop–Tree Duality P.3	5.37307E-4	8.6E-9	-6.68103E-4	8.6E-9
LoopTools P.4	-5.61370E-7	0	-1.01665E-6	0
Loop–Tree Duality P.4	-5.61371E-7	7.2E-10	-1.01666E-6	7.2E-10

Point 3 $p_1 = \{10.51284, 6.89159, -7.40660, -2.85795\}$

$p_2 = \{6.45709, 2.46635, 5.84093, 1.22257\}$

$m_1 = m_2 = m_3 = 0.52559$

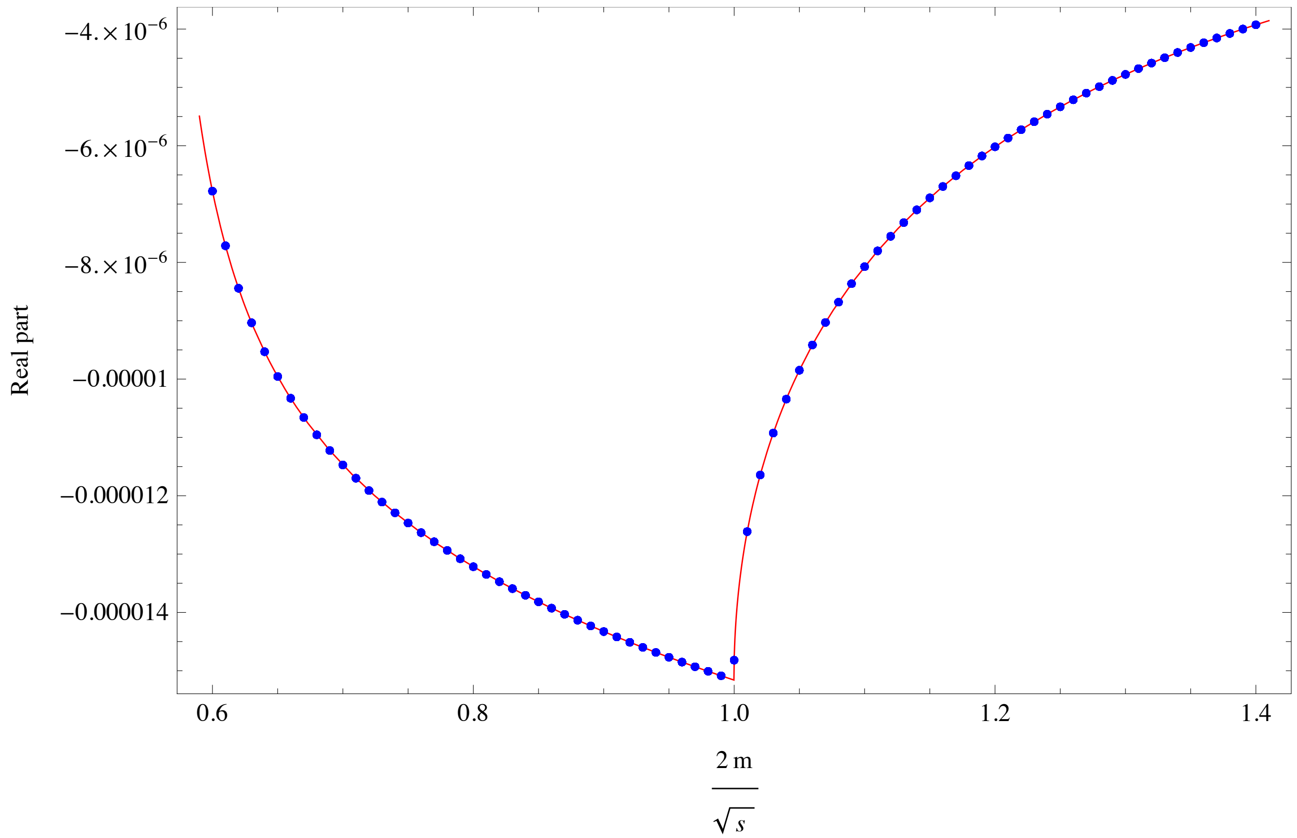
Point 4 $p_1 = \{95.77004, 31.32025, -34.08106, -9.38565\}$

