A Lattice Calculation of Parton Distributions

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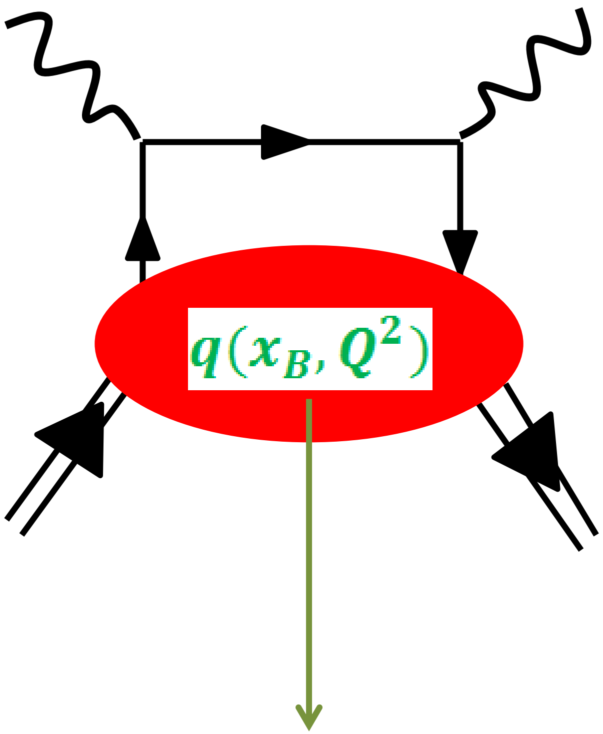
Main References

- X. Ji, “Parton Physics on a Euclidean Lattice,” PRL 110 (2013) 262002.
- X. Xiong, X. Ji, J.-H. Zhang and Y. Zhao,
  “One loop matching for parton distributions:Nonsinglet case,”PRD90 (2014) 014051.
- H.-W. Lin, J.-W. Chen, S. D. Cohen and X. Ji,
  “Flavor Structure of the Nucleon Sea from Lattice QCD,” Phys. Ver. D91 (2015) 054510.
- C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, k. Hadjiyiannakou, K. Jansen, FS and C. Wiese, “A Lattice Calculation of Parton Distributions,” arXiv:1504.07455.
- J.-W. Chen, S. D. Cohen, X. Ji, H.-W. Lin and J.-H Zhang, “Nucleon Helicity and Transversity Parton Distributions from Lattice QCD’, arXiv:1603.066664.

Outline

- Quark distributions
- Quasi-quark distributions
- Relation between quark and quasi-quark distributions
- Calculation of the matrix elements of the quasi distributions
- Results

In the Bjorken limit: $Q^2, \nu \to \infty, x_B = \dfrac{Q^2}{2P \cdot q}$ *fixed*



Parton distributions

$$q(x_B, Q^2)$$

$\downarrow$

Q^2 dependence comes from perturbative QCD

$\downarrow$

Q^2 not infinite but if large enough, the handbag diagram with pQCD corrections describes well the data

Light cone quark distributions

QCD

$\downarrow$

OPE

$\downarrow$

$$\int_0^1 dx_B x_B^{n-2} F_i(x_B, Q^2) \sim \sum_j C_n^{i,j}(Q^2) \left\langle P \middle| \mathcal{O}^{i,j}_{\mu_1 \cdots \mu_n} \middle| P \right\rangle \frac{q^{\mu_1} \ldots q^{\mu_n}}{Q^{2n}}$$

$$\left\langle P \middle| \mathcal{O}^{i}_{\mu_1 \cdots \mu_n} \middle| P \right\rangle = a_n^i \big(\underbrace{P_{\mu_1} \cdots P_{\mu_n}}_{\text{Leading}} + \underbrace{P^2 g_{\mu_1 \mu_2} P_{\mu_3} \cdots P_{\mu_n}}_{\text{Subleading } (P^2/Q^2)} + \cdots \big)$$

Moments of quark distributions

Leading

Subleading (P^2/Q^2)

Can not be disregarded if $Q^2 \sim P^2$

The x dependence of the distributions

Inverse Mellin transform

$$a_n = \int dx\, x^{n-1} q(x) \qquad q(x) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn\, x^{-n} a_n$$

And taking $\boldsymbol{\mu_1} = \boldsymbol{\mu_2} = \cdots = \boldsymbol{\mu_n} = +$

$$q(x) = \frac{1}{2\pi}\int d\xi^- e^{-ixp^+\xi^-} \langle P|\bar{\psi}(\xi^-)\Gamma\mathcal{L}(\xi^-,0)\psi(0)|P\rangle$$

$$= e^{-ig\int_0^{\xi^-} d\eta^- A^+(\eta^-)}$$

- Light cone correlations in the nucleon rest frame
- Equivalent to distributions in the IMF
- Light cone dominated: $\xi^2 = t^2 - z^2 \sim 0$
- Not calculable on Euclidian Lattice as $t^2 + z^2 \sim 0$

Moments of quark distributions in terms of matrix element of local operators can be calculated in Lattice QCD

$$\int dx\, x^{n-1} q(x) = \langle N | \mathcal{O}^{\{\mu_1 \dots \mu_n\}} | N \rangle,$$

$$\mathcal{O}^{\{\mu_1 \dots \mu_n\}} = \bar{\psi} \left(\gamma^{\{\mu_1} i \overleftrightarrow{D}^{\mu_2} \dots i \overleftrightarrow{D}^{\mu_n\}} \right) \frac{\tau^a}{2} \psi.$$

