Relation between the on-shell and $\overline{\mathrm{MS}}$ quark mass at four loops

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Quark mass definitions

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} G_{\mu\nu}^2 + \sum_q \bar{\psi}_q \left(\not{D} - m_q \right) \psi_q$$

- OS mass
- $\overline{\rm MS}$ mass
- PS mass [Beneke'98]
- 1S mass [Hoang,Smith,Stelzer,Willenbrock'99]
- RS mass [Pineda'01]
- kinetic mass [Bigi,Shifman,Uraltsev,Vainshtein'97]
- ...

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In this talk:

- heavy quarks: c, b, t
- $\overline{\rm MS}$-OS relation to 4 loops
- precise relation between $m^{\rm PS,1S,RS}$ and $m^{\overline{\rm MS}}$
- How precise can we determine the top pole mass?

Not in this talk: “Which mass is extracted at LHC?”

Top and bottom quark mass

- top quark mass
  - Tevatron/LHC March 2014: $m_t^{\rm OS} = 173.34 \pm 0.27 \pm 0.71$ GeV
    CMS September 2015: $m_t^{\rm OS} = 172.44 \pm 0.13 \pm 0.47$ GeV
    ➪ convert to $\overline{\rm MS}$ top mass
  - threshold scan at ILC
    ➪ determine in a first step m_t^{PS} or m_t^{1S} or ...
    ➪ convert to $\overline{\rm MS}$ top mass
- bottom quark mass
  Example: m_b from sum rules: $\mathcal{M}_n \equiv \int {\rm d}s \frac{R_b(s)}{s^{(n+1)}}$
  - $\overline{\rm MS}$: m_b from low-moments SRs [Chetyrkin et al.'09, ...]
    $m_b^{\overline{\rm MS}}(m_b) = 4.163 \pm 0.016$ GeV
  - PS mass: m_b from Υ SRs [Beneke,Maier,Piclum,Rauh'15; ...]
    $m_b^{\rm PS}(2GeV) = 4.532^{+0.013}_{-0.039}$ GeV
    ➪ convert to $\overline{\rm MS}$ bottom mass

Example for threshold mass: PS mass

1. defined via relation to pole mass

[Beneke'98]

$$m^{\rm PS}(\mu_f) = m^{\rm OS} - \delta m(\mu_f)$$

$$\delta m(\mu_f) = -\frac{1}{2}\int_{|\vec{q}\,|<\mu_f} \frac{{\rm d}^3 q}{(2\pi)^3} V(\vec{q}\,)$$

$$= \mu_f \frac{C_F \alpha_s}{\pi}\left\{1 + \frac{\alpha_s}{4\pi} a_1 + \left(\frac{\alpha_s}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s}{4\pi}\right)^3 a_3\right\}$$

$V(\vec{q}\,)$: static potential
known to 3 loops
[Smirnov,Smirnov,Steinhauser'09; Anzai,Kiyo,Sumino'09]

➪ $m^{\rm PS} - m^{\rm OS}$ relation known to N^{3}LO (top: $\mu_f = 80$ GeV)

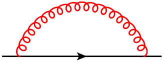
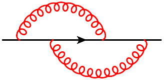
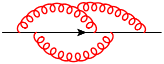
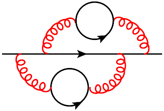
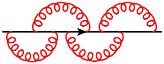
2. use $m^{\rm OS} - m^{\overline{\rm MS}}$ relation

$$m^{\rm OS} = m^{\overline{\rm MS}}\left(1 + \frac{\alpha_s}{\pi} c_m^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 c_m^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 c_m^{(3)} + \left(\frac{\alpha_s}{\pi}\right)^4 c_m^{(4)} + \ldots\right)$$

needed to 4 loops

Z_m^{OS}: known results

$$m^{\mathrm{bare}} = Z_m^{\mathrm{OS}} m^{\mathrm{OS}} = Z_m^{\overline{\mathrm{MS}}} m^{\overline{\mathrm{MS}}}$$