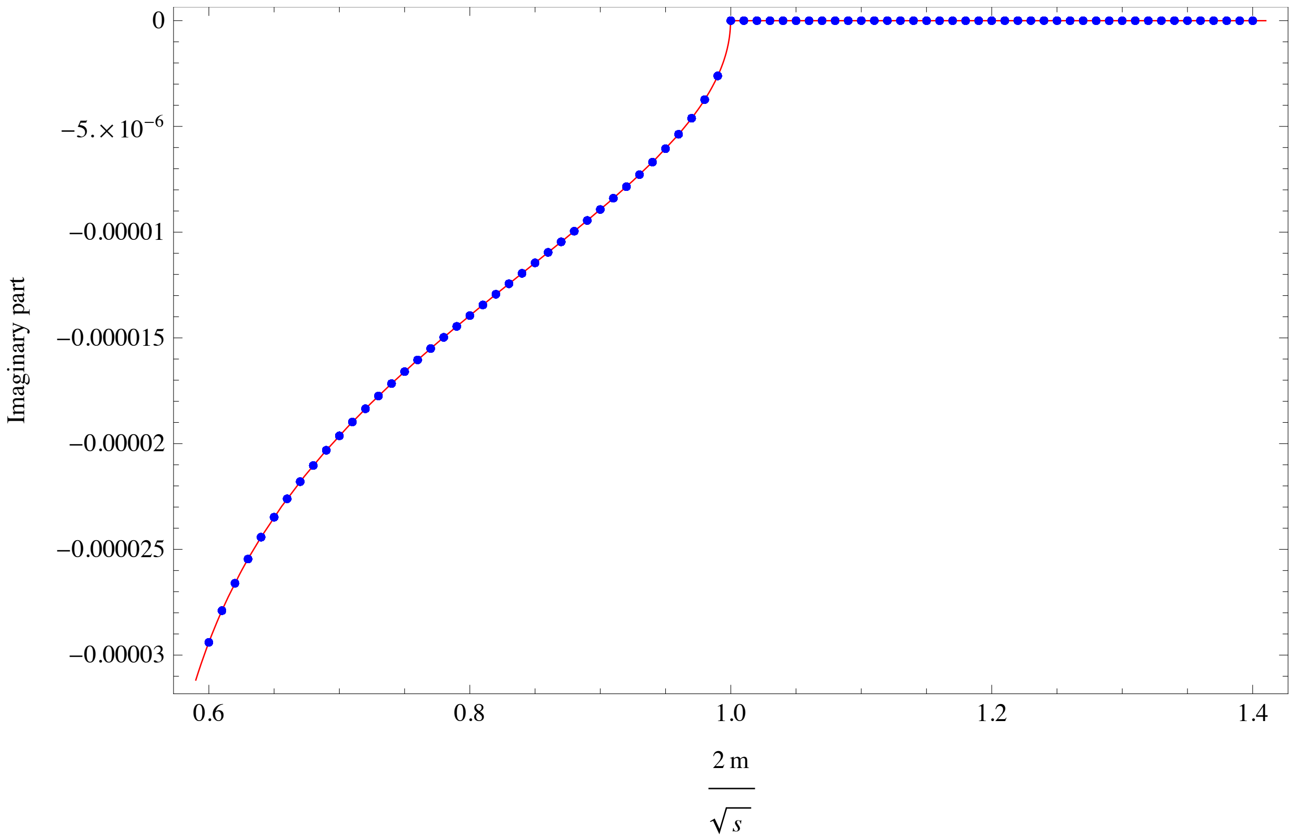
$p_2 = \{94.54738, -53.84229, 67.11107, 45.56763\}$

$m_1 = 83.02643, m_2 = 76.12873, m_3 = 55.00359$

<1 to 15 seconds for 4 digits accuracy

Scalar triangles





All internal masses equal

The red curve is from running LoopTools

Scalar boxes

	Real Part	Real Error	Imaginary Part	Imaginary Error
LoopTools P.7	-2.38766E-10	0	-3.03080E-10	0
Loop–Tree Duality P.7	-2.38798E-10	8.2E-13	-3.03084E-10	8.2E-13
LoopTools P.8	-4.27118E-11	0	4.49304E-11	0
Loop–Tree Duality P.8	-4.27127E-11	5.3E-14	4.49301E-11	5.3E-14
LoopTools P.9	6.43041E-11	0	1.61607E-10	0
Loop–Tree Duality P.9	6.43045E-11	8.4E-15	1.61607E-10	8.4E-15
LoopTools P.10	-4.34528E-11	0	3.99020E-11	0
Loop–Tree Duality P.10	-4.34526E-11	3.5E-14	3.99014E-11	3.5E-14

Point 7 $p_1 = \{62.80274, -49.71968, -5.53340, -79.44048\}$
$p_2 = \{48.59375, -1.65847, 34.91140, 71.89564\}$
$p_3 = \{76.75934, -19.14334, -17.10279, 30.22959\}$
$m_1 = m_2 = m_3 = m_4 = 9.82998$