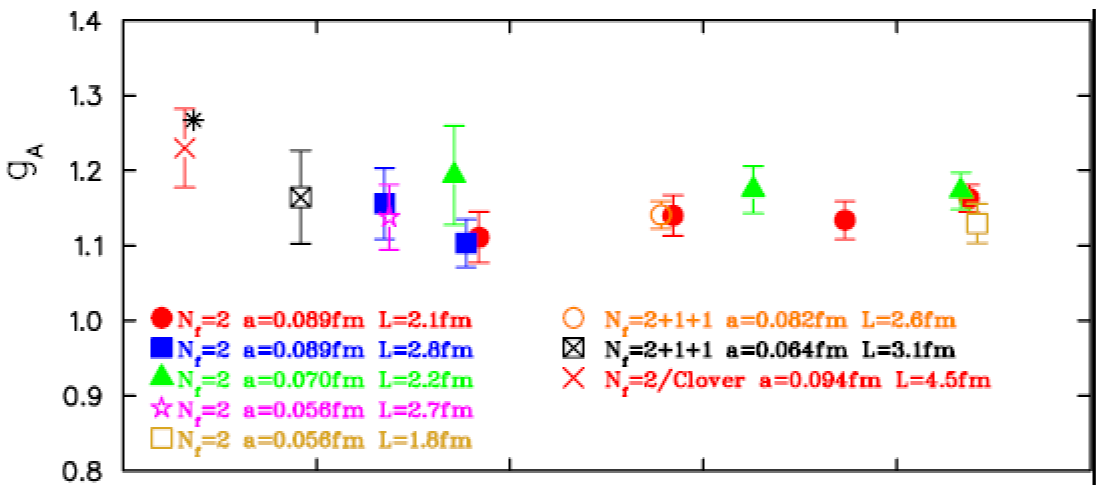
Example – isovector quark momentum fraction ($q(x) = u(x) - d(x)$):

$$\langle x \rangle_{u-d} = \int dx\, x \, \left(q(x) + \bar{q}(x) \right).$$

- If a sufficient number of moments are calculated, one can reconstruct the x dependence of the distributions

- Hard to simulate high order derivatives on the lattice

- Nevertheless, the first few moments as well as charges can and have been calculated



$$g_A = \int_0^1 dx\,\left(\Delta u(x) - \Delta d(x)\right)$$

Source: C. Alexandrou et al. 1405.5213

Quasi Distributions

Matrix elements $\langle P|O^{\mu_1\mu_2\ldots\mu_n}|P\rangle = 2a_n^{(0)}\Pi^{\mu_1\mu_2\ldots\mu_n}$ $\quad P=(P_0,0,0,P_3)$.

where $$\Pi^{\mu_1\mu_2\ldots\mu_n} = \sum_{j=0}^{k}(-1)^j\frac{(2k-j)!}{2^j(2k)!}\{g\ldots gP\ldots P\}_{k,j}(P^2)^j$$

Setting $\mu_1=\mu_2=\ldots=\mu_{2k}=3$

$$\Pi^{3\ldots3} = \sum_{j=0}^{k}(-1)^j\frac{(2k-j)!}{2^j(2k)!}\frac{(2k)!}{2^j j!(2k-2j)!}(-1)^j(P_3^2)^{k-j}(M^2)^j$$

$$\boxed{\langle P|O^{3\ldots3}|P\rangle = 2\tilde{a}_{2k}^{(0)}(P_3)^{2k}\sum_{j=0}^{k}\mu^j\begin{pmatrix}2k-j\\ j\end{pmatrix} \equiv 2\tilde{a}_{2k}(P_3)^{2k}}$$

With $\boxed{\mu = M^2/4(P_3)^2}$

Defining: $$\tilde{a}_n(\Lambda, P_3) = \int_{-\infty}^{+\infty} x^{n-1}\tilde{q}(x, \Lambda, P_3)dx$$

Mellin transformation implies in

$$\tilde{q}(x, \Lambda, P_3) = \int_{-\infty}^{\infty} \frac{dz}{4\pi} e^{-izk_3} \langle P|\bar{\psi}(0, z)\gamma^3 W(z)\psi(0, 0)|P\rangle.$$

Parton momentum $k_3 = xP_3$

Wilson line $W(z) = e^{-ig\int_0^z A_3(z')dz'}$

- Nucleon moving with finite momentum in the z direction
- Pure spatial correlation
- Can be simulated on a lattice

What are these quasi-distributions? Do they have a partonic interpretation?

The light cone distributions:

$$x = \frac{k^+}{P^+}$$

$$0 \leq x \leq 1$$

Distributions can be defined in an Infinite Momentum Frame: P_3, P^+ goes to infinite

Quasi distributions:

P_3 large but finite

Some constituents can be moving backward or even with momentum greater than P_3

$x < 0$ or $x > 1$ is possible

Usual partonic interpretation is lost

But they can be related to each other!

Matching Condition

- Relating finite to infinite momentum
- Axial gauge $A_3 = 0$
- UV divergence regulate with $|k_T| \leq \Lambda \sim \dfrac{1}{a}$
- Renormalization scale μ

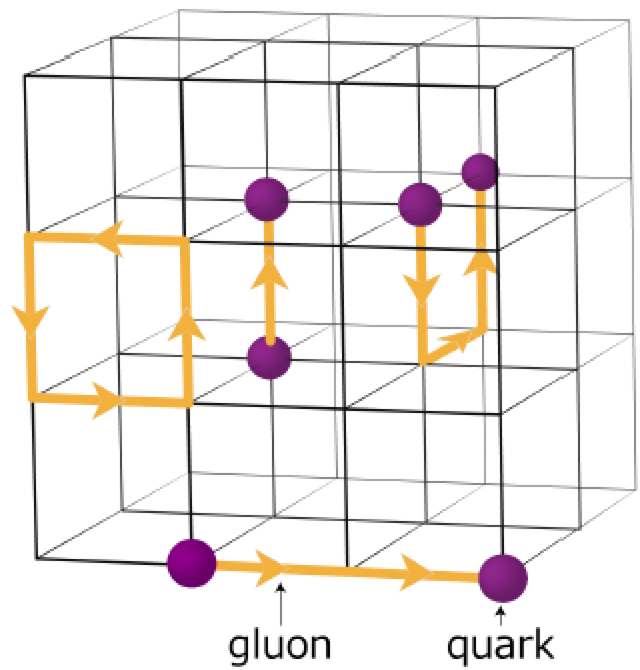
$$\boxed{\begin{aligned} q(x,\mu) \;&=\; \tilde{q}(x,\Lambda,P_3) - \frac{\alpha_s}{2\pi}\tilde{q}(x,\Lambda,P_3)\delta Z_F^{(1)}\left(\frac{\mu}{P_3},\frac{\Lambda}{P_3}\right) \\ &\quad -\frac{\alpha_s}{2\pi}\int_{-1}^{1} Z^{(1)}\left(\frac{x}{y},\frac{\mu}{P_3},\frac{\Lambda}{P_3}\right)\tilde{q}(y,\Lambda,P_3)\frac{dy}{|y|} + \mathcal{O}(\alpha_s^2) \end{aligned}}$$

Desired quantity

From the lattice

From pQCD. How to calculate them?

- We introduce a 4D hypercubic lattice:
  - ⋆ quark fields on lattice sites,
  - ⋆ gluon fields on lattice links.
- Gauge invariant objects:
  - ⋆ Wilson loops,
  - ⋆ quarks and antiquarks connected with a gauge link.
- Lattice as a regulator:
  - ⋆ UV cut-off – inverse lat. spac. a^{-1},
  - ⋆ IR cut-off – inverse lat. size L^{-1}.
- Remove the regulator:
  - ⋆ continuum limit $a \to 0$,
  - ⋆ infinite volume limit $L \to \infty$.



Source: JICFuS, Tsukuba

The Wilson twisted mass fermion action for the 2 light (u, d quarks) is given in the so-called twisted basis by: [R. Frezzotti, P. Grassi, G.C. Rossi, S. Sint, P. Weisz, 2000-2004]

$$S_l[\psi, \bar{\psi}, U] = a^4 \sum_x \bar{\chi}_l(x)\big(D_W + m_{0,l} + i\mu_l\gamma_5\tau_3\big)\chi_l(x),$$

where:

- D_W – Wilson-Dirac operator,
- $m_{0,l}$ and μ_l – bare untwisted and twisted light quark masses,
- the matrix τ^3 acts in flavour space,
- $\chi_l = (\chi_u, \chi_d)$ is a 2-component vector in flavour space, related to the one in the physical basis by a chiral rotation with angle ω:

$$\psi = e^{i\gamma_5\tau_3\omega/2}\chi.$$

With maximal twist, $\omega = \pi/2$, automatic $\mathrm{O}(a)$-improvement is achieved.

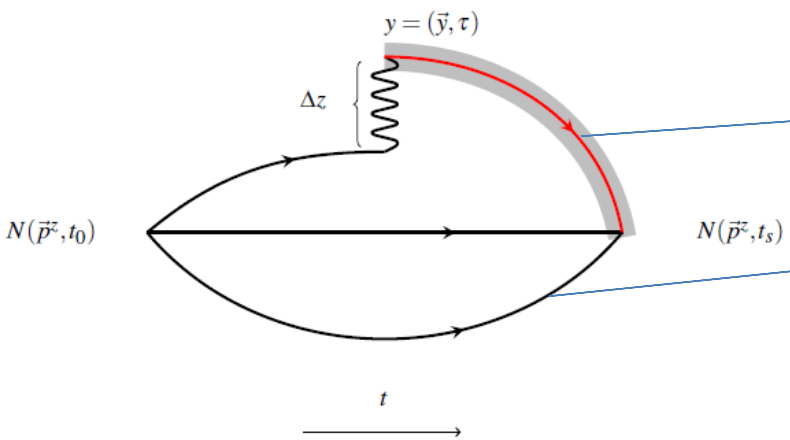
We want: $$h(P_3, z) = \langle P|\overline{\psi}(z)\gamma_3 W_3(z,0)\psi(0)|P\rangle$$

Let : $$C^{3\mathrm{pt}}(t,\tau,0) = \left\langle N_\alpha(\vec{P},t)\mathcal{O}(\tau)\overline{N}_\alpha(\vec{P},0)\right\rangle$$

$$N_\alpha(\vec{P},t) = \Gamma_{\alpha\beta}\sum_{\vec{x}} \mathrm{e}^{i\vec{P}\vec{x}}\epsilon^{abc}u^a_\beta(x)\left(d^{b^T}(x)\mathcal{C}\gamma_5 u^c(x)\right)$$

$$\mathcal{O}(z,\tau,Q^2=0) = \sum_{\vec{y}} \overline{\psi}(y+z)\gamma_3 W_3(y+z,y)\psi(y)$$

All to all propagators needed



Stochastic source method is used

Point source method is used

Flavour structure: u - d

Extraction of the matrix elements

$$\frac{C^{3pt}(t,\tau,0;P_3)}{C^{2pt}(t,0;P_3)} = \frac{-iP_3}{E} h(P_3,z), \quad 0 \ll \tau \ll t$$

$8a, 10a$ Source – sink separation

$32^3 \times 64$ Lattice

$\beta = \frac{6}{g_0^2} = 1.95$ $a \approx 0.082\, fm$ $N_f = 2+1+1$

Maximally twisted mass ensemble: $a\mu = 0.0055 \Rightarrow m_{PS} \approx 370\ MeV$

$$P_3 = \frac{2\pi}{L}, \frac{4\pi}{L}, \cdots$$

Configurations

1000 gauge configurations
15 point source forward propagators
02 Stochastic propagators

30000 measurements

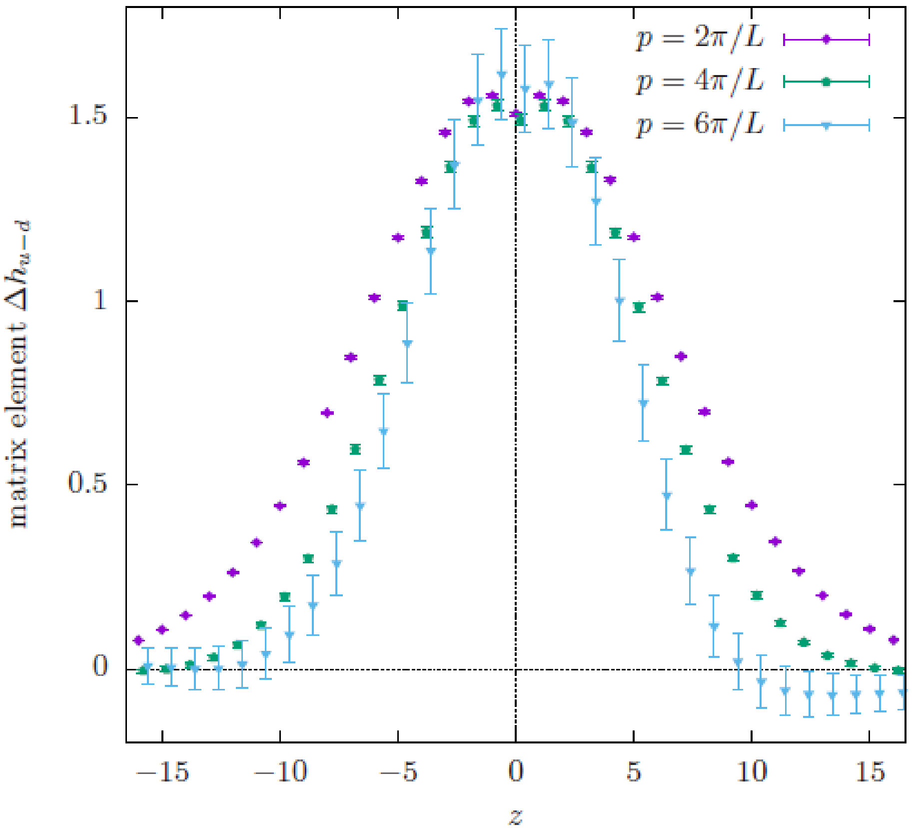
Operators

Unpolarized γ_3

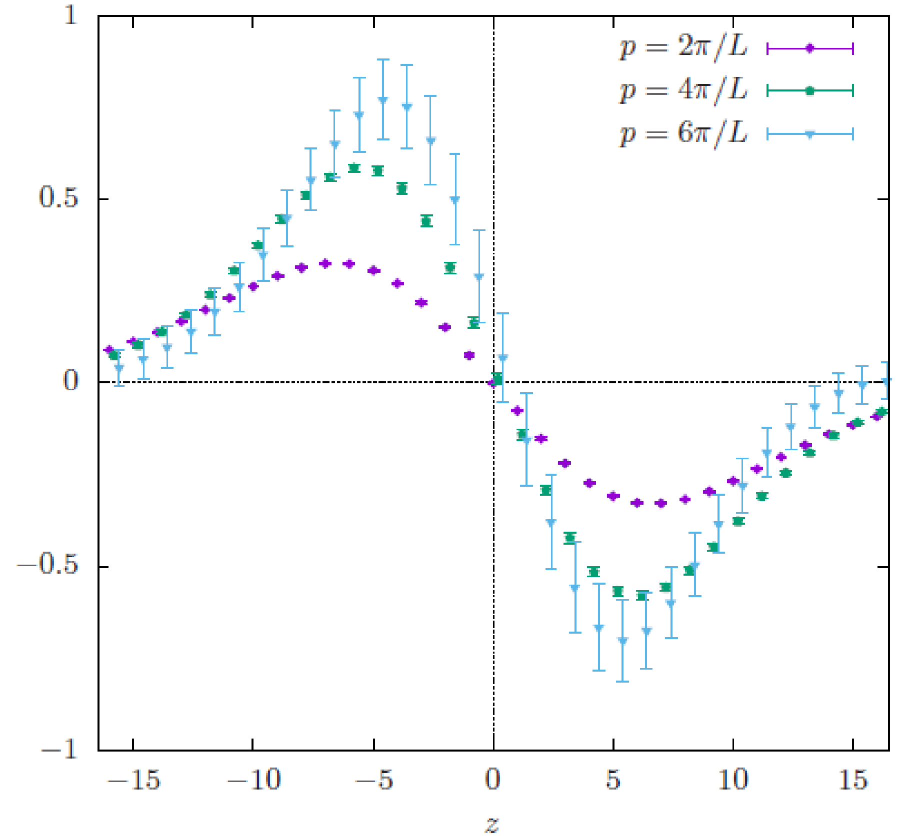
Helicity $\gamma_3\gamma_5$

Transversity $\gamma_3\gamma_j\gamma_5$

Matrix Elements



Real part



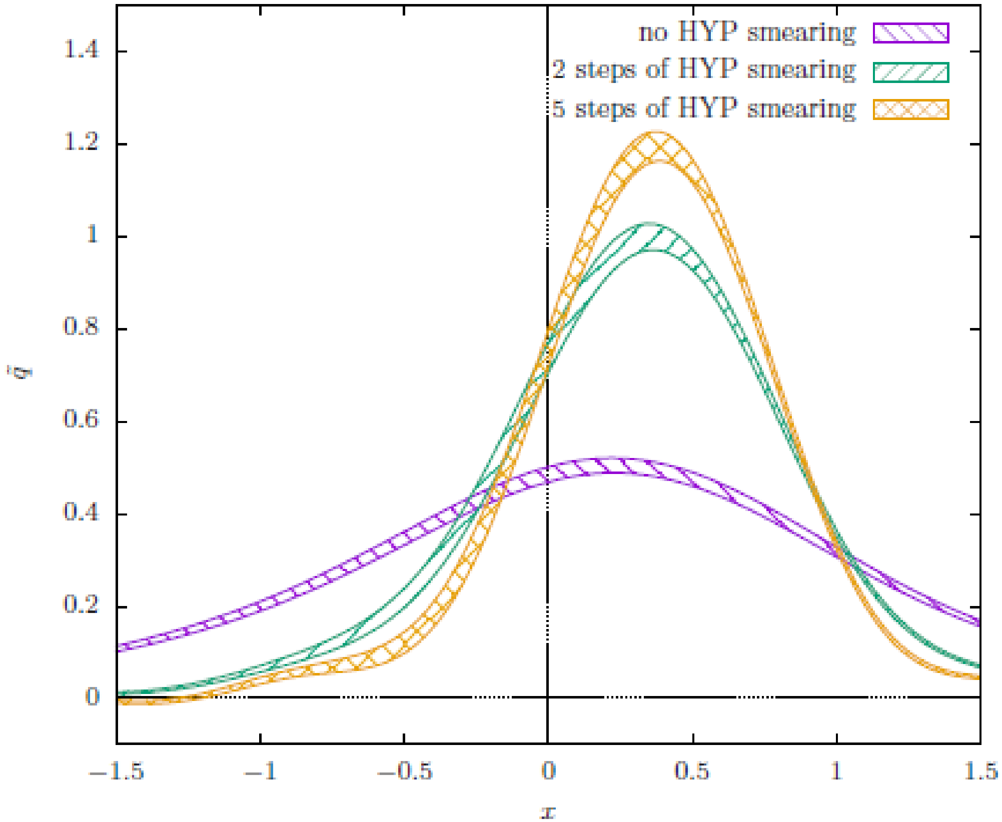
Imaginary part

5 steps of HYP Smearing

HYP Smearing

It replaces a given gauge link with some average over neighbouring links, i.e. ones from hypercubes attached to it

Crude substitute for renormalization



Parameters

$$\alpha_s = 6/(4\pi\beta) \approx 0.245$$

$$\Lambda = 1/a \cong 2.5\,\text{GeV}$$

$$\boxed{P_3 = \frac{4\pi}{L}}$$

What is the minimum Bjorken x ?

$$\vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

Largest momentum $|\vec{p}| = \frac{\pi}{a}$

Smallest momentum $|\vec{p}| = \frac{2\pi}{L}$

The injected momentum can be distributed at most between a/L lattice points

$$x = \frac{k_3}{P_3} \longrightarrow x_{min} = \frac{1}{L/a}$$

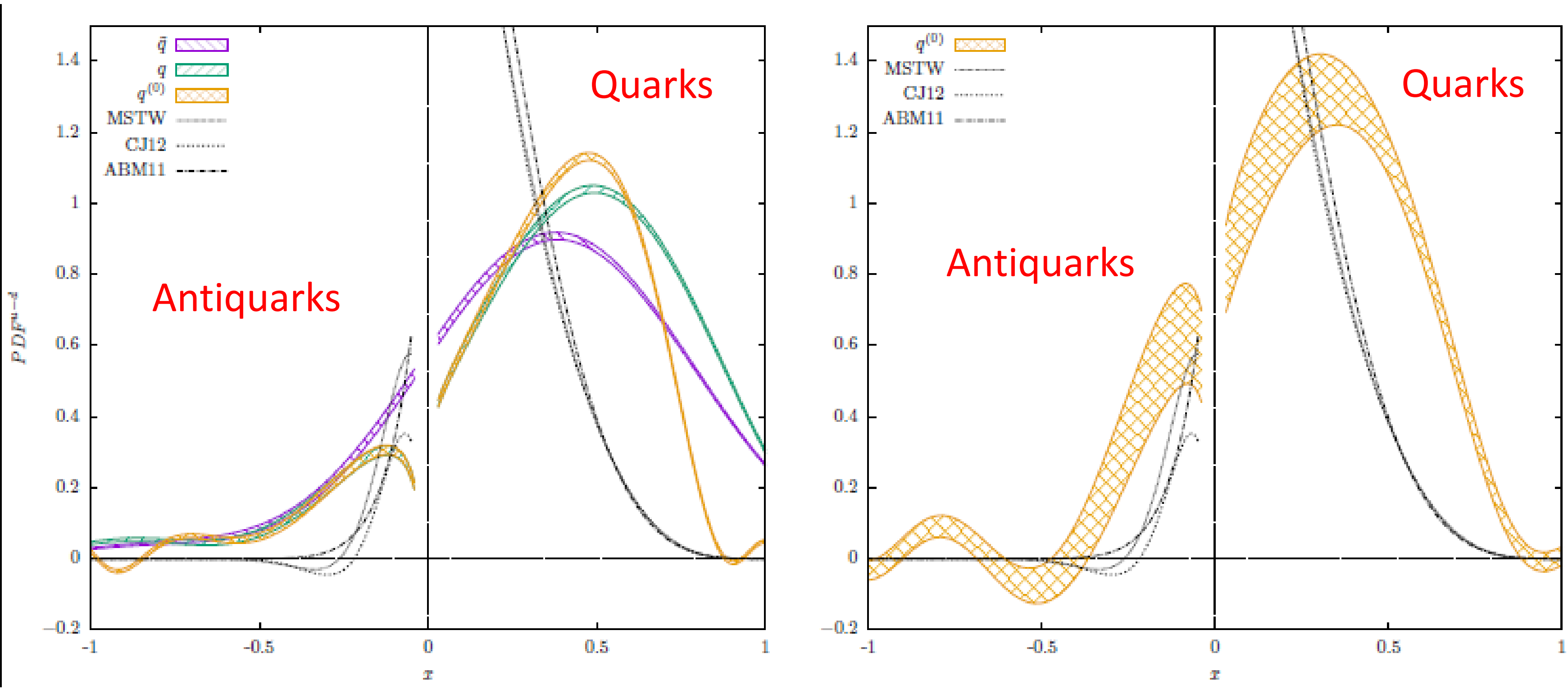
In terms of the injected momentum,

$$x_{min} \sim \frac{\Lambda_{QCD}}{P_3}$$

Present approach is valid at intermediate and large x: cut imposed by the Lattice spacing

$u(x) - d(x)$

Crossing relation: $\bar{q}(x) = -q(-x)$

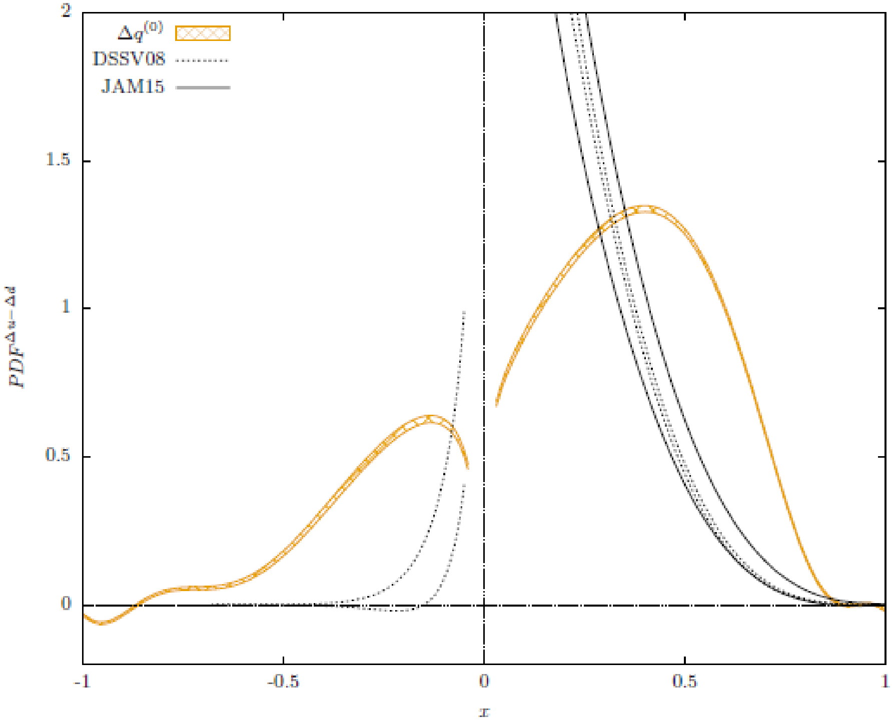


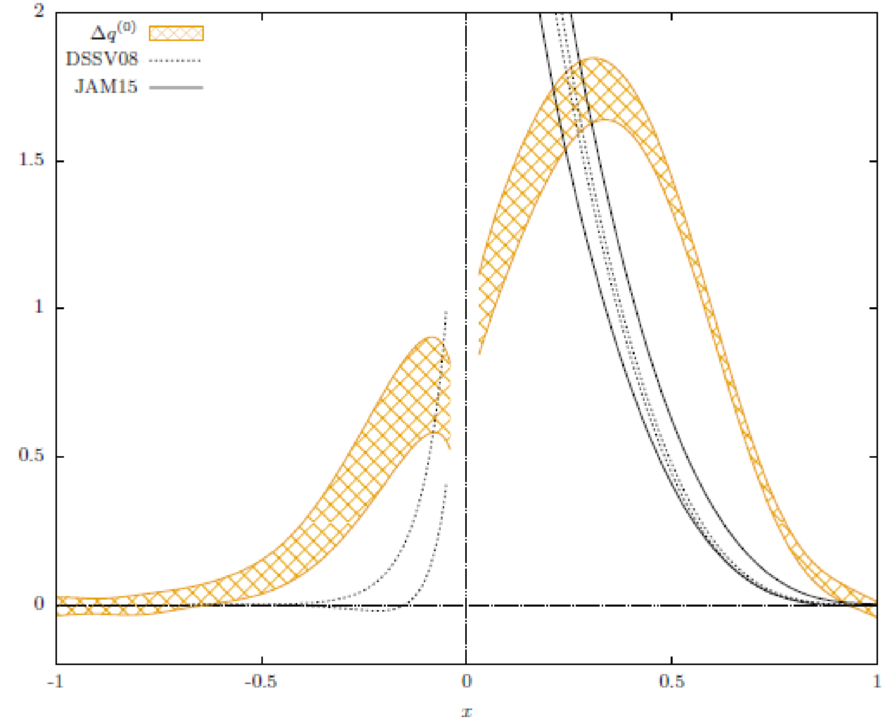
$P_3 = \frac{4\pi}{L}$ $\qquad$ $P_3 = \frac{6\pi}{L}$

$$\Delta u(x) - \Delta d(x)$$

Crossing relation: $\Delta\bar{q}(x) = \Delta q(-x)$

$$\Rightarrow \boxed{\Delta\bar{u}(x) > \Delta\bar{d}(x)}$$





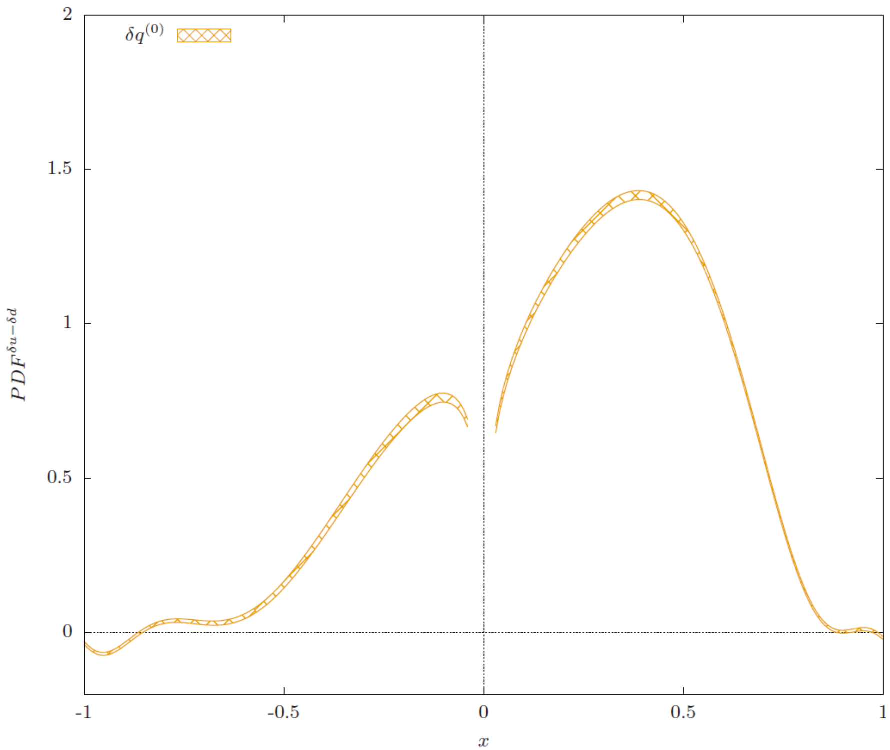
$$\boxed{P_3 = \frac{4\pi}{L}}$$

$$\boxed{P_3 = \frac{6\pi}{L}}$$

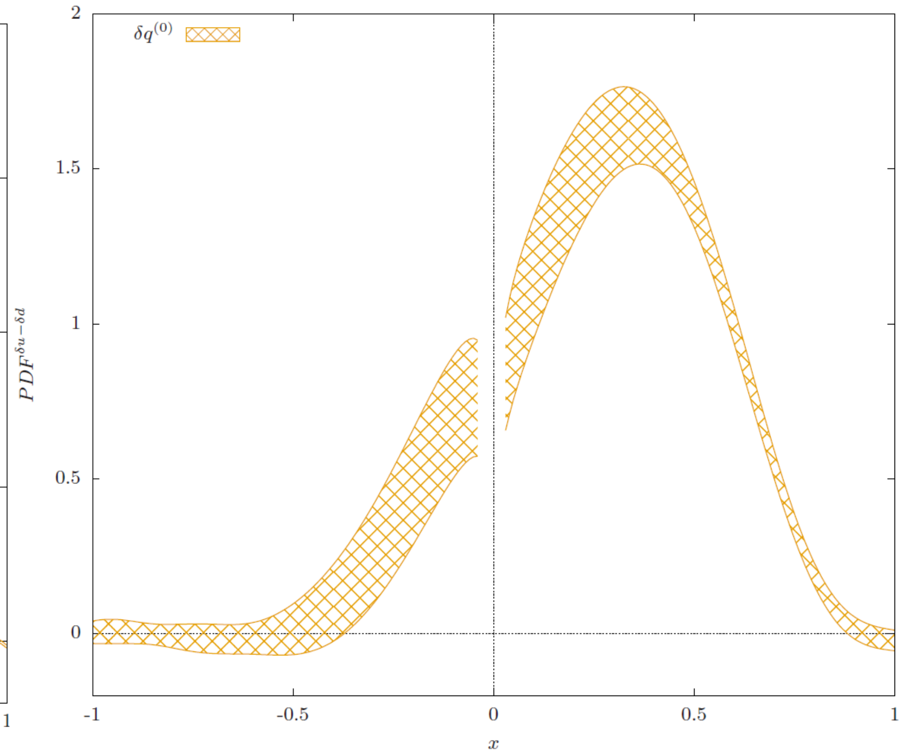
$$\boldsymbol{\delta u(x) - \delta d(x)}$$

Crossing relation: $\delta\overline{q}(x) = -\delta q(-x)$

$\Rightarrow$ $\boxed{\delta\overline{d} > \delta\overline{u}}$



$\boxed{P_3 = \frac{4\pi}{L}}$

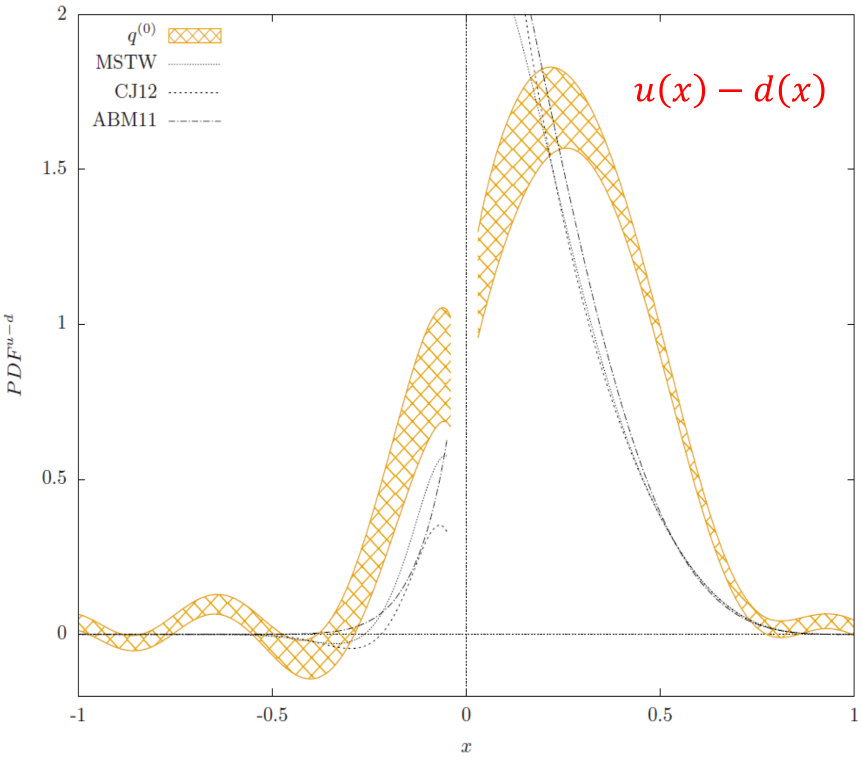


$\boxed{P_3 = \frac{6\pi}{L}}$

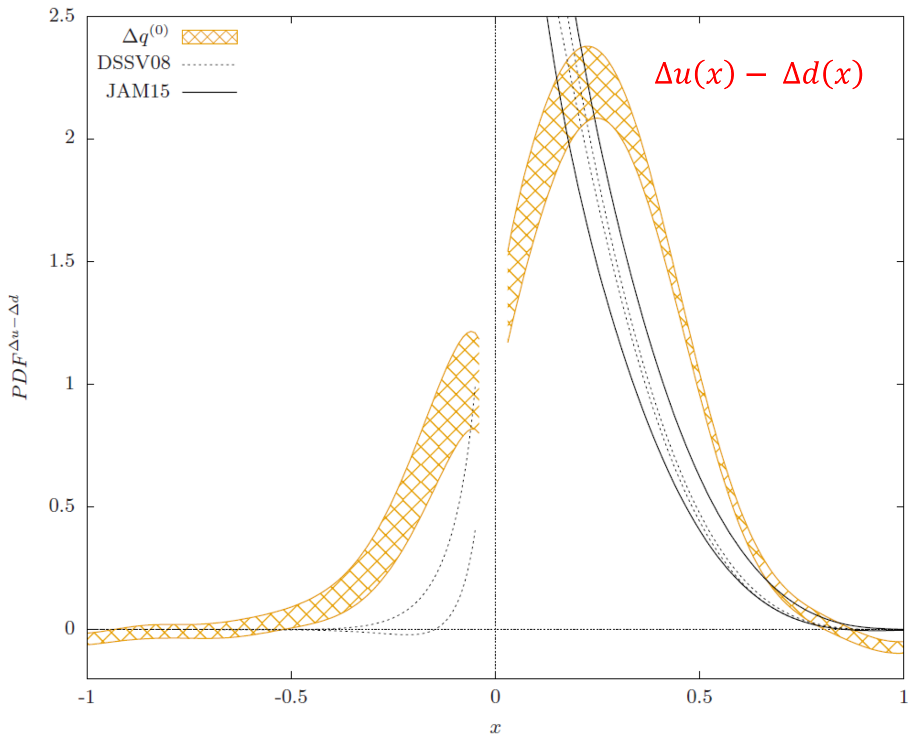
Mixed setup scheme

Matrix elements calculated with $P_3 = \frac{6\pi}{L}$

Fourier transformation and matching using $P_3 = \frac{8\pi}{L}$

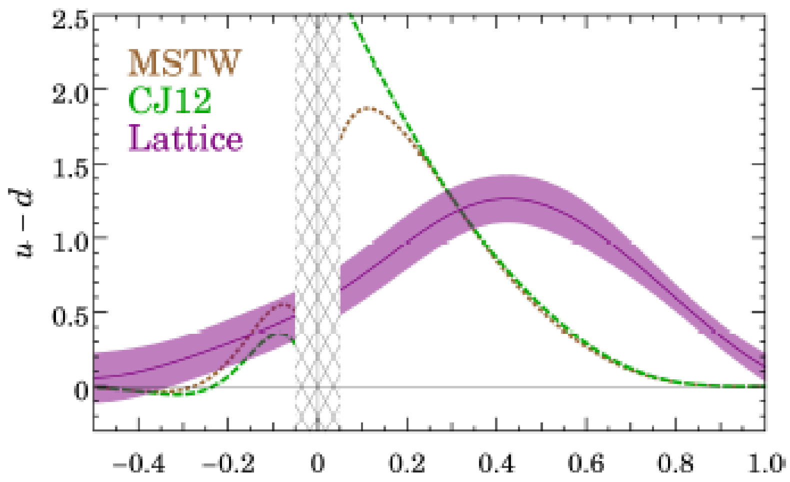


Unpolarized distributions