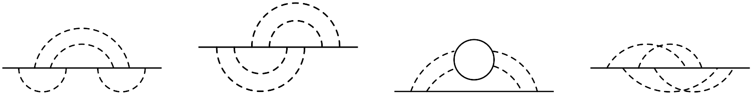
-  [Tarrach'81]
-  [Gray,Broadhurst,Grafe,Schilcher'90]
-  [Chetyrkin,Steinhauser'99; Melnikov, v. Ritbergen'00; Marquard,Mihaila,Piclum,Steinhauser'07]
- n_l^2  [Lee,Marquard,Smirnov,Smirnov,Steinhauser'13]
- full  [Marquard,Smirnov,Smirnov,Steinhauser'15]

- $Z_m^{\overline{\mathrm{MS}}}$ known to 4 loops [Chetyrkin'97; Larin,van Ritbergen,Vermaseren'97]
  (5 loops: [Baikov,Chetyrkin,Kühn'14])
- electroweak corrections

[Hempfling,Kniehl'94; Jegerlehner,Kalmykov'03; Faisst,Kühn,Veretin'04; Martin'05; Eiras,Steinhauser'05; Martin'16]

Some technical details

- generation of amplitudes: `QGRAF` [Nogueira'91]
  manipulation/transformation to `FORM`: `q2e/exp`
  [Harlander,Seidelsticker,Steinhauser'97;Seidelsticker'97]
- map to $\sim$ 100 families (topologies)



- reduce to master integrals (MIs): `FIRE5` [Smirnov'14] and `crusher` [Marquard,Seidel]
- minimize MIs over all topologies: `tsort` [Pak'11]
  ➪ 386 4-loop OS MIs
- compute MIs (analytic, numerical with `FIESTA` [Smirnov'14'15], Mellin Barnes)

Note: $m^{\rm OS} - m^{\overline{\rm MS}}$ relation in terms of MIs known analytically
➪ systematic improvement possible

4-loop coefficient

$$m^{\overline{\rm MS}} = m^{\rm OS}\left(1 + \frac{\alpha_s}{\pi} z_m^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 z_m^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 z_m^{(3)} + \left(\frac{\alpha_s}{\pi}\right)^4 z_m^{(4)} + \dots\right)$$

[Marquard,Smirnov,Smirnov,Steinhauser'15]

charm

$$z_m^{(4)}\Big|_{n_l=3} = -1744.8 \pm 21.5 - 703.48\, l_{\rm OS} - 122.97\, l_{\rm OS}^2 - 14.234\, l_{\rm OS}^3 - 0.75043\, l_{\rm OS}^4$$

bottom

$$z_m^{(4)}\Big|_{n_l=4} = -1267.0 \pm 21.5 - 500.23\, l_{\rm OS} - 83.390\, l_{\rm OS}^2 - 9.9563\, l_{\rm OS}^3 - 0.514033\, l_{\rm OS}^4$$

top

$$z_m^{(4)}\Big|_{n_l=5} = -859.96 \pm 21.5 - 328.94\, l_{\rm OS} - 50.856\, l_{\rm OS}^2 - 6.4922\, l_{\rm OS}^3 - 0.33203\, l_{\rm OS}^4$$

$$l_{\rm OS} = \ln(\mu^2/M^2)$$

- $\ln(\mu^2/M^2)$ known from RGE
- constant term: uncertainty $< 3\%$

$\overline{\mathrm{MS}}$−OS relation for top and bottom

$$
\begin{aligned}
m_t^{\mathrm{OS}} &= m_t^{\overline{\mathrm{MS}}}\left[1 + 0.4244\,\alpha_s + 0.8345\,\alpha_s^2 + 2.375\,\alpha_s^3 + (8.49 \pm 0.25)\,\alpha_s^4\right] \\
&= 163.643 + 7.557 + 1.617 + 0.501 + (0.195 \pm 0.005)\ \mathrm{GeV}
\end{aligned}
$$

$\overline{\mathrm{MS}}$−OS relation for top and bottom

$$m_t^{\mathrm{OS}} = m_t^{\overline{\mathrm{MS}}}\left[1 + 0.4244\,\alpha_s + 0.8345\,\alpha_s^2 + 2.375\,\alpha_s^3 + (8.49 \pm 0.25)\,\alpha_s^4\right]$$
$$= 163.643 + 7.557 + 1.617 + 0.501 + (0.195 \pm 0.005)\ \mathrm{GeV}$$

$$m_b^{\mathrm{OS}} = m_b^{\overline{\mathrm{MS}}}\left[1 + 0.4244\,\alpha_s + 0.9401\,\alpha_s^2 + 3.045\,\alpha_s^3 + (12.57 \pm 0.38)\,\alpha_s^4\right]$$
$$= 4.163 + 0.401 + 0.201 + 0.148 + (0.138 \pm 0.004)\ \mathrm{GeV}$$

$$m_c^{\mathrm{OS}} = m_c^{\overline{\mathrm{MS}}}(3\ \mathrm{GeV})$$
$$\times\left(1 + 1.133\,\alpha_s + 3.119\,\alpha_s^2 + 10.98\,\alpha_s^3 + (51.29 \pm 0.52)\,\alpha_s^4\right)$$
$$= 0.986 + 0.286 + 0.202 + 0.182 + (0.217 \pm 0.002)\ \mathrm{GeV}$$

- top: good/reasonable convergence
- bottom, charm: no convergence

top quark: renormalon ambiguity

$$m_t^{\mathrm{OS}} \;=\; m_t^{\overline{\mathrm{MS}}} \left(1 + \sum_{k\geq 1} r_{k-1}\alpha_s^k\right)$$

$$r_k \stackrel{k\to\infty}{\longrightarrow} N\,(2\beta_0)^k\Gamma(k+1+b)\left(1 + \frac{s_1}{k+b} + \frac{s_2}{(k+b)(k+b+1)} + \ldots\right)$$

$b = \beta_1/(2\beta_0)$

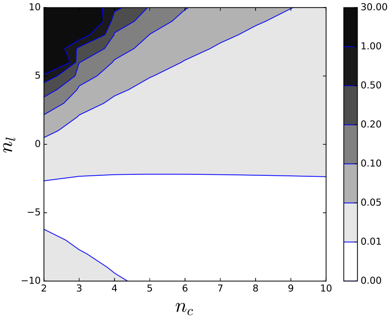
[Beneke,Braun'94; Beneke'95]

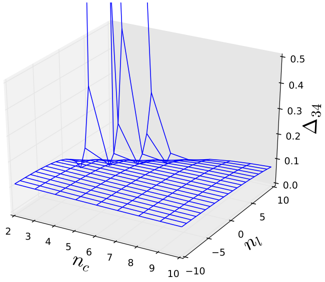
1. determine N from exact (3- and) 4-loop coefficient

top quark: renormalon ambiguity

$$m^{\rm OS} = m^{\overline{\rm MS}}\left(1 + \sum r_k \alpha^k\right)$$

$\Delta_{34}(n_c, n_l)$: consistency of determination of N from 3 and 4 loops





top: $n_c = 3, n_l = 5$

PRELIMINARY

[Nason'15; Beneke,Marquard,Nason,Steinhauser'16]

top quark: renormalon ambiguity

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$$r_k \stackrel{k\to\infty}{\longrightarrow} N\,(2\beta_0)^k\Gamma(k+1+b)\left(1+\frac{s_1}{k+b}+\frac{s_2}{(k+b)(k+b+1)}+\ldots\right)$$

1. determine N from exact (3- and) 4-loop coefficient
2. consider $m_t^{\rm OS}(n) = m_t^{\overline{\rm MS}}\left(1+\sum_{k=1}^{n} r_{k-1}\alpha_s^k\right)$
   and determine n such that $m_t^{\rm OS}(n+1) - m_t^{\rm OS}(n)$ is minimal
3. $$m_t^{\rm OS} = m_t^{\overline{\rm MS}}\left(1+\sum_{k=1}^{4} r_{k-1}\alpha_s^k\right) + \delta^{(5+)}m_t^{\rm OS}$$

$$\delta^{(5+)}m_t^{\rm OS} = 0.2xx^{+0.04}_{-0.02}(N) \pm 0.07(\text{last term})\ \text{GeV}$$

➪ final uncertainty about (below?) 100 MeV!

PRELIMINARY

[Nason'15; Beneke,Marquard,Nason,Steinhauser'16]

$\overline{\text{MS}}-$ threshold top mass relation

	$m_t^{\overline{\text{MS}}}(m_t^{\overline{\text{MS}}})$			
input	$m^{\text{PS}} =$	$m^{\text{1S}} =$	$m^{\text{RS}} =$	
#loops	168.204	172.227	171.215	
1	164.311	165.045	164.847	
2	163.713	163.861	163.853	1-2 GeV

$\overline{\text{MS}}$− threshold top mass relation

input	$m^{\text{PS}} =$	$m^{\text{1S}} =$	$m^{\text{RS}} =$	
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1	164.311	165.045	164.847	
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3	163.625	163.651	163.663	$\lesssim$ 200 MeV

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4	163.643	163.643	163.643	$\lesssim$ 20 MeV

$\overline{\text{MS}}$ − threshold top mass relation

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	$m_t^{\overline{\text{MS}}}(m_t^{\overline{\text{MS}}})$			
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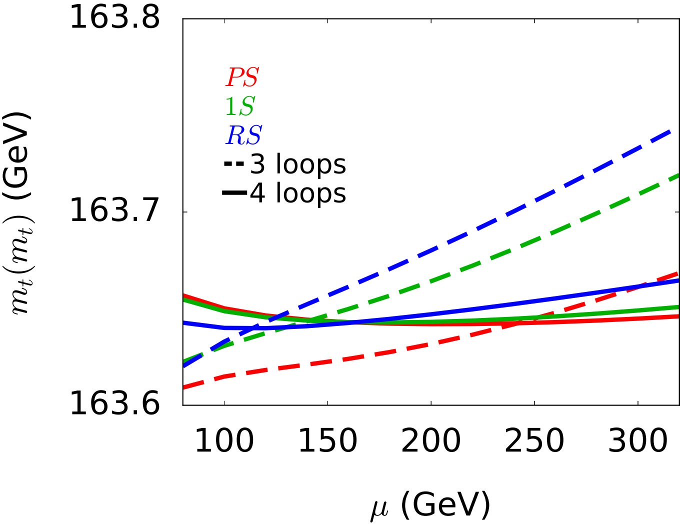
2	163.713	163.861	163.853	1-2 GeV
3	163.625	163.651	163.663	$\lesssim$ 200 MeV
4	163.643	163.643	163.643	$\lesssim$ 20 MeV
4 ($\times$1.03)	163.637	163.637	163.637	

- 3 loops: $\lesssim$ 200 MeV
- 4 loops: $\{18, 8, 20\}$ MeV
- 3% uncertainty $\widehat{=}$ 6 MeV
- $\delta m_t^{\text{ILC}} \lesssim$ 50 MeV

} ➪ $\{9, 7, 11\}$ MeV uncertainty in the $m_t^{\overline{\text{MS}}} - m_t^{\text{thr}}$ relation

$\overline{\mathrm{MS}}-$ threshold top mass relation

$$m^{\mathrm{thr}} \longrightarrow m_t^{\overline{\mathrm{MS}}}(\mu) \longrightarrow m_t^{\overline{\mathrm{MS}}}(m_t^{\overline{\mathrm{MS}}})$$



$\overline{\rm MS}-$ threshold bottom mass relation

	$m_b^{\overline{\rm MS}}(m_b^{\overline{\rm MS}})$			
input	$m^{\rm PS}=$	$m^{\rm 1S}=$	$m^{\rm RS}=$	
#loops	4.483	4.670	4.365	
1	4.266	4.308	4.210	
2	4.191	4.190	4.172	$\lesssim$ 110 MeV
3	4.161	4.154	4.158	$\lesssim$ 40 MeV
4	4.163	4.163	4.163	$\lesssim$ 9 MeV
4 ($\times$1.03)	4.159	4.159	4.159	

- 3 loops: $\approx$ 40 MeV
- 4 loops: $\{2, 9, 5\}$ MeV
- 3% uncertainty: 4 MeV

} ➪ $\{4, 6, 5\}$ MeV uncertainty in the $m_b^{\overline{\rm MS}} - m_b^{\rm thr}$ relation

- N³LO extractions: $\delta m_b \approx 10 - 20$ MeV
- "OS-$\overline{\rm MS}$" – "PS-OS"

$$m_b^{\rm PS} = 4.163 + (0.401 - 0.192) + (0.201 - 0.121) + (0.148 - 0.115) + (0.138 - 0.140)$$

Conclusions

- 4-loop $\overline{\mathrm{MS}}$-OS relation for heavy quarks
- 4-loop contribution to pole mass: 200 MeV
- Precise $m^{\overline{\mathrm{MS}}}$-$m^{\mathrm{thr}}$ relations to N^{3}LO
- Renormalon analysis for top quark
  ⇨ uncertainty (probably) below 100 MeV