The University of Manchester

New insights into Coulomb gluons

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In collaboration with
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JHEP 1512 (2015) 091 & arXiv:1602.00623 DIS16

Outline

Some progress in including Coulomb gluons and colour interference in partons showers

- Coulomb gluons and the breaking of *strict* collinear factorisation of QCD (squared) amplitudes.

- Coulomb gluons in 'gaps between jets' → Super-leading logarithms.
  - but there is (was) an ordering (evolution-) variable problem.

- Colour interference (evolution) of soft corrections from first principles.

Introduction: one-loop soft gluon correction

$$k^\mu << \sqrt{2p_i \cdot p_j}$$

Colour operator acting on $\big|2\big\rangle$

Hard process: a vector colour + spin

$$ig_s^2\mu^{2\epsilon}\mathbf{T}_i\cdot\mathbf{T}_j\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-p_i\cdot p_j}{[p_j\cdot k\pm i0][-p_i\cdot k\pm i0][k^2+i0]}\big|2\big\rangle$$

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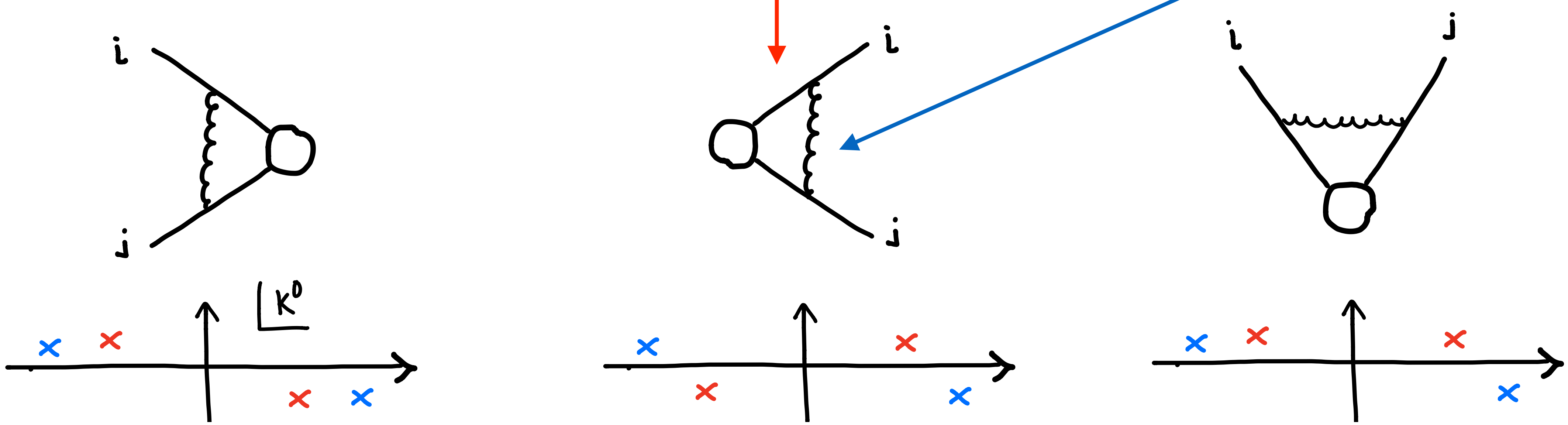

$i: \quad -i0$

$j: \quad -i0$

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Introduction: one-loop soft gluon correction

After contour integration:

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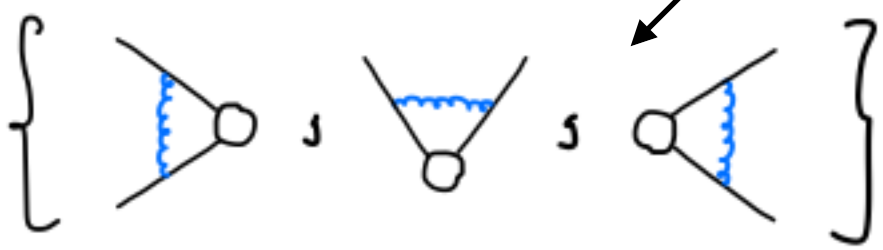
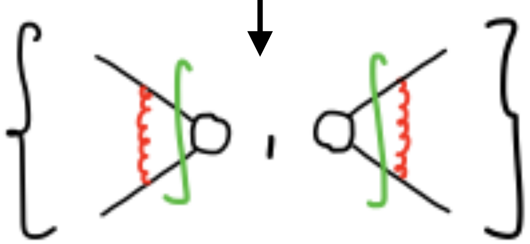


(On-shell gluon: Purely real)



(Coulomb gluon: Purely imaginary)

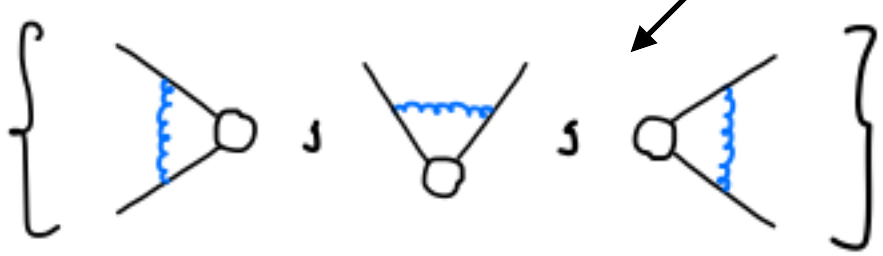
$$\tilde{\delta}_{ij} = \begin{cases} 1 & \text{if } i,j \text{ in }, \\ 1 & \text{if } i,j \text{ out }, \\ 0 & \text{otherwise.} \end{cases}$$

Introduction: one-loop soft gluon correction

After contour integration:

$$k^\mu = \alpha p_i^\mu + \beta p_j^\mu + (k_T^{(ij)})^\mu$$
$$\Rightarrow \alpha = \beta = 0$$

$$g_s^2 \mu^{2\epsilon} \mathbf{T}_i \cdot \mathbf{T}_j p_i \cdot p_j \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left[\frac{(2\pi)\delta(k^2)\theta(k^0)}{[p_j \cdot k][p_i \cdot k]} + i\tilde{\delta}_{ij} \frac{(2\pi)^2 \delta(p_i \cdot k)\delta(-p_j \cdot k)}{2[k^2]} \right] |2\rangle$$



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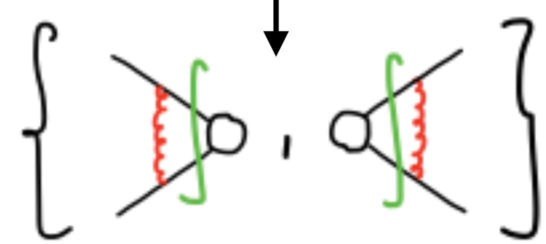

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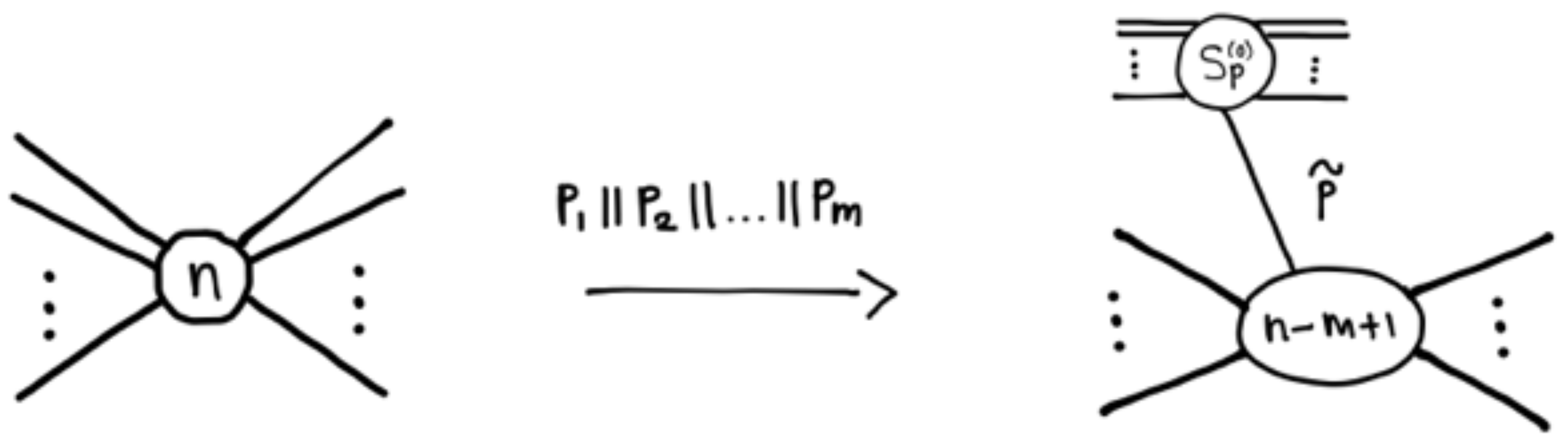
Tree-level collinear factorisation

For a general on-shell scattering:



$$|n\rangle \simeq S_P^{(0)}\,|n-m+1\rangle$$

($\tilde{P}$: Single parton $(1+\ldots+m)$)

Colour + Spin operator.
Depends only on collinear partons

$$S_P^{(0)}\left(\{C\},\tilde{P}\right)$$

Depends only on non-collinear partons

$$|n-m+1\left(\{NC,\tilde{P}\}\right)\rangle$$

Hence, universal (process-independent) contributions at cross section level.

Generalised factorisation beyond tree level

Catani, De Florian & Rodrigo JHEP 1207 (2012) 026

This collinear factorisation generalises to all orders

$$|n\rangle \simeq Sp\,|n-m+1\rangle$$

$$Sp = Sp^{(0)} + Sp^{(1)} + Sp^{(2)} + \dots$$

but

$$Sp^{(\ell)} = Sp^{(\ell)}(\tilde{P}, \{C\}, \{NC\})$$

Violation of strict (process-independent) factorisation!

One loop

The problem first seeds at this order, and is specific to hadron-hadron collisions:

$$S_P^{(1)} = S_{P_f}^{(1)}(\tilde{P}, \{C\}) + \widetilde{\hat{\Delta}}(\tilde{P}, \{C\}, \{NC\})\, S_P^{(0)}$$

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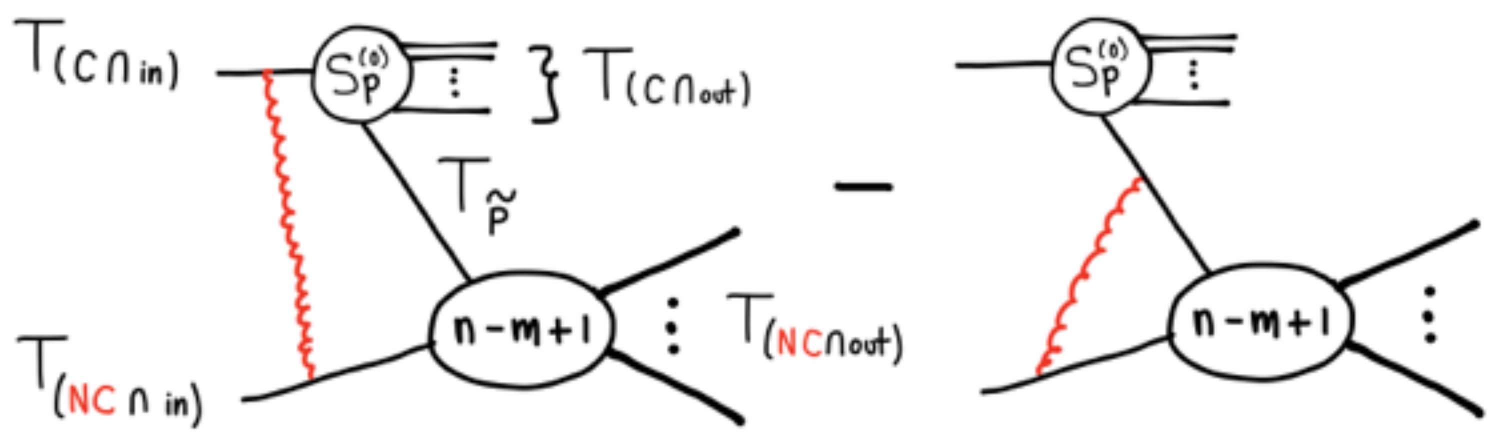
$$S_P^{(1)} = S_{P_f}^{(1)}(\tilde{P}, \{C\}) + \tilde{\Delta}(\tilde{P}, \{C\}, \{NC\})\, S_P^{(0)}$$

$$\tilde{\Delta}\, S_P^{(0)} |\cdots\rangle =$$



$$\tilde{\Delta} = \tilde{\Delta}(\{NC\}) \iff T_{(C\cap in)},\ T_{(C\cap out)},\ T_{(NC\cap in)},\ T_{(NC\cap out)} \neq 0$$