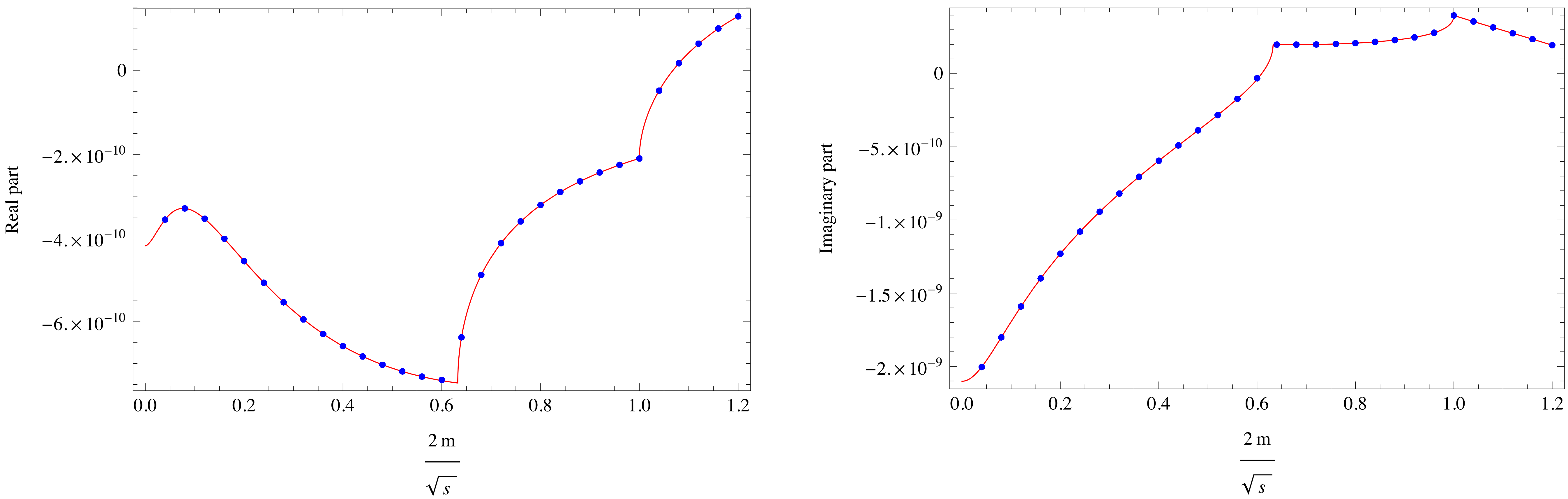
Point 8 $p_1 = \{98.04093, 77.37405, 30.53434, -81.88155\}$
$p_2 = \{73.67657, -53.78754, 13.69987, 14.20439\}$
$p_3 = \{68.14197, -36.48119, 59.89499, -81.79030\}$
$m_1 = 81.44869, m_2 = 94.39003, m_3 = 57.53145, m_4 = 0.40190$

Point 9 $p_1 = \{90.15393, -60.44028, -18.19041, 42.34210\}$
$p_2 = \{75.27949, 86.12082, 19.15087, -95.80345\}$
$p_3 = \{14.34134, 2.00088, 87.56698, 39.80553\}$
$m_1 = m_2 = 21.23407, m_3 = m_4 = 81.40164$

Point 10 $p_1 = \{56.88939, 87.04163, -34.62173, -42.86104\}$
$p_2 = \{92.86718, -91.88334, 59.75945, 38.70047\}$
$p_3 = \{55.98527, -35.20008, 9.02722, 82.97219\}$
$m_1 = m_3 = 67.88777, m_2 = m_4 = 40.77317$

<1 to 20 seconds for 4 digits accuracy

Scalar boxes



All internal masses equal

The red curve is from running LoopTools

Scalar pentagons

	Real Part	Real Error	Imaginary Part	Imaginary Error
LoopTools P.13	1.02350E-11	0	1.40382E-11	0
Loop–Tree Duality P.13	1.02353E-11	1.0E-16	1.40385E-11	1.0E-16
LoopTools P.14	7.46345E-15	0	-9.13484E-15	0
Loop–Tree Duality P.14	7.46309E-15	6.1E-18	-9.13444E-15	6.1E-18
LoopTools P.15	6.89836E-15	0	2.14893E-15	0
Loop–Tree Duality P.15	6.89848E-15	6.5E-18	2.14894E-15	6.5E-18

Point 13
$p_1 = \{1.58374, 6.86200, -15.06805, -10.63574\}$
$p_2 = \{7.54800, -3.36539, 34.57385, 27.52676\}$
$p_3 = \{43.36396, -49.27646, -25.35062, -17.68709\}$
$p_4 = \{22.58103, 38.31530, -14.67581, -3.08209\}$
$m_1 = m_2 = m_3 = m_4 = m_5 = 2.76340$

Point 14
$p_1 = \{-93.06712, -36.37997, -27.71460, 38.42206\}$
$p_2 = \{-46.33465, -11.90909, 32.33395, 46.42742\}$
$p_3 = \{8.41724, -83.92296, 56.21715, 34.04937\}$
$p_4 = \{-15.23696, 71.33931, 48.68306, -53.67870\}$
$m_1 = 59.10425, m_2 = 60.25099, m_3 = 76.79109$
$m_4 = 65.27606, m_5 = 5.99925$

Point 15
$p_1 = \{-32.14401, -64.50445, 46.04455, -75.56462\}$
$p_2 = \{-96.90340, -27.60002, -71.50486, 86.25541\}$
$p_3 = \{-37.95135, 46.18586, 25.67520, -71.38501\}$
$p_4 = \{-87.67870, 66.66463, -36.20151, -27.37362\}$
$m_1 = m_2 = m_3 = 79.63229, m_4 = m_5 = 51.70237$

<1 to 30 seconds for 4 digits accuracy

Tensor diagrams

- In general, tensor one-loop diagrams do not present a priori an extra difficulty for the Loop-Tree Duality. The run times seem to increase only a bit in order to get the same accuracy as in the scalar diagrams case.

Tensor pentagons

	Rank	Tensor Pentagon	Real Part	Imaginary Part	Time [s]
P16	2	LoopTools	-1.86472×10^{-8}		
		SecDec	$-1.86471(2) \times 10^{-8}$		45
		LTD	$-1.86462(26) \times 10^{-8}$		1
P17	3	LoopTools	1.74828×10^{-3}		
		SecDec	$1.74828(17) \times 10^{-3}$		550
		LTD	$1.74808(283) \times 10^{-3}$		1
P18	2	LoopTools	-1.68298×10^{-6}	$+i\ 1.98303 \times 10^{-6}$	
		SecDec	$-1.68307(56) \times 10^{-6}$	$+i\ 1.98279(90) \times 10^{-6}$	66
		LTD	$-1.68298(74) \times 10^{-6}$	$+i\ 1.98299(74) \times 10^{-6}$	36
P19	3	LoopTools	-8.34718×10^{-2}	$+i\ 1.10217 \times 10^{-2}$	
		SecDec	$-8.33284(829) \times 10^{-2}$	$+i\ 1.10232(107) \times 10^{-2}$	1501
		LTD	$-8.34829(757) \times 10^{-2}$	$+i\ 1.10119(757) \times 10^{-2}$	38

$(\ell \cdot p_3) \times (\ell \cdot p_4)$

$(\ell \cdot p_3) \times (\ell \cdot p_4) \times (\ell \cdot p_5)$

Tensor hexagons

	Rank	Tensor Hexagon	Real Part	Imaginary Part	Time[s]
P20	1	SecDec	$-1.21585(12) \times 10^{-15}$		36
		LTD	$-1.21552(354) \times 10^{-15}$		6
P21	3	SecDec	$4.46117(37) \times 10^{-9}$		5498
		LTD	$4.461369(3) \times 10^{-9}$		11
P22	1	SecDec	$1.01359(23) \times 10^{-15}$	$+i\ 2.68657(26) \times 10^{-15}$	33
		LTD	$1.01345(130) \times 10^{-15}$	$+i\ 2.68633(130) \times 10^{-15}$	72
P23	2	SecDec	$2.45315(24) \times 10^{-12}$	$-i\ 2.06087(20) \times 10^{-12}$	337
		LTD	$2.45273(727) \times 10^{-12}$	$-i\ 2.06202(727) \times 10^{-12}$	75
P24	3	SecDec	$-2.07531(19) \times 10^{-6}$	$+i\ 6.97158(56) \times 10^{-7}$	14280
		LTD	$-2.07526(8) \times 10^{-6}$	$+i\ 6.97192(8) \times 10^{-7}$	85

$$(\ell \cdot p_4) \times (\ell \cdot p_5) \times (\ell \cdot p_6)$$

P24

$$p_1 = (-70.26380, 96.72681, 21.66556, -37.40054)$$
$$p_2 = (-13.45985, 2.12040, 3.20198, 91.44246)$$
$$p_3 = (-62.59164, -29.93690, -22.16595, -58.38466)$$
$$p_4 = (-67.60797, -83.23480, 18.49429, 8.94427)$$
$$p_5 = (-34.70936, -62.59326, -60.71318, 2.77450)$$
$$m_1 = 94.53242, m_2 = 64.45092, m_3 = 74.74299,$$
$$m_4 = 10.63129, m_5 = 31.77881, m_6 = 23.93819$$

Conclusions & Outlook

- The Loop-Tree Duality has many appealing theoretical properties

- Here we have shown numerical results from a first implementation of the method suitable for computing one-loop Feynman diagrams

- The method seems to excel in cases where we have many legs and many different scales as the increase of the run time is mild

- Near future: attack two-loop multi-scale diagrams