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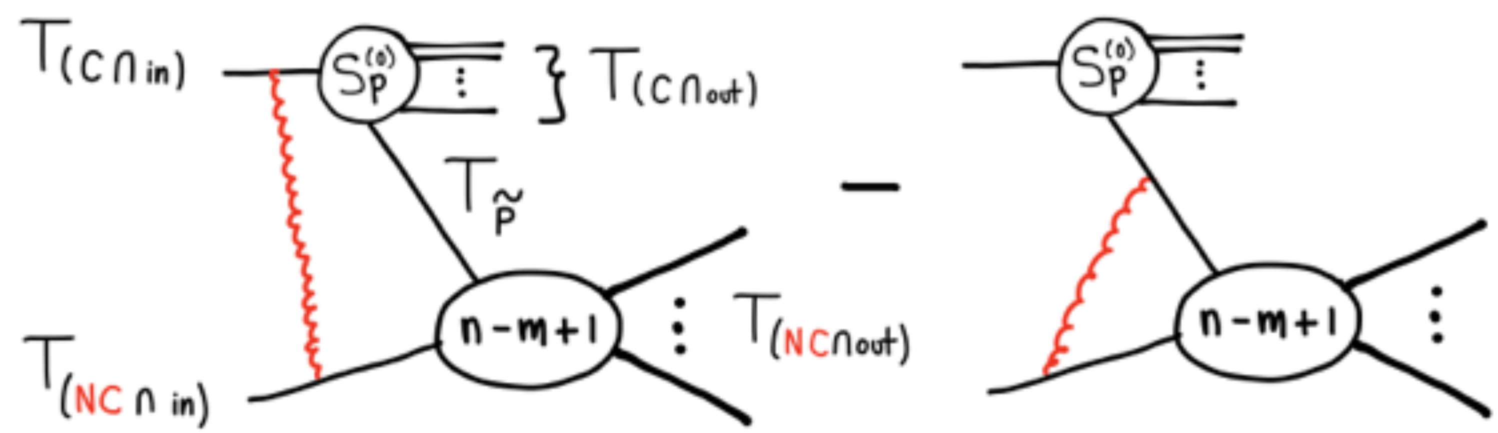
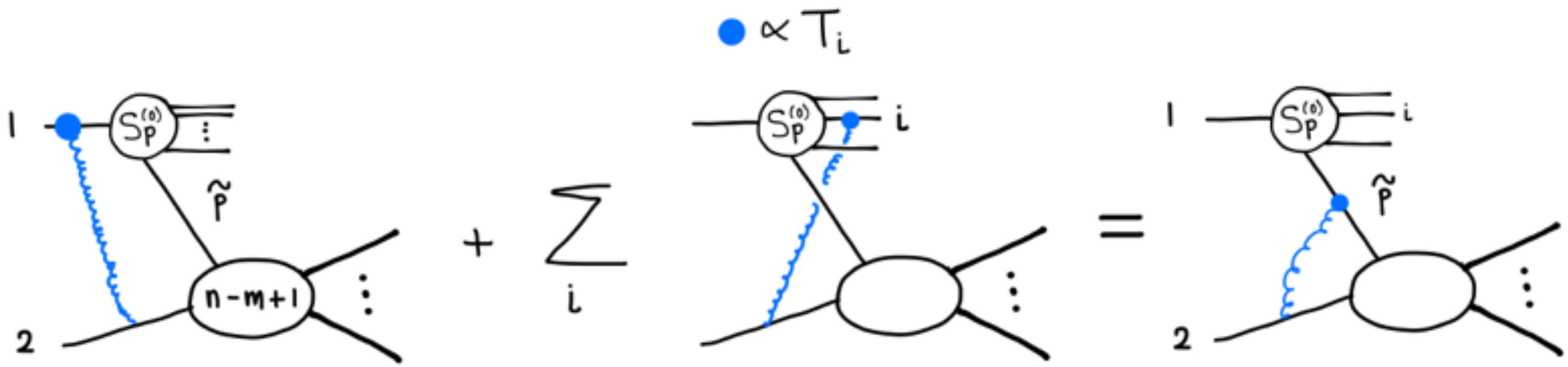


$$\tilde{\Delta} = \tilde{\Delta}(\{NC\}) \iff T_{(C\cap in)},\ T_{(C\cap out)},\ T_{(NC\cap in)},\ T_{(NC\cap out)} \neq 0$$

However, these violations cancel at cross section level:

$$\tilde{\Delta} + \tilde{\Delta}^{+} = 0$$

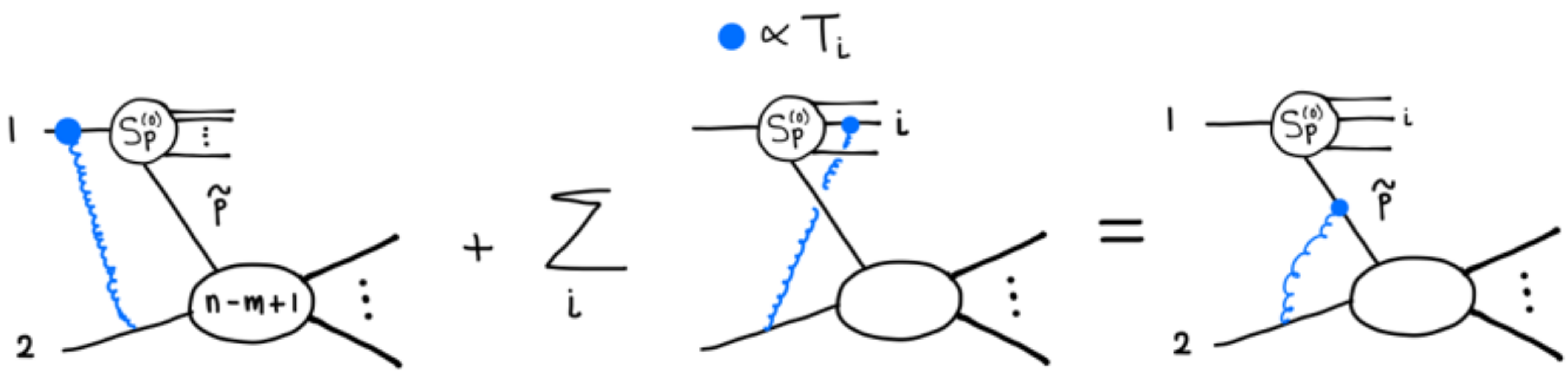
Coulomb gluons and colour coherence



$$T_2\cdot\left(T_1+\sum_i T_i\right) Sp^{(0)}\,|n-m+1\rangle = Sp^{(0)}\left[T_2\cdot T_{\tilde{p}}\,|n-m+1\rangle\right]$$

Effectively, correction to

Coulomb gluons and colour coherence



$$T_2\cdot\left(T_1+\sum_i T_i\right) Sp^{(0)}\,|n\text{-}m\text{+}1\rangle = Sp^{(0)}\left[T_2\cdot T_{\tilde{p}}\,|n\text{-}m\text{+}1\rangle\right]$$

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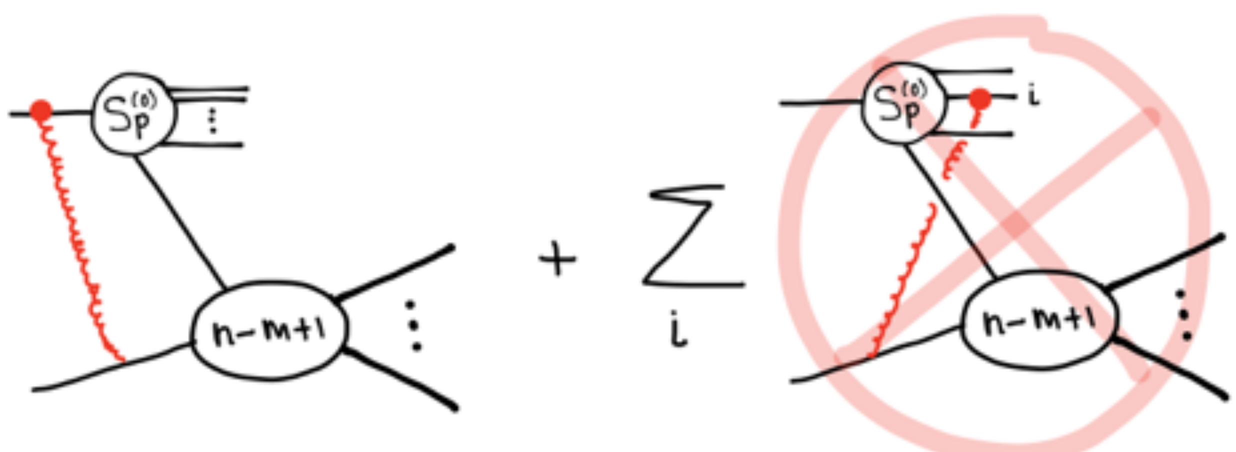
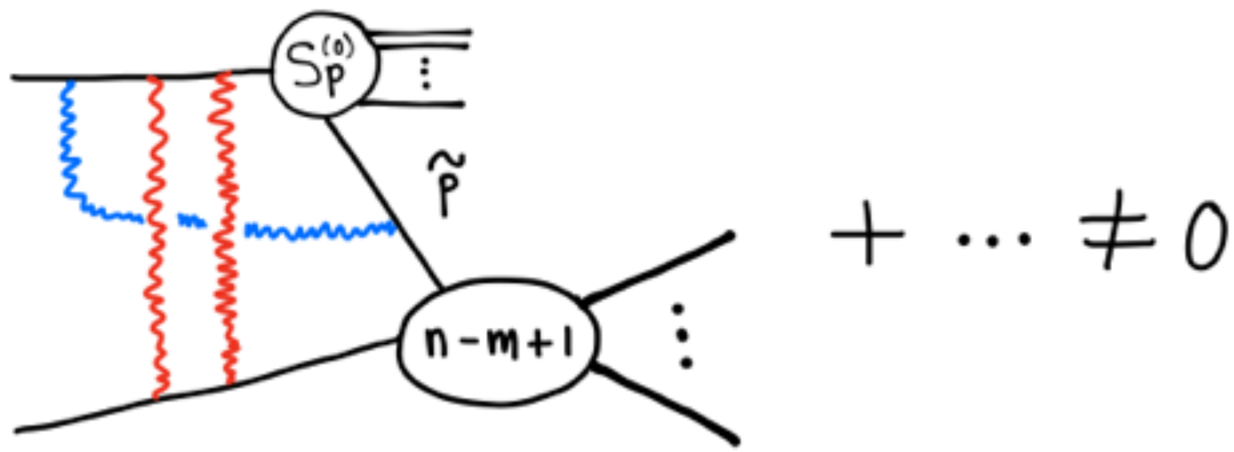


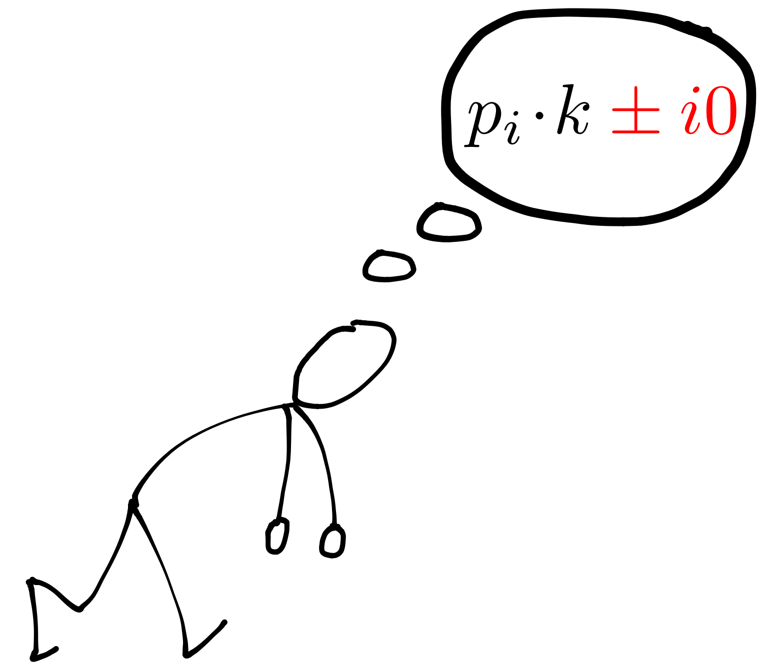
$$= T_1\cdot T_2\, Sp^{(0)}\,|n\text{-}m\text{+}1\rangle$$

Unavoidable colour correlation

Three loops (no escape)

Forshaw, Seymour, Siodmok JHEP 1211 (2012) 066
Catani, De Florian & Rodrigo JHEP 1207 (2012) 026

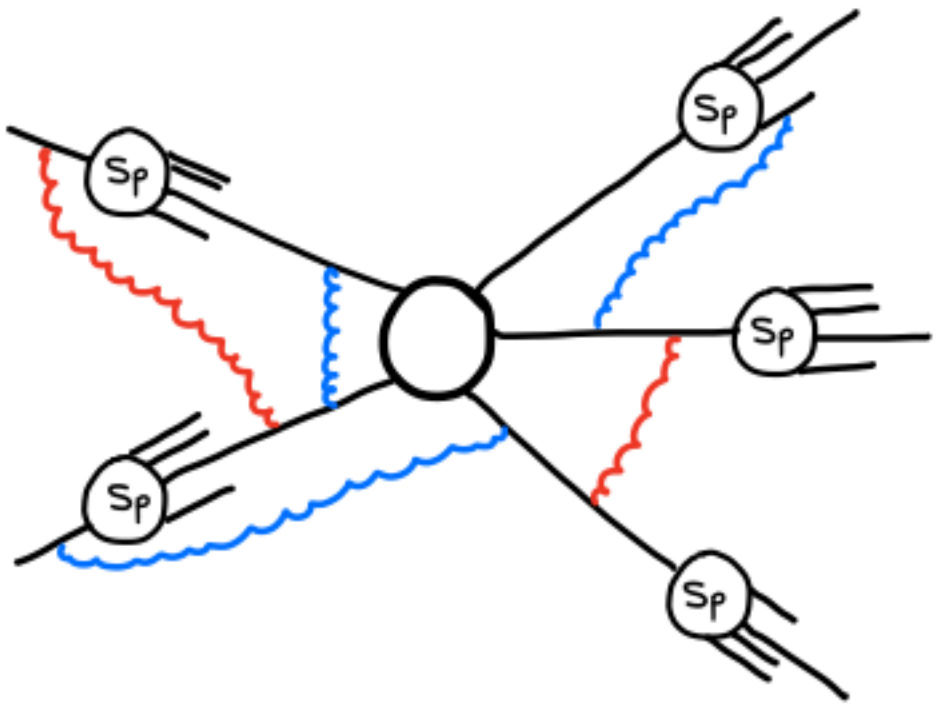




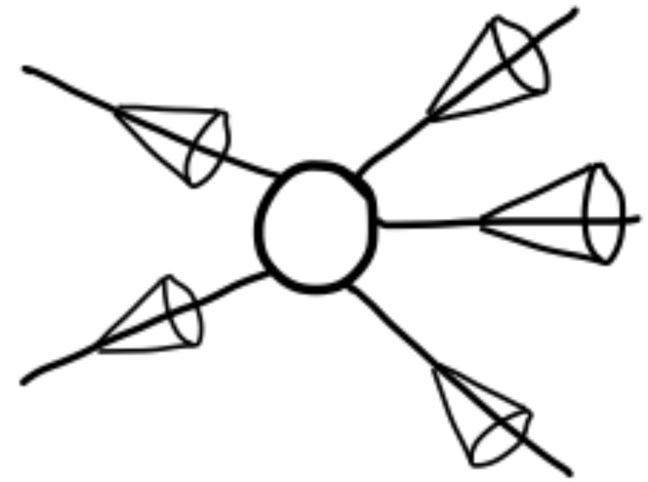
- Violation of strict factorisation no longer vanish at squared amplitude level (N^4LO= Hard process + 1 collinear + 3 loops).

Coulomb gluons and factorisation

Conclusion: coherence allows one to “unhook” on-shell gluons and recover collinear factorisation. But it fails for Coulomb gluons:



instead of

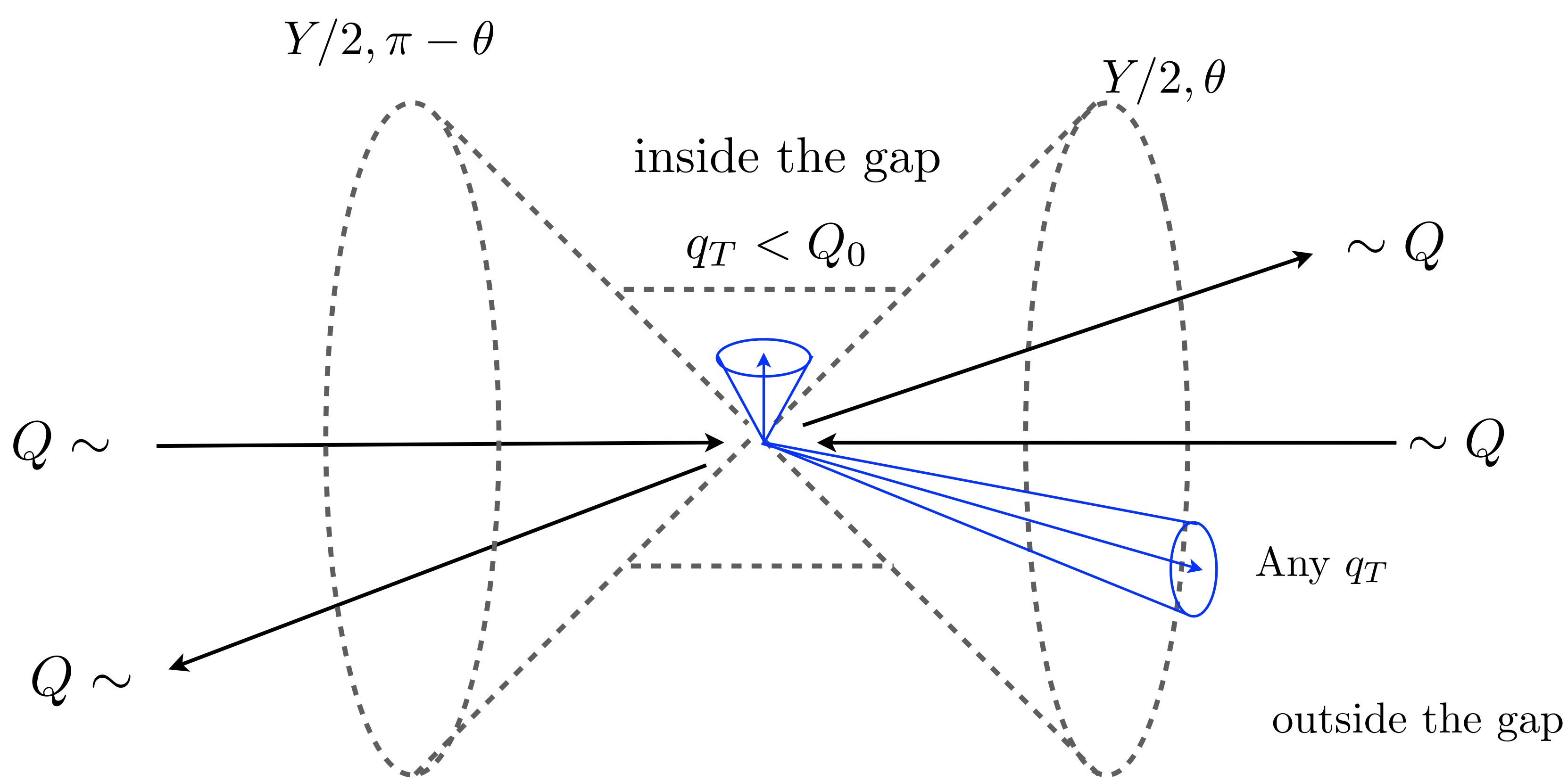


Can we make sense of this nested structure?

Worth remarking: the violations of collinear factorisation discussed here are of soft origin & should cancel for sufficiently inclusive observables!

Concrete case: 'gaps between jets'

(Forshaw, Kyrieleis & Seymour hep /0604094 ; /0808.1269)



Soft corrections

$$Q_0 \ll q^\mu \ll Q$$

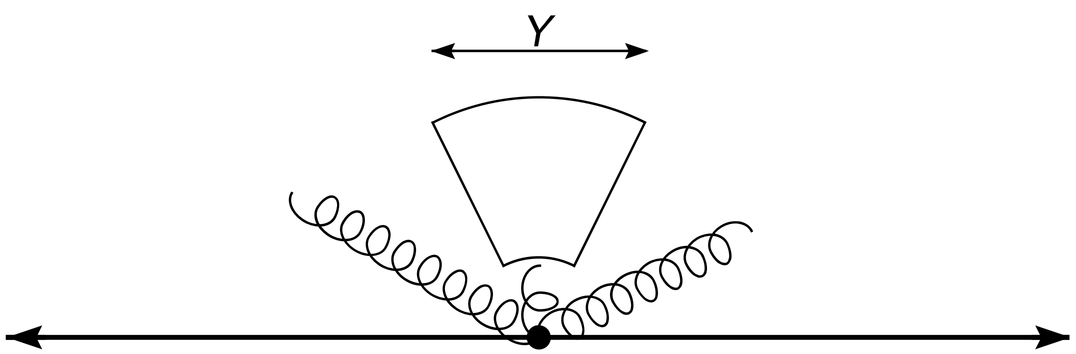
$$d\sigma_m = |\mathcal{M}(q_1, ...q_m)|^2 \; \phi \; dPS$$

Large logs

$$\ln^n \frac{Q^2}{Q_0^2} \gg 1$$

Structure of logarithmic corrections

Radiation outside the gap cancels?



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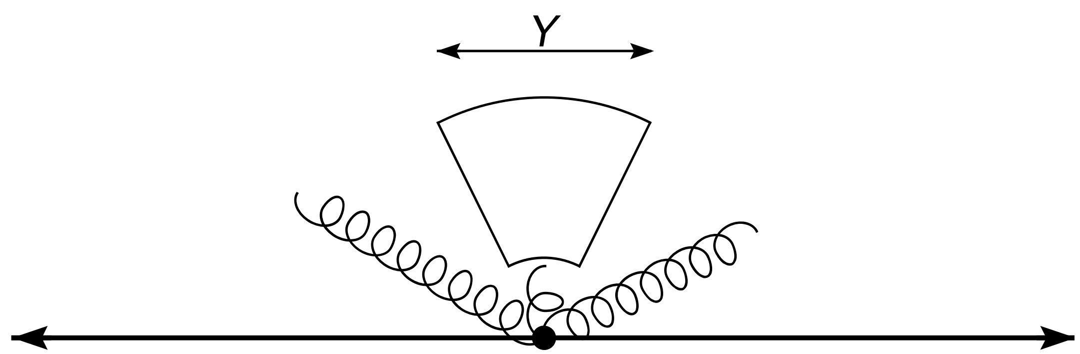
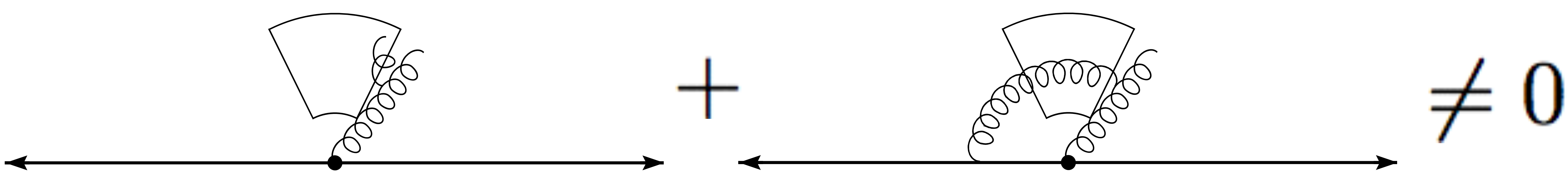


It does, but what about radiation from outside?



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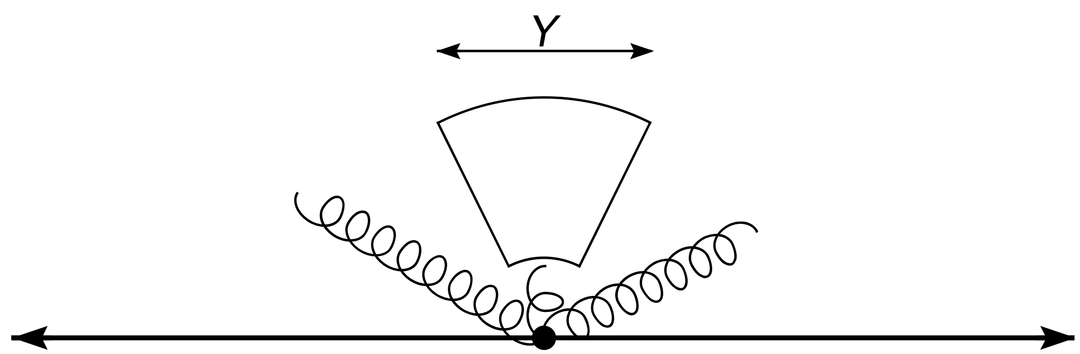

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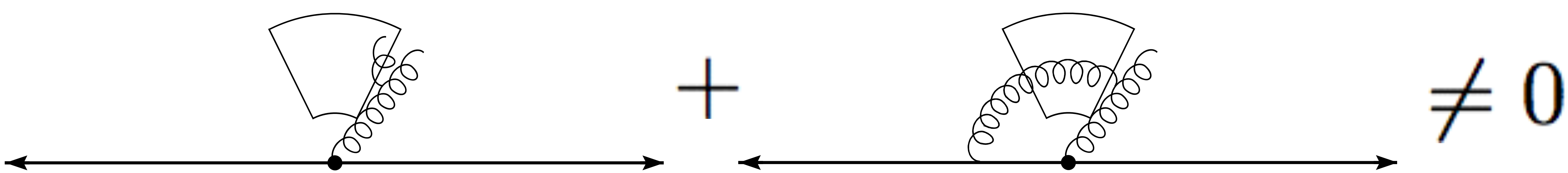


Additional, also leading, non-global logarithms

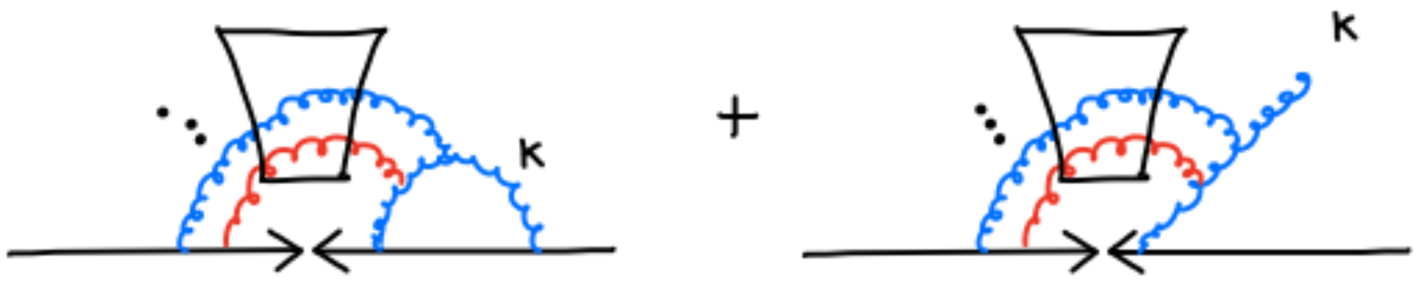
- M.Dasgupta & G.P.Salam, JHEP 0203:017,2002

$$\sigma(Q, Q_0; y) \sim \sigma_0(Q) \left(1 + \ldots f_n(Y) \left(\alpha_s \ln \frac{Q}{Q_0}\right)^n + \ldots\right)$$

A surprise: super-leading logs

(Forshaw, Kyrieleis, Seymour JHEP 0608 (2006) 059 & 0809 (2008) 128)

- Included Coulomb gluons & colour interference. Claim: The leading logs (in Q/Q_0) due to one gluon outside of the gap is:



$$\sigma_{1-\mathrm{out}} = \frac{-2\alpha_s}{\pi} \int_{Q_0}^{Q} \frac{\mathrm{d}k_T}{k_T} \int_{\mathrm{out}} \frac{\mathrm{d}y\mathrm{d}\phi}{2\pi} \left[\Omega_V + \Omega_R\right]$$

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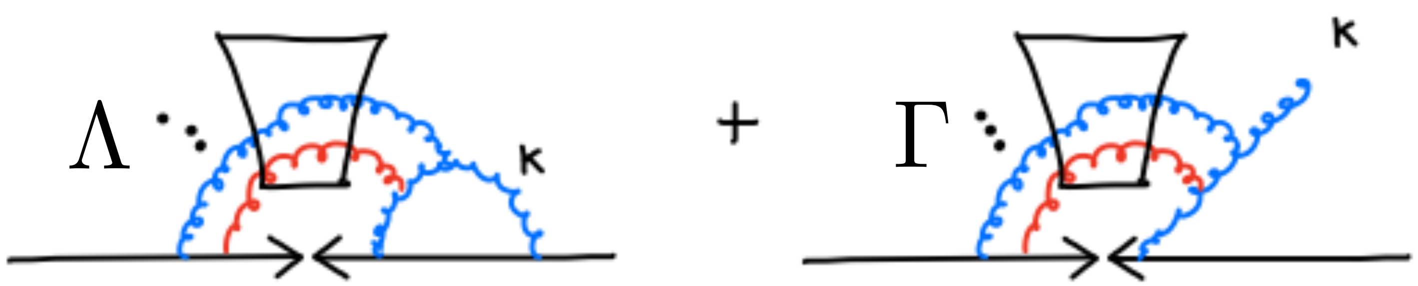

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Key assumption: Real and virtual radiation ordered (evolved) in "kT"

$$\Omega_R = \left| e^{-\frac{2\alpha_s}{\pi}\int_{Q_0}^{k_T} \frac{\mathrm{d}k''_T}{k''_T}\Gamma} \mathbf{D}(k) e^{-\frac{2\alpha_s}{\pi}\int_{k_T}^{Q} \frac{\mathrm{d}k'_T}{k'_T}\Lambda} |4\rangle \right|^2$$

BN: cancellation below inclusivity scale Q_0

$$\left| e^{-\frac{2\alpha_s}{\pi}\int_{Q_0}^{Q} \frac{\mathrm{d}k'_T}{k'_T}\Lambda} |4\rangle \right|^2 \propto 0\text{-out} + \Omega_V + 2\text{-out} + \ldots$$

Exponentiation of one-loop corrections: on-shell inside of the gap + Coulomb part

A surprise: super-leading logs

- Gluon (k) integrated everywhere outside gap:
  - Final-state collinear singularity → cancel
  - Initial-state singularity? NO → Super-leading logarithm

$$\sigma_{1,\mathrm{out=hardest}} = \left(\frac{2\alpha_s}{\pi}\right)^4 \int_{Q_0}^{Q} \frac{dk_T}{k_T} \left(2\ln\frac{Q}{k_T}\right) \left(\int_{Q_0}^{k_T} \frac{dk'_T}{k'_T}\right)^3 \frac{Y\pi^2}{3}$$

$$\left\langle m_0 \middle| t_1^a \left[[t_1\cdot t_4, t_1\cdot t_2], t_1\cdot t_2 \right] - t_1^{a\dagger} \left[[T_1\cdot T_4, T_1\cdot T_2], T_1\cdot T_2 \right] t_1^a \middle| m_0 \right\rangle.$$

$$\sim \sigma_0 (\frac{2\alpha_s}{\pi})^4 \pi^2 Y \ln^5 \frac{Q}{Q_0} + \mathcal{O}\left(\ln^4 \frac{Q}{Q_0}\right)$$

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JHEP 1211 (2012) 066

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$$\sim \sigma_0 (\frac{2\alpha_s}{\pi})^4 \pi^2 Y \ln^5 \frac{Q}{Q_0} + \mathcal{O}\left(\ln^4 \frac{Q}{Q_0}\right)$$

→And it is believed that $\sim \alpha_s^n \ln^{2n-3} \frac{Q}{Q_0} \quad n \geq 4$

(Keates & Seymour
JHEP 0904 (2009) 040)

Summary/ So..is factorisation violated in GBJ?

- Violations of strict collinear factorisation are of soft origin. Then, a Bloch-Nordsieck mechanism should cancel the singularities below Q_0, the scale of inclusivity (Forshaw, Seymour, Siodmok JHEP 1211 (2012) 066).

Summary/ So..is factorisation violated in GBJ?

- Violations of strict collinear factorisation are of soft origin. Then, a Bloch-Nordsieck mechanism should cancel the singularities below Q_0, the scale of inclusivity (Forshaw, Seymour, Siodmok JHEP 1211 (2012) 066).

- Hence, for 'gaps between jets' (an insufficiently inclusive observable) first factorise at $\mu \sim Q_0$, and then evolve up to Q.

$$\sigma_{ab}(p_a, p_b) = \int_0^1 \mathrm{d}\eta_1\, \mathrm{d}\eta_2\, f_{i/a}(\eta_1, \mu^2)\, f_{j/b}(\eta_2, \mu^2)\, \sigma_{ij}(\eta_1 p_a, \eta_2 p_b, \mu^2) \; + \mathcal{O}\left(\frac{1}{\mu^2}\right)$$

LO: OK

NLO: OK

N2LO: OK (Hermitian)

$\tilde{\Delta} + \tilde{\Delta}^{+} = 0$

N3LO: OK (Trace=0)

N4LO: Factorisation violated! → Super-leading logs.

All orders proof is required!

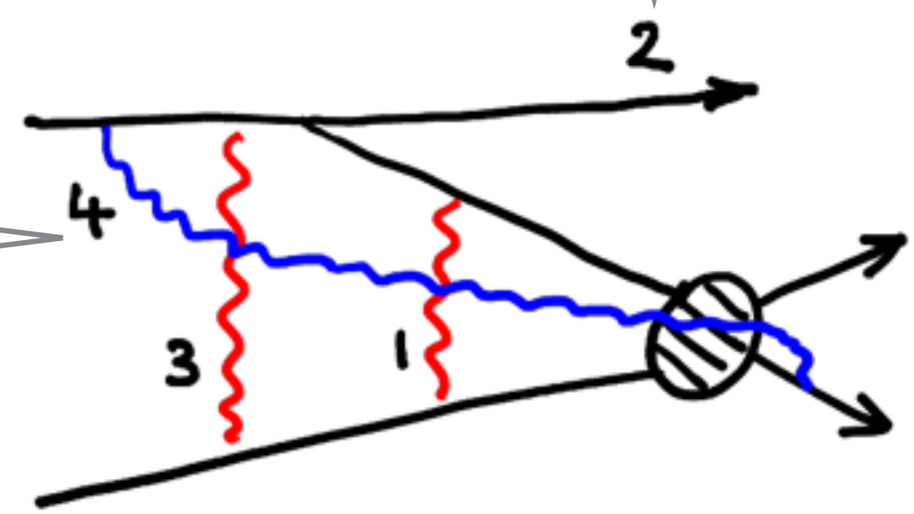
The ordering variable problem

Banfi, Salam, Zanderighi arXiv:1001.4082

– Under similar assumptions, found super-leading logs in wide class of event shapes but

Collinear & out of the gap

Eikonal & central
$k_{4T} \sim E_4$



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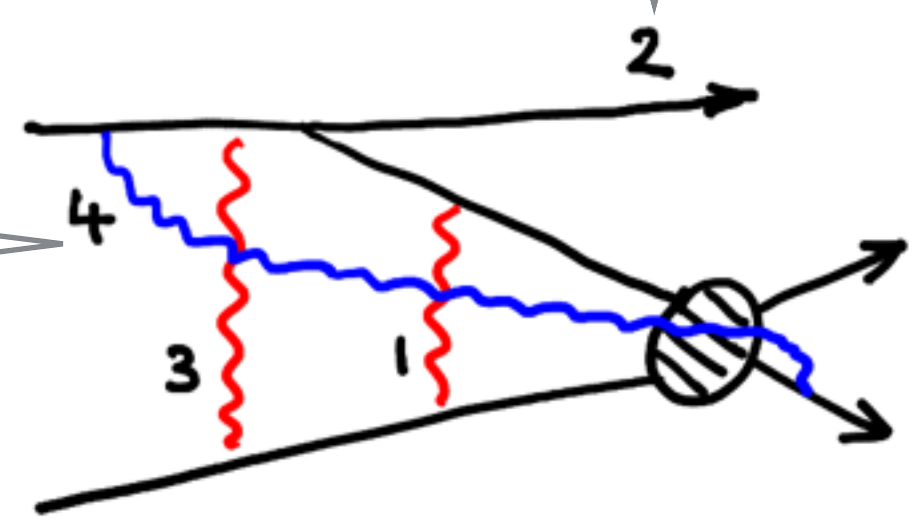

$\underline{k_t \text{ ordering}}$: $k_{t1} \gg k_{t2} \gg k_{t3}$

$$\alpha_s^4 \int_{Q_0}^{Q} \frac{\mathrm{d}k_{t1}}{k_{t1}} \int_{Q_0}^{k_{t1}} \frac{\mathrm{d}k_{t2}}{k_{t2}} \int_{k_{t2}/Q}^{1} \frac{\mathrm{d}\theta_2}{\theta_2} \int_{Q_0}^{k_{t2}} \frac{\mathrm{d}k_{t3}}{k_{t3}} \int_{Q_0}^{k_{t3}} \frac{\mathrm{d}k_{t4}}{k_{t4}} = 2\,\frac{\alpha_s^4}{5!} L^5$$

$\underline{\text{virtuality ordering}}$: $k_{t1} \gg t_2 = k_{t2}\theta_2 \gg k_{t3}$

$$\alpha_s^4 \int_{Q_0}^{Q} \frac{\mathrm{d}k_{t1}}{k_{t1}} \int_{Q_0}^{k_{t1}} \frac{\mathrm{d}t_2}{t_2} \int_{\sqrt{t_2/Q}}^{1} \frac{\mathrm{d}\theta_2}{\theta_2} \int_{Q_0}^{t_2} \frac{\mathrm{d}k_{t3}}{k_{t3}} \int_{Q_0}^{k_{t3}} \frac{\mathrm{d}k_{t4}}{k_{t4}} = \frac{\alpha_s^4}{5!} L^5$$

Choice of ordering variable is crucial!

$\underline{\text{energy ordering}}$: $k_{t1} \gg E_2 = k_{t2}/\theta_2 \gg k_{t3}$

$$\alpha_s^4 \int_{Q_0}^{Q} \frac{\mathrm{d}k_{t1}}{k_{t1}} \int_{Q_0}^{k_{t1}} \frac{\mathrm{d}E_2}{E_2} \int_{0}^{1} \frac{\mathrm{d}\theta_2}{\theta_2} \int_{Q_0}^{E_2} \frac{\mathrm{d}k_{t3}}{k_{t3}} \int_{Q_0}^{k_{t3}} \frac{\mathrm{d}k_{t4}}{k_{t4}} = \infty$$

Coulomb gluons and colour evolution

Coulomb gluons are related to the violations of strict collinear factorisation and super-leading logs but

- Which is the correct evolution variable?
- Can we derive it from first principles?

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- Which is the correct evolution variable?
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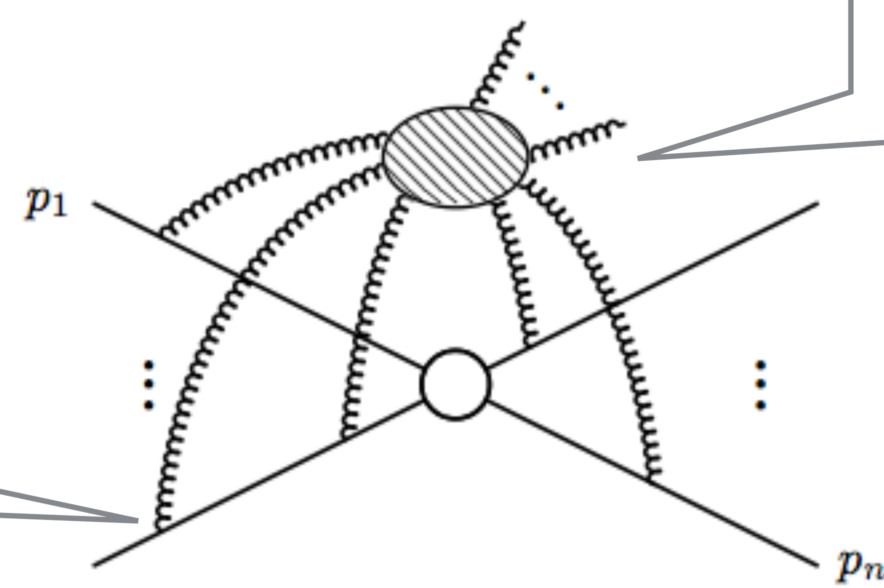
One-loop diagrammatic calculation assuming only that all gluon are soft, but not relative softness (JHEP 1512 (2015) 091 & 1602.00623)

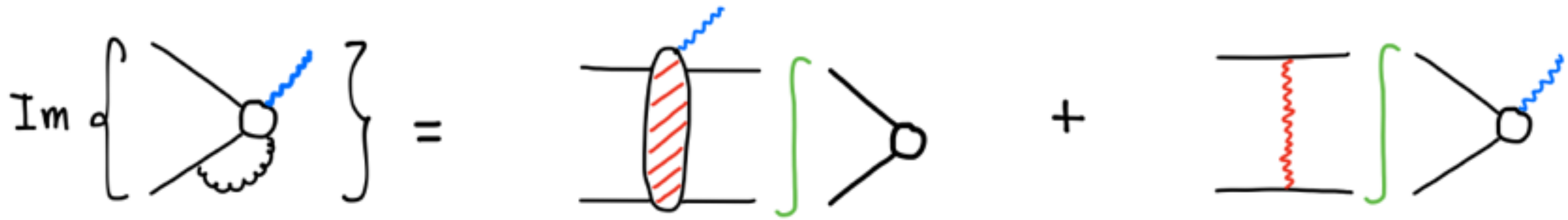
Exact rules for 3 & 4 -gluon vertices & ghost



Eikonal approximation

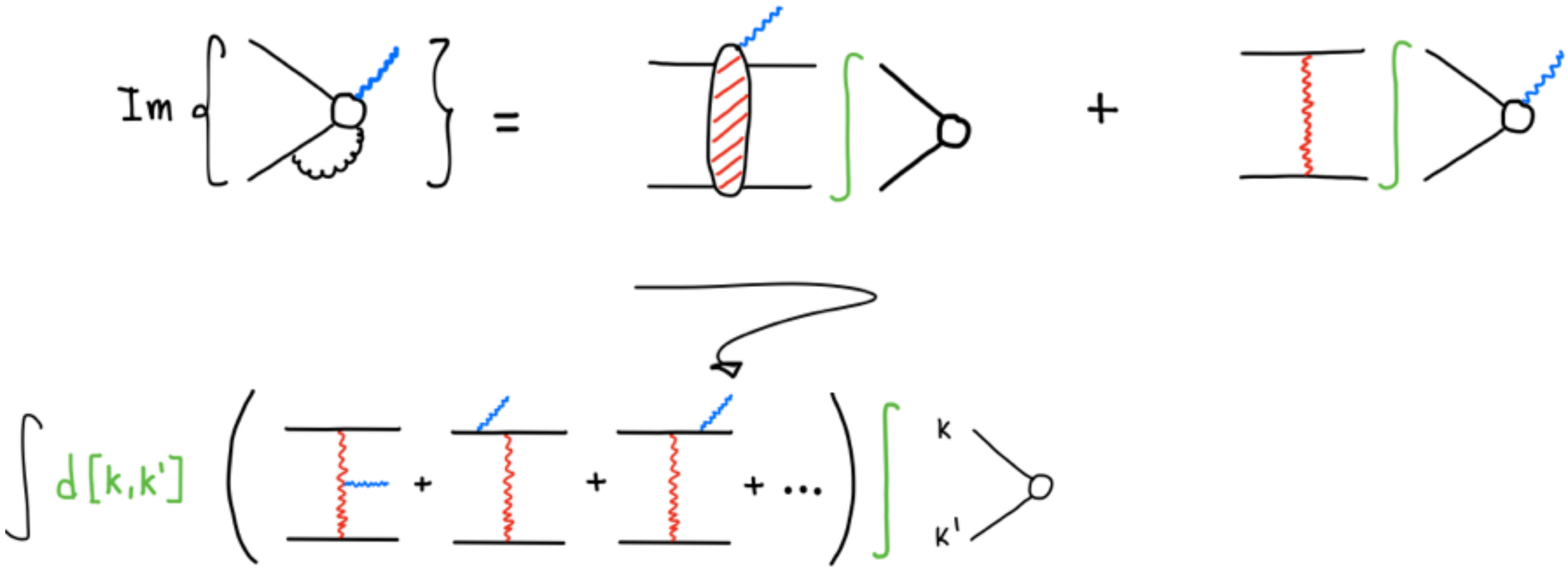
Ordering in the simplest case (DY): cutting rules

- Gauge invariantly: Imaginary (Coulomb) part of loop integrals can be written as



Ordering in the simplest case (DY): cutting rules

- Gauge invariantly: Imaginary (Coulomb) part of loop integrals can be written as



On-shell

$2 \to 3$

On-shell

$2 \to 0$

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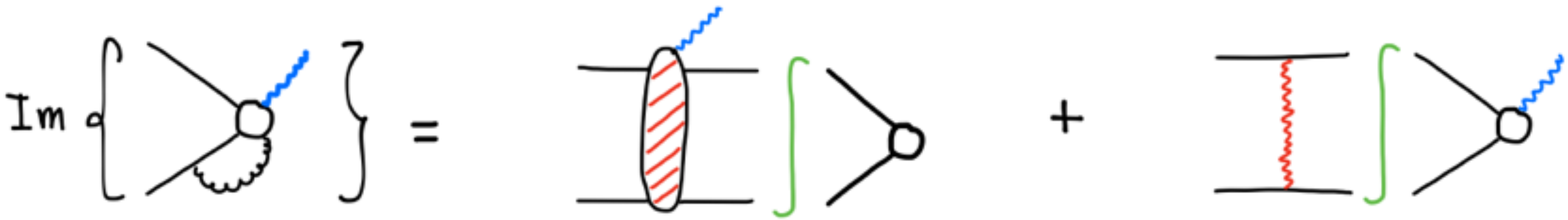
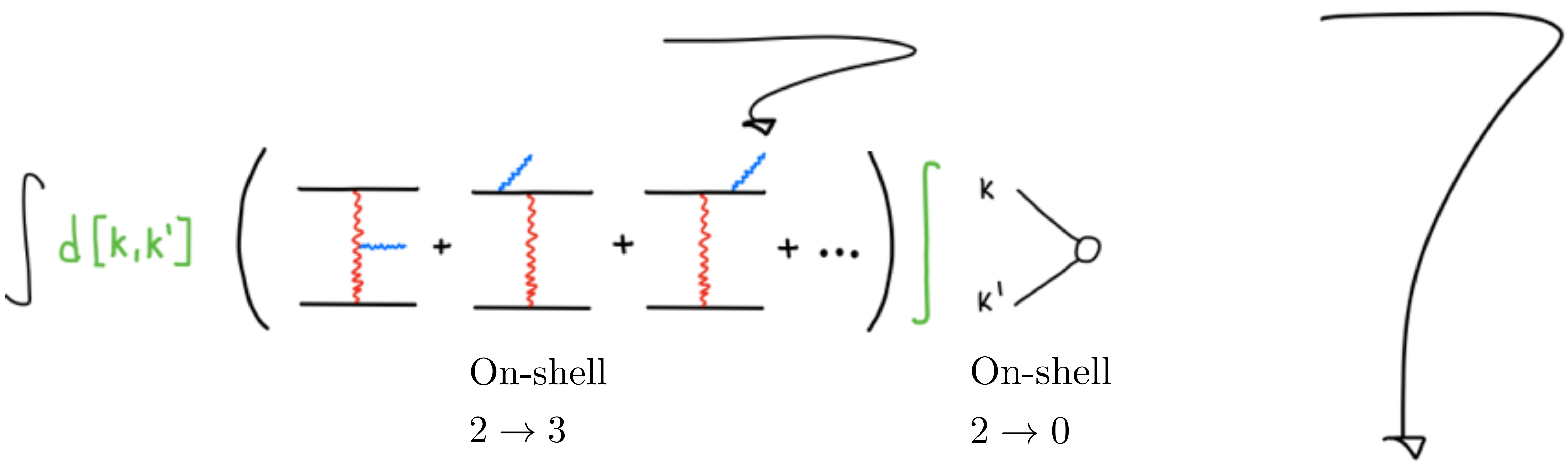
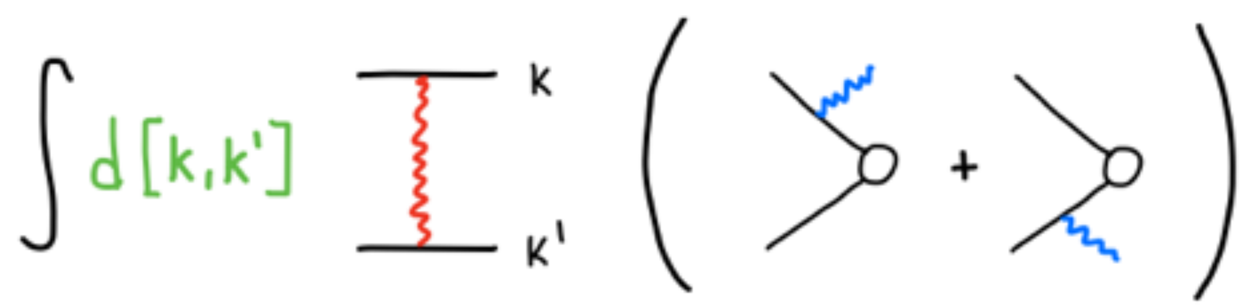




On-shell $2 \to 3$

On-shell $2 \to 0$

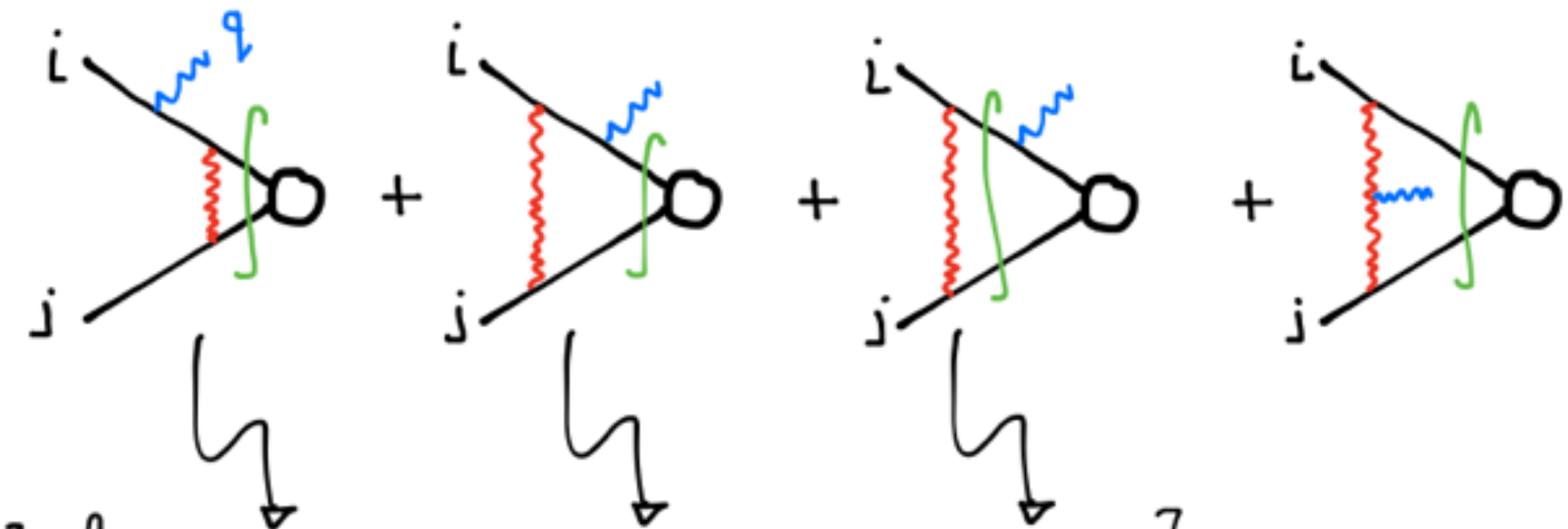
Coulomb contributions:
Product of on-shell scattering amplitudes.



On-shell $2 \to 2$

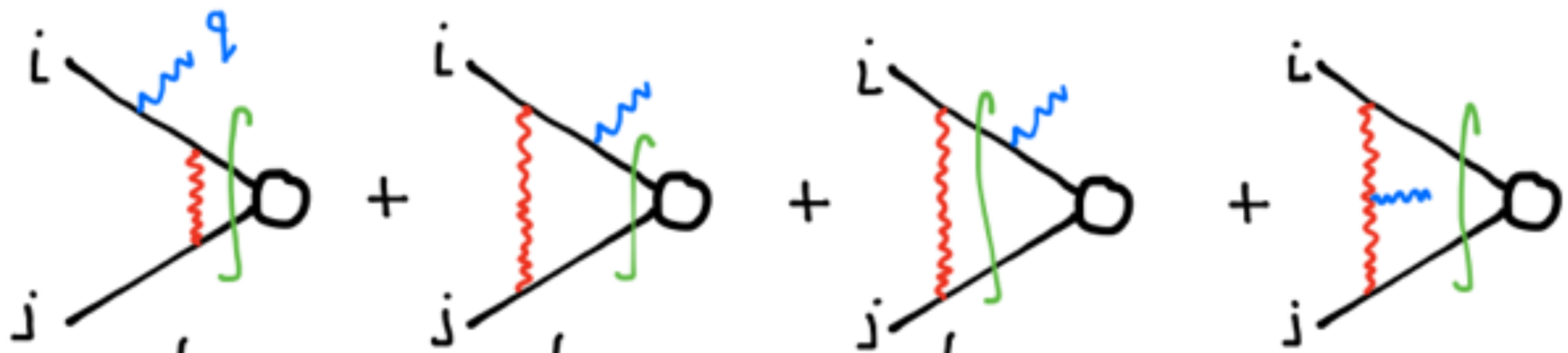
On-shell $2 \to 1$

Ordering in the simplest case (DY)



$$i\pi \frac{p_i\cdot \varepsilon}{p_i\cdot q}\int_0^{Q^2}\frac{dk_T^2}{k_T^2}\left\{ T_i(T_i\cdot T_j) - T_i\cdot T_j\, T_i + T_i\cdot T_j\, T_i \right\}|2\rangle$$

Ordering in the simplest case (DY)



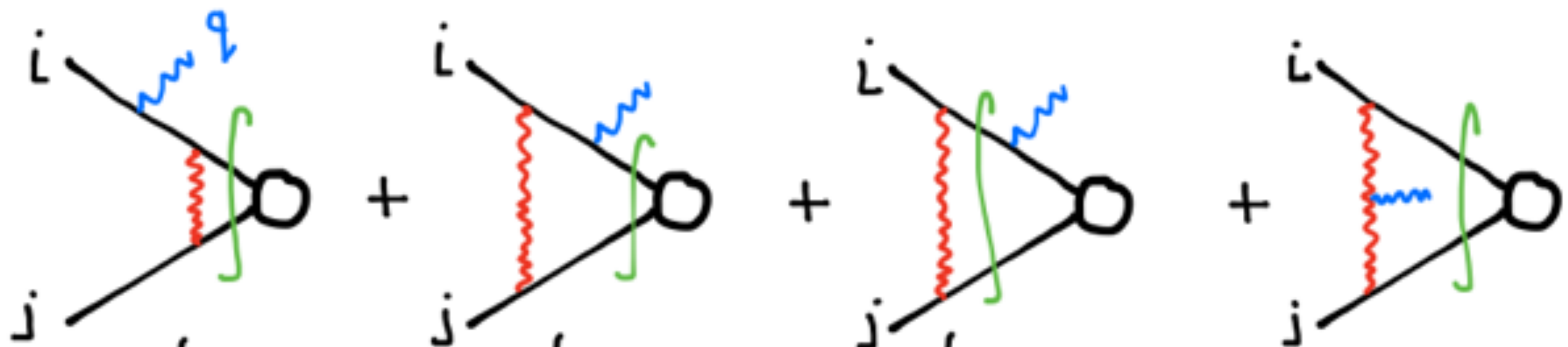
$$ i\pi \frac{p_i\cdot\epsilon}{p_i\cdot q}\int_0^{Q^2}\frac{dk_T^2}{k_T^2}\left\{ T_i(T_i\cdot T_j) - T_i\cdot T_j\; T_i + T_i\cdot T_j\; T_i \right\}|2\rangle $$

Decomposition in colour and spin

$$ -i\pi\frac{p_i\cdot\epsilon}{p_i\cdot q}\underbrace{\int\frac{dk_T^2}{k_T^2}\frac{q_T^2}{k_T^2+q_T^2}}_{\simeq\int_0^{q_T^2}\frac{dk_T^2}{k_T^2}}\left[T_i\cdot T_j\; T_i - T_i\; T_i\cdot T_j\right]|2\rangle $$

$$ +(i\leftrightarrow j) $$

Ordering in the simplest case (DY)



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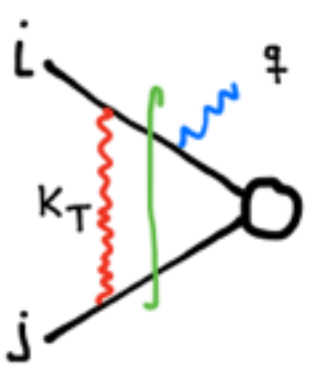
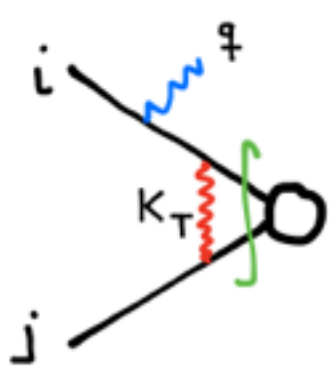
$\simeq \displaystyle\int_0^{q_T^2}\frac{dk_T^2}{k_T^2}$



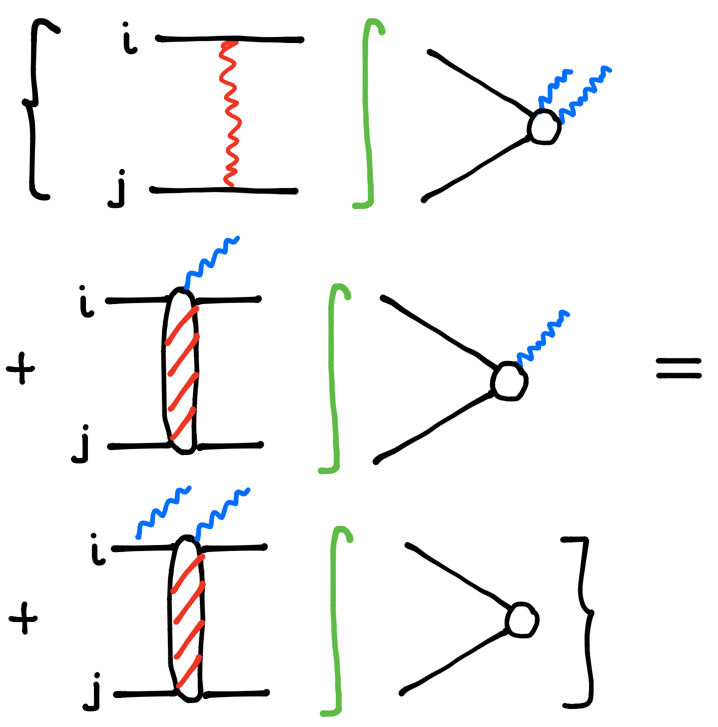
$+ \displaystyle\int_{q_T^2}^{Q^2}\frac{dk_T^2}{k_T^2}$



i.e. non-trivial test of k_T ordering!

Ordering in two emission case

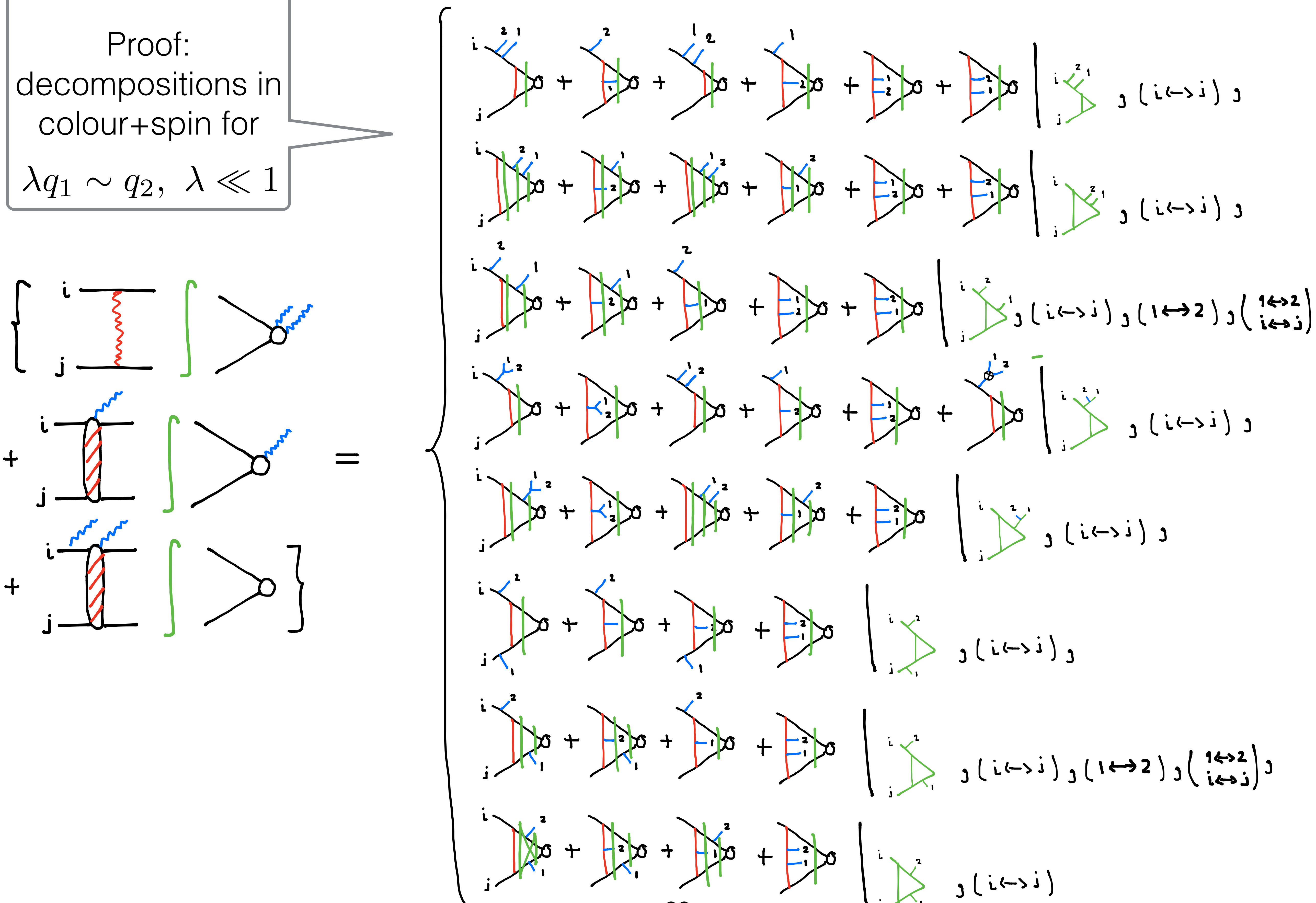
(JHEP 1512 (2015) 091)



Ordering in two emission case

(JHEP 1512 (2015) 091)

Proof: decompositions in colour+spin for $\lambda q_1 \sim q_2,\ \lambda \ll 1$



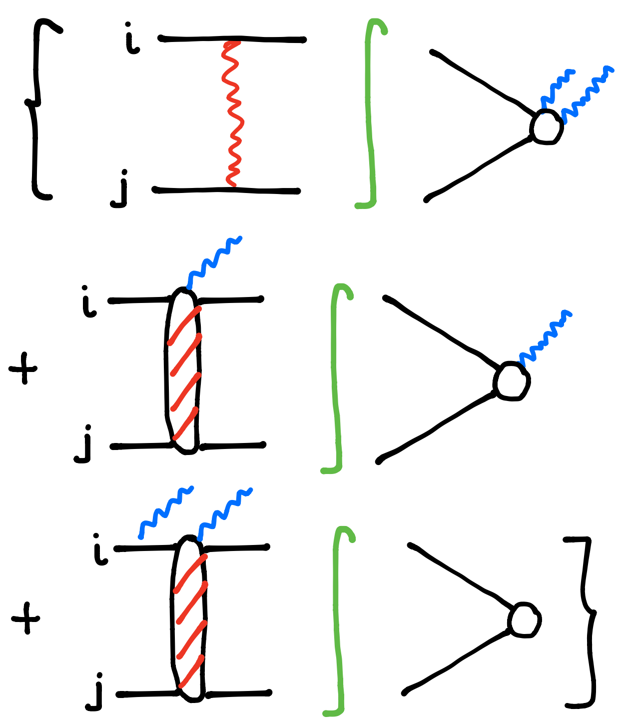
Ordering in two emission case

(JHEP 1512 (2015) 091)

Proof: decompositions in colour+spin for

$\lambda q_1 \sim q_2, \;\; \lambda \ll 1$

This is the ordering assumption used in the calculation of the super-logs in `gaps between jets'



$$= \Big[\mathbf{C}(0, q_2^{(ij)})\, \mathbf{J}^{(0)}(q_2)\, \mathbf{J}^{(0)}(q_1) + \mathbf{J}^{(0)}(q_2)\, \mathbf{C}(q_2^{(ij)}, q_1^{(ij)})\, \mathbf{J}^{(0)}(q_1) + \mathbf{J}^{(0)}(q_2)\, \mathbf{J}^{(0)}(q_1)\, \mathbf{C}\left(q_1^{(ij)}, \sqrt{2p_i \cdot p_j}\right) \Big] |2^{(0)}\rangle,$$

$$\mathbf{C}(a,b) = -i\pi \frac{\alpha}{2\pi} \mathbf{T}_i \cdot \mathbf{T}_j \int_{a^2}^{b^2} \frac{d(k^{(ij)})^2}{(k^{(ij)})^2}.$$

Coulomb gluons and colour evolution

Our fixed order calculations suggest that the one-loop amplitude of a general hard scattering with N soft-gluon emissions (ordered in softness $q_i\lambda \sim q_{i+1}$) is

$$\left|n_N^{(1)}\right\rangle = \sum_{m=0}^{N}\sum_{i=2}^{p}\sum_{j=1}^{i-1} \mathbf{J}^{(0)}(q_N)\cdots\mathbf{J}^{(0)}(q_{m+1})\,\mathbf{I}_{ij}(q_{m+1}^{(ij)},q_m^{(ij)})\mathbf{J}^{(0)}(q_m)\cdots\mathbf{J}^{(0)}(q_1)\,|n_0^{(0)}\rangle$$
$$+\sum_{m=1}^{N}\sum_{j=1}^{n+m-1}\sum_{k=1}^{n+m-1} \mathbf{J}^{(0)}(q_N)\cdots\mathbf{J}^{(0)}(q_{m+1})\,\mathbf{I}_{n+m,j}(q_{m+1}^{(ij)},q_m^{(jk)})\,\mathbf{d}_{jk}(q_m)\mathbf{J}^{(0)}(q_{m-1})\cdots\mathbf{J}^{(0)}(q_1)|n_0^{(0)}\rangle,$$

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\begin{aligned}
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&+\sum_{m=1}^{N}\sum_{j=1}^{n+m-1}\sum_{k=1}^{n+m-1} \mathbf{J}^{(0)}(q_N)\cdots\mathbf{J}^{(0)}(q_{m+1})\,\mathbf{I}_{n+m,j}(q_{m+1}^{(ij)},q_m^{(jk)})\,\mathbf{d}_{jk}(q_m)\mathbf{J}^{(0)}(q_{m-1})\cdots\mathbf{J}^{(0)}(q_1)|n_0^{(0)}\rangle,
\end{aligned}
$$

where the virtual insertion operator:

$$
\mathbf{I}_{ij}(a,c) = \mathbf{I}_{ij}(a,b) + \mathbf{I}_{ij}(b,c), \qquad \mathbf{I}_{ij}(0,b) = \frac{\alpha_s}{2\pi}\frac{c_\Gamma}{\epsilon^2}\,\mathbf{T}_i\cdot\mathbf{T}_j\left[\left(\frac{b^2}{4\pi\mu^2}\right)^{-\epsilon}\left(1+i\pi\epsilon\,\tilde{\delta}_{ij}-\epsilon\ln\frac{2p_i\cdot p_j}{b^2}\right)\right],
$$

describes the non-emission evolution of partons i and j from b to a.

$$
q^\mu = \alpha p_i^\mu + \beta p_j^\mu + (q_T^{(ij)})^\mu
$$

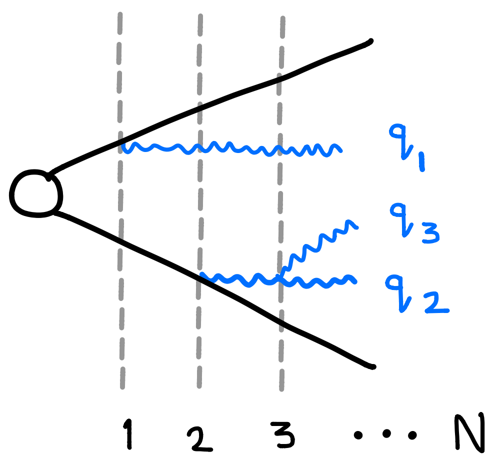
Key point: Ordering variable: dipole kT

$$
\sum_j \mathbf{d}_{ij}(q) = \sum_j \mathbf{T}_j\frac{p_j\cdot\varepsilon}{p_j\cdot q} = \mathbf{J}^{(0)}(q)
$$

- Gauge invariant.
- Correct IR poles
- Interpretation: ordered evolution!
- …

Diagrammatics of dipole kT evolution

1.- Successively insert emissions on external legs



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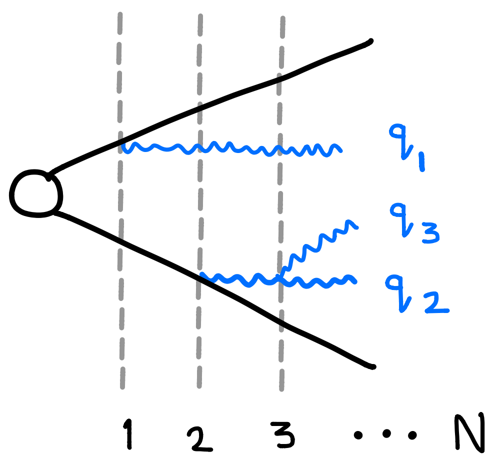
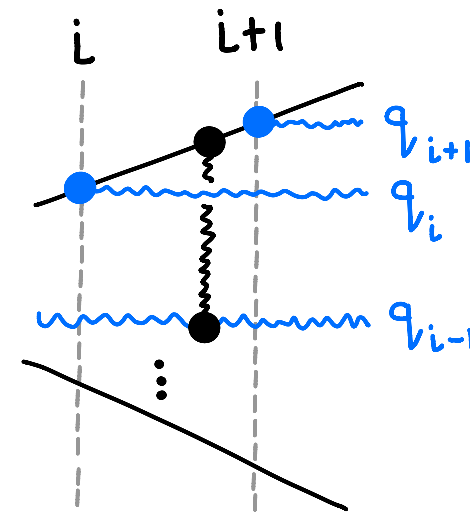
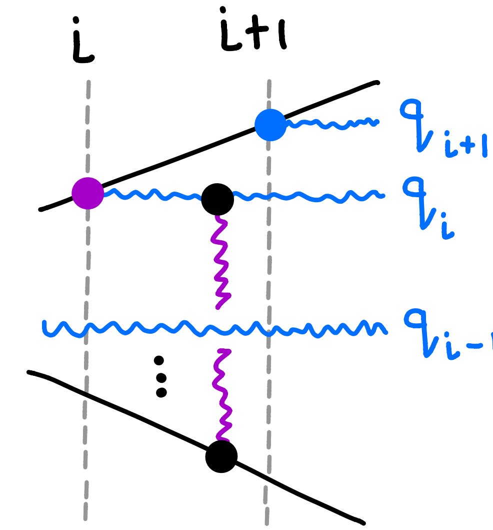


2.- Insert virtual exchanges

Case a)



Case b)



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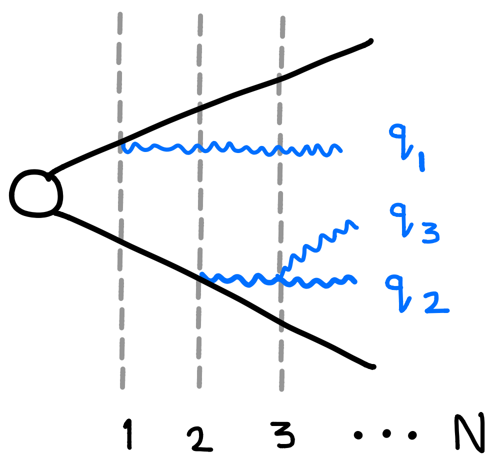

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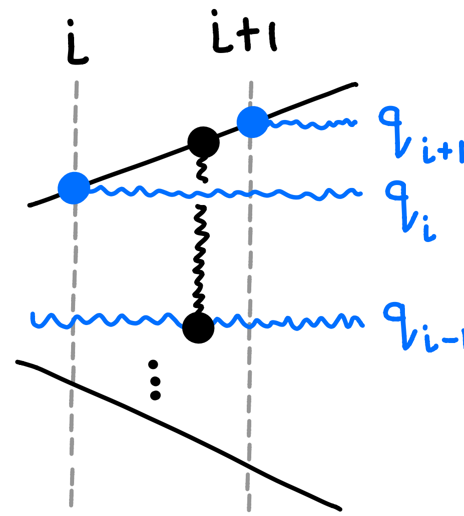

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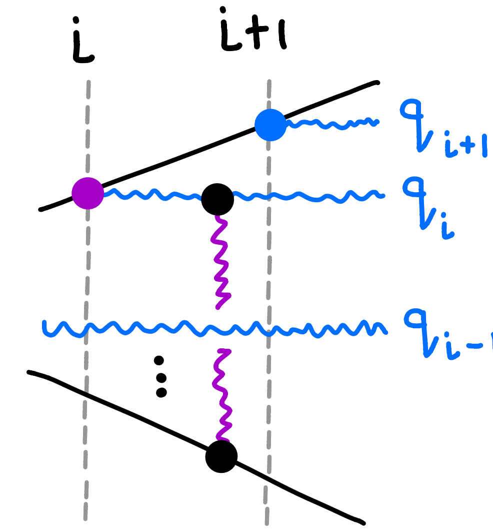
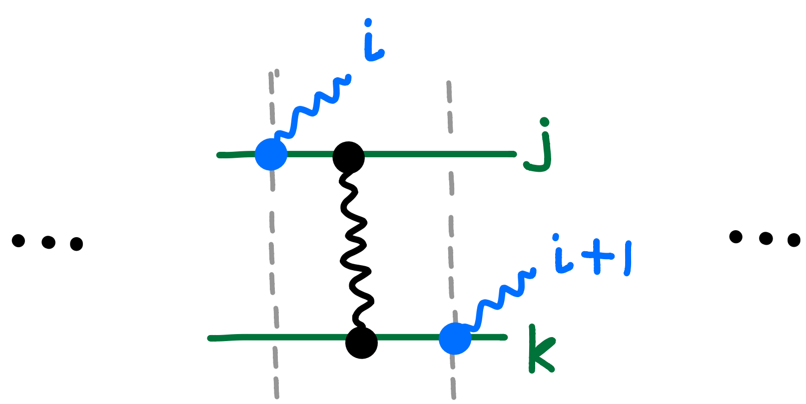
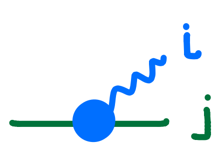


3.- and apply effective rules:

a)





$$= T_j \frac{p_j \cdot \varepsilon_i}{p_j \cdot q_i}$$

$$= I_{jk}\left(q_{i+1}^{(jk)}, q_i^{(jk)}\right)$$

Diagrammatics of dipole kT evolution

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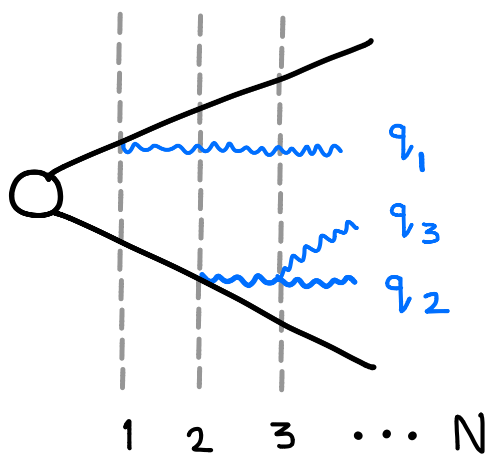

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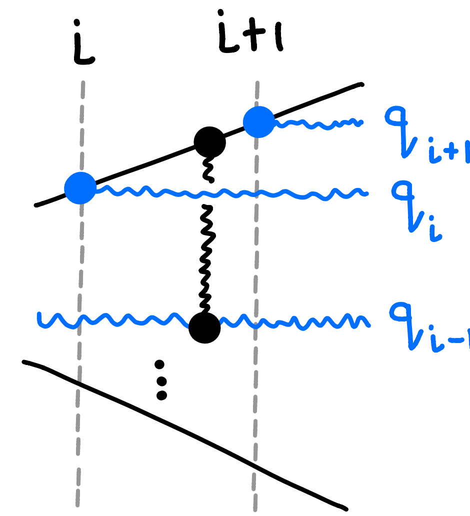

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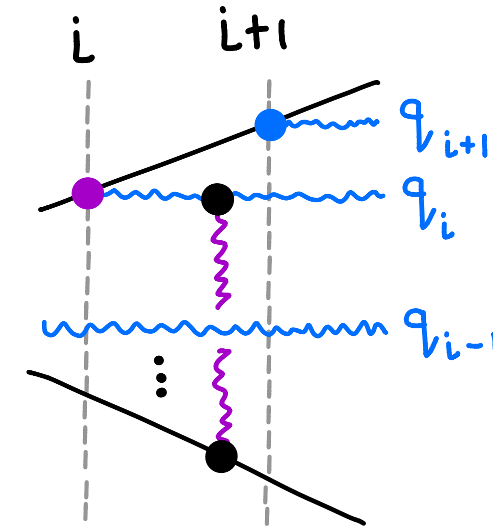

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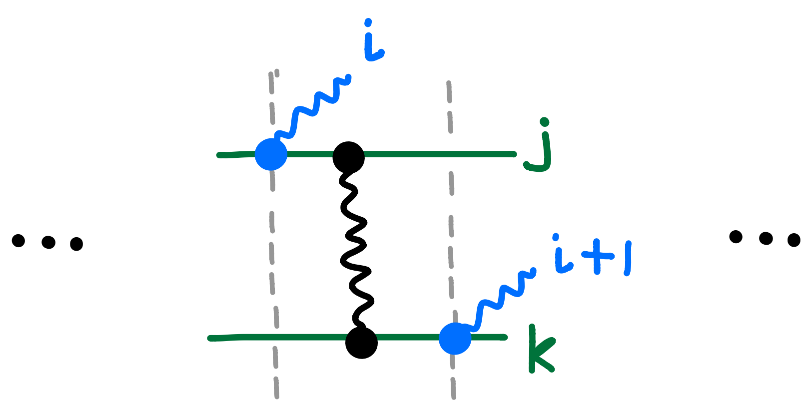
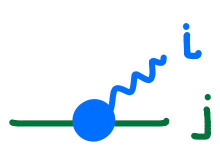
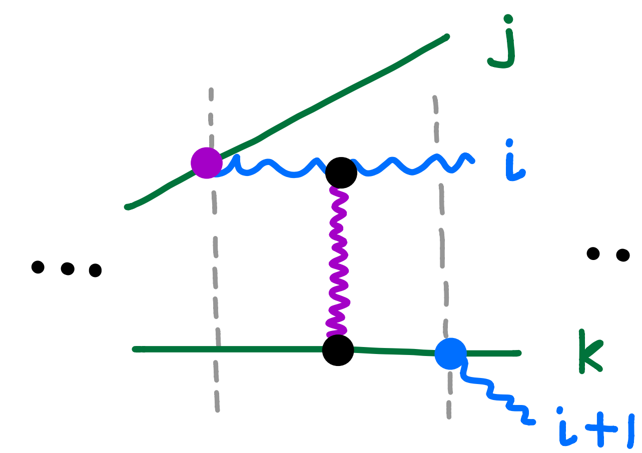




$$= T_j \frac{p_j \cdot \varepsilon_i}{p_j \cdot q_i}$$

$$= I_{jk}\left(q_{i+1}^{(jk)}, q_i^{(jk)}\right)$$

b)



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Diagrammatics of dipole kT evolution

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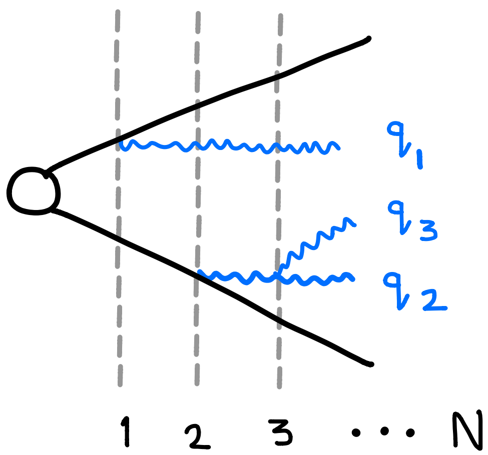

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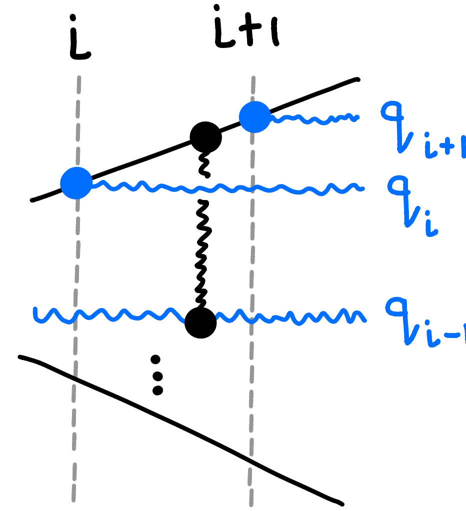
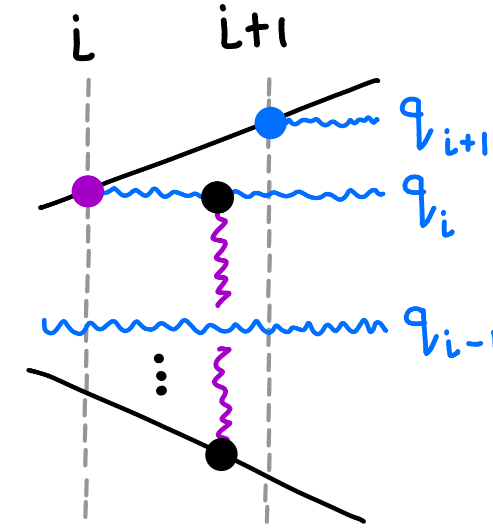


Case b)



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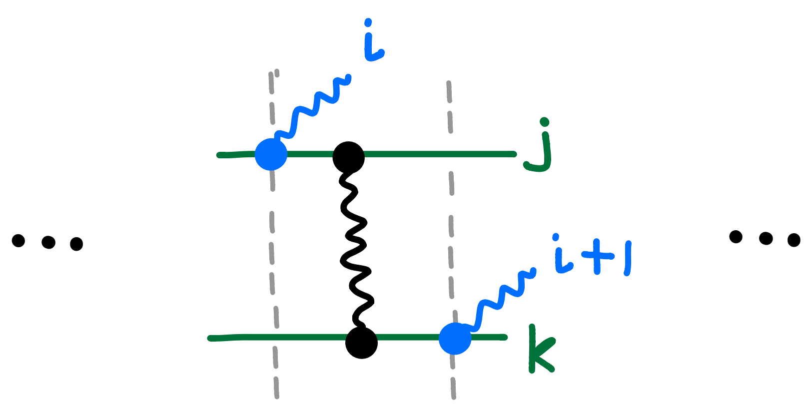
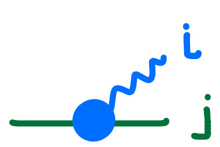


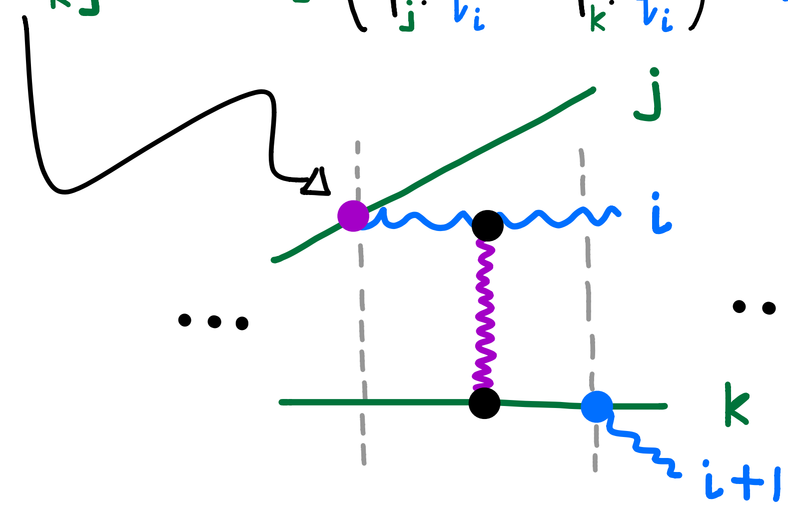


$$= T_j \frac{p_j \cdot \varepsilon_i}{p_j \cdot q_i}$$

$$= I_{jk}\left(q_{i+1}^{(jk)},\, q_i^{(jk)}\right)$$

b)

$$\bullet = d_{kj}(i) = T_j \left(\frac{p_j}{p_j \cdot q_i} - \overbrace{\frac{p_k}{p_k \cdot q_i}}^{\text{Gauge invariance}} \right) \cdot \varepsilon_i$$



$$= I_{ik}\left(q_{i+1}^{(ik)},\, q_i^{(jk)}\right)$$

Non-emission evolution operator

$$\mathbf{I}_{ij}(a,b) = \frac{\alpha_s}{2\pi} c_\Gamma \mathbf{T}_i \cdot \mathbf{T}_j \int \mathrm{d}(k^{(ij)})^2 (k^{(ij)})^{-2\epsilon} \left[\int_{\ln \sqrt{2} p_j^- / k^{(ij)}}^{\ln \sqrt{2} p_i^+ / k^{(ij)}} \mathrm{d}y \frac{p_i \cdot p_j}{2[p_j \cdot k][p_i \cdot k]} - \frac{i\pi\delta_{ij}}{(k^{(ij)})^2} \right]$$

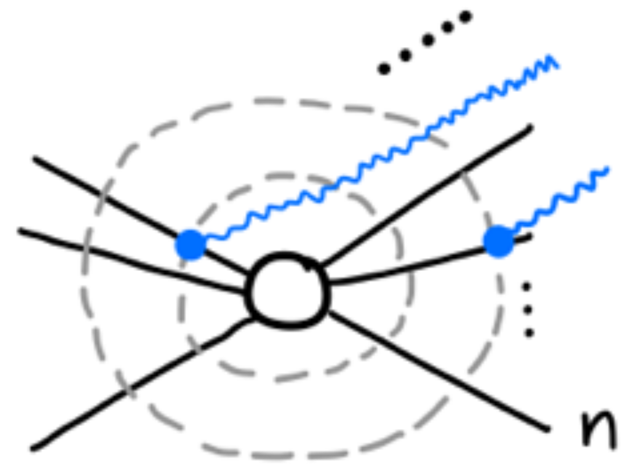
$$\times\, \theta(a < k^{(ij)} < b)$$

This is the same one-loop operator that appears at NLO but kT ordered!

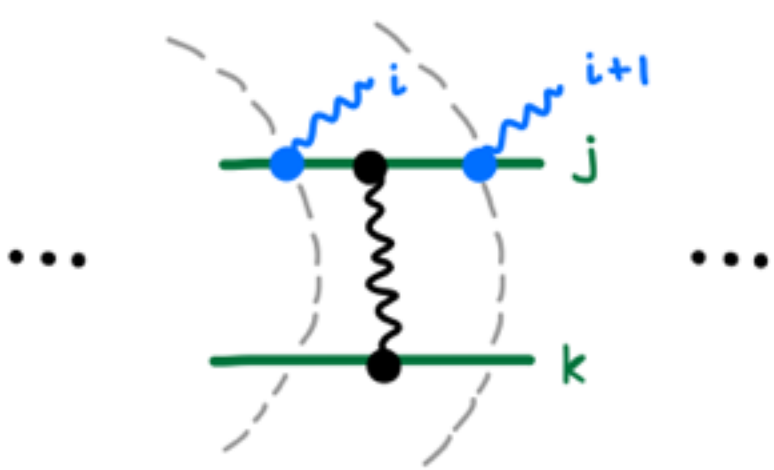
$$k^\mu = \alpha p_i^\mu + \beta p_j^\mu + (k^{(ij)})^\mu$$

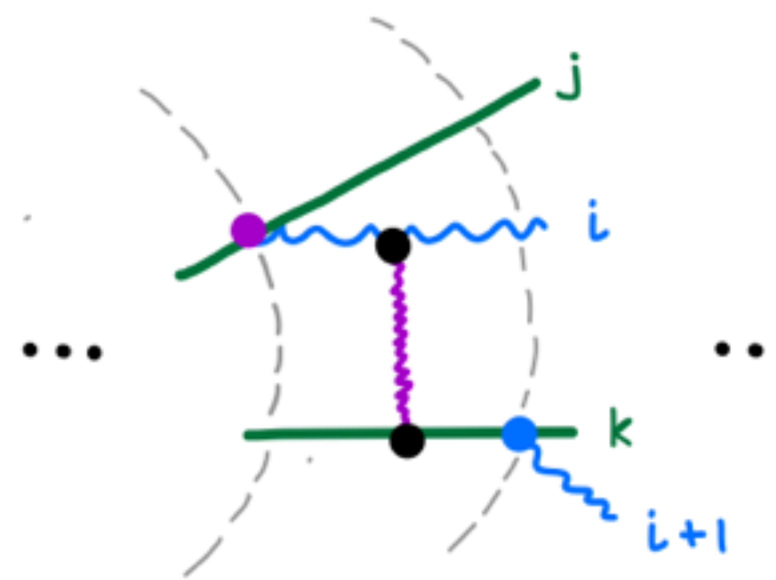
Diagrammatics of dipole kT evolution

For a general scattering $|n\rangle$ we need spheres



The effective rules are the same:





Loop-expanded approach

The dipole kt evolution approach is equivalent to a Catani-Grazzini loop-expansion (RAM , Forshaw & Seymour hep-ph/0007142):

$$\left|n_N\right\rangle = (g_s\mu^\epsilon)^N \, \mathbf{J}(q_N)\cdots\mathbf{J}(q_1)\left|n_0\right\rangle$$

$$\mathbf{J} = \mathbf{J}^{(0)} + \mathbf{J}^{(1)}, \qquad \left|n\right\rangle = \left|n_0^{(0)}\right\rangle + \left|n_0^{(1)}\right\rangle,$$

$$\mathbf{J}^{(1)}(q_a) = \frac{1}{2}\sum_{j\neq i}^{n+a-1}\frac{\alpha_s}{2\pi}\frac{c_\Gamma}{\epsilon^2}\left(\frac{(-p_i\cdot q_a - i0)(-p_j\cdot q_a - i0)}{(-p_i\cdot p_j - i0)8\pi}\right)^{-\epsilon}\mathbf{T}_{q_a}\cdot\mathbf{T}_i\;\mathbf{d}_{ij}(q_a).$$

Dipole kT

Manifestly, analytic function of Lorentz invariants! This approach has been studied for e+e- annihilation (Feige & Schawartz PhD thesis & Phys.Rev. D90 (2014)) and is used in a recent resummation of non-global logs (1501.03754).

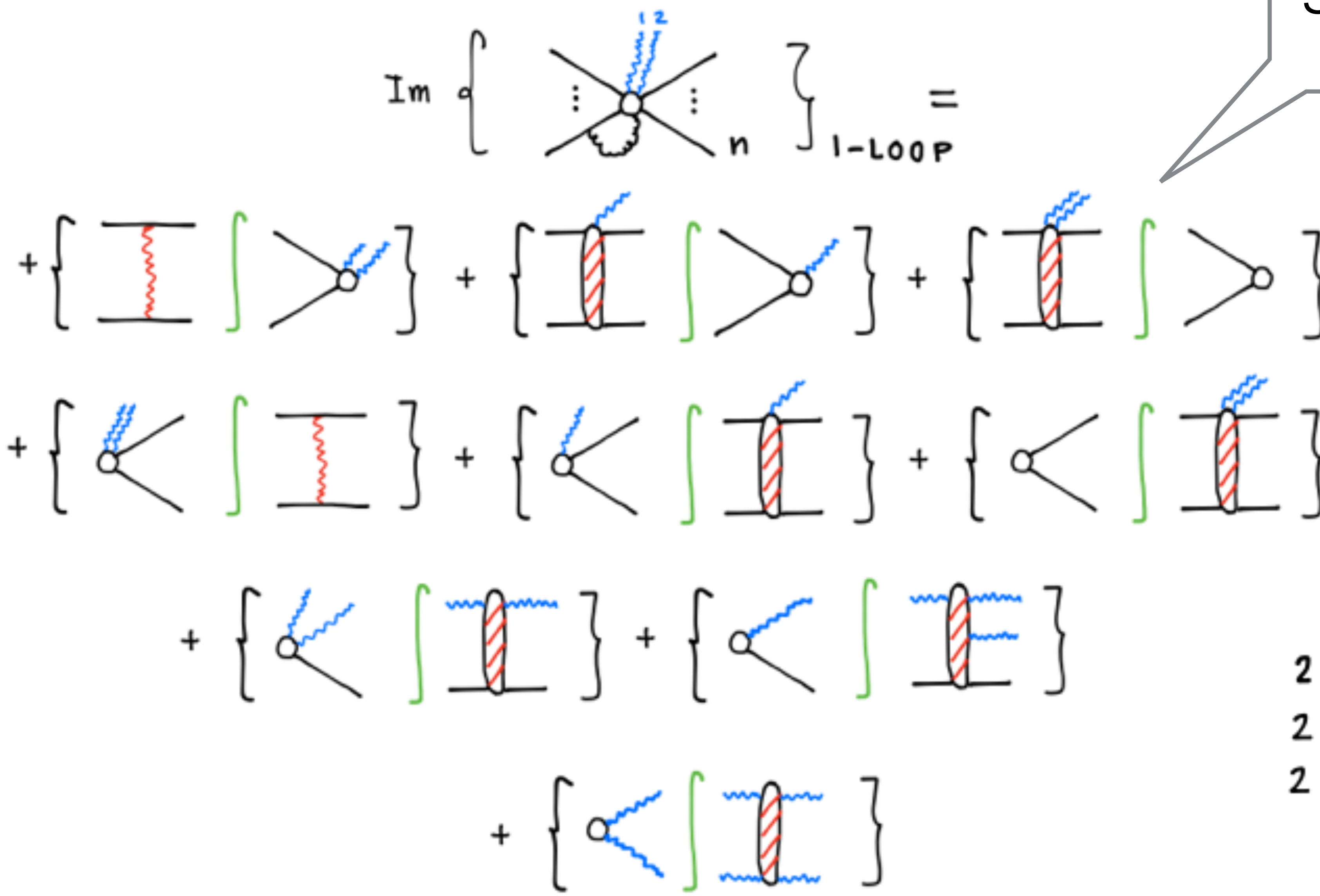
Summary / Conclusions

- Coulomb gluons distinguish initial and final partons and hence introduce colour (quantum) interference.
- Play an essential role in the colour evolution of hard processes
  - super-leading logarithms
  - violation of strict factorisation
- Can be incorporated at *amplitude level* as an evolution in dipole transverse momentum
- Can evolution approach be developed into a parton-shower like algorithm?

Two emissions case

Again, gauge invariantly the amplitude the sum of product of on-shell scattering amplitudes (assuming all gluons are soft but not relative softness):

Set of sets of cut graphs

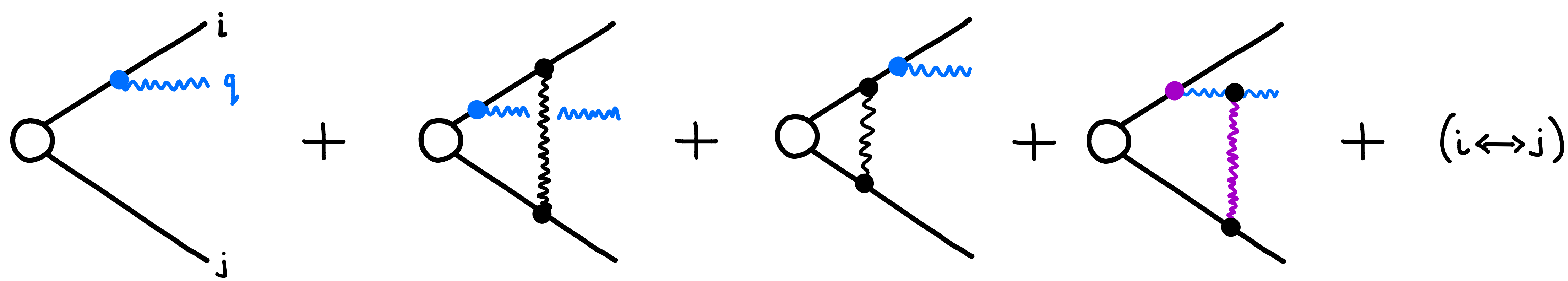


$\vert n\rangle$	GRAPHS
$2 \to 0$	105
$2 \to 2$	826
$2 \to 5$	6625

This time, ordering is a feature of particular kinematics, e.g. when emissions are ordered in softness $q_2 \sim \lambda q_1$.

(Relation to) Loop-expanded approach

There is a different way of organising these contributions:



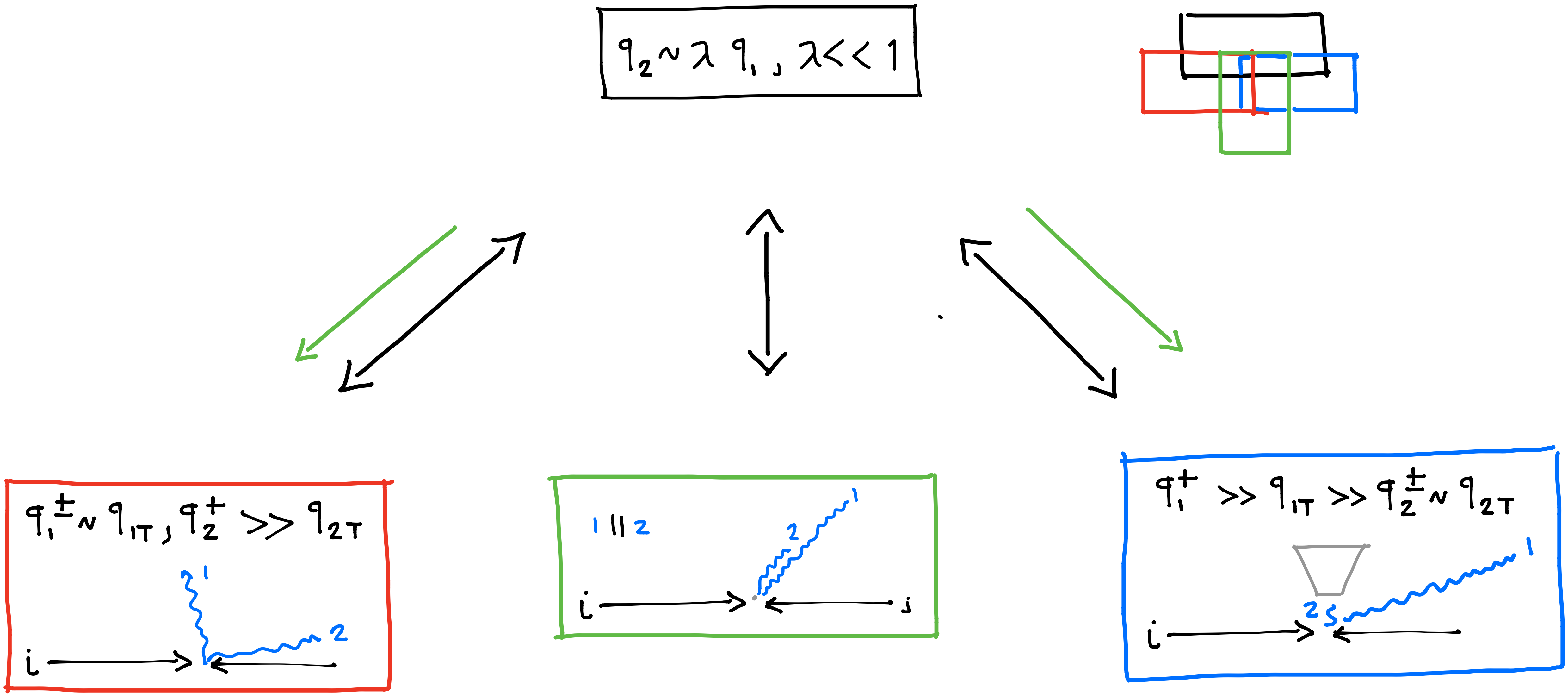
$$= \left(J^{(0)} + I_{ij}(0, q^{(ij)})\, J^{(0)} + J^{(0)} \underbrace{I_{ij}(q^{(ij)}, \sqrt{2 p_i \cdot p_j})}_{\left[- I_{ij}(0, q^{(ij)}) + I_{ij}(0, \sqrt{2 p_i \cdot p_j}) \right]} + I_{q_j}(0, q^{(ij)})\, d_{ji} + I_{q_i}(0, q^{(ij)})\, d_{ij} \right) |2^{(0)}\rangle$$

One-loop soft-gluon emission current (Catani & Grazzini Nucl.Phys. B591 (2000))

Independent loop-expansion!

$$= \left(J^{(0)} + J^{(1)} \right) \left(|2^{(0)}\rangle + |2^{(1)}\rangle \right) + (\text{2-Loop})$$

An interesting property



Limits commute

Can be derived from

Paper in preparation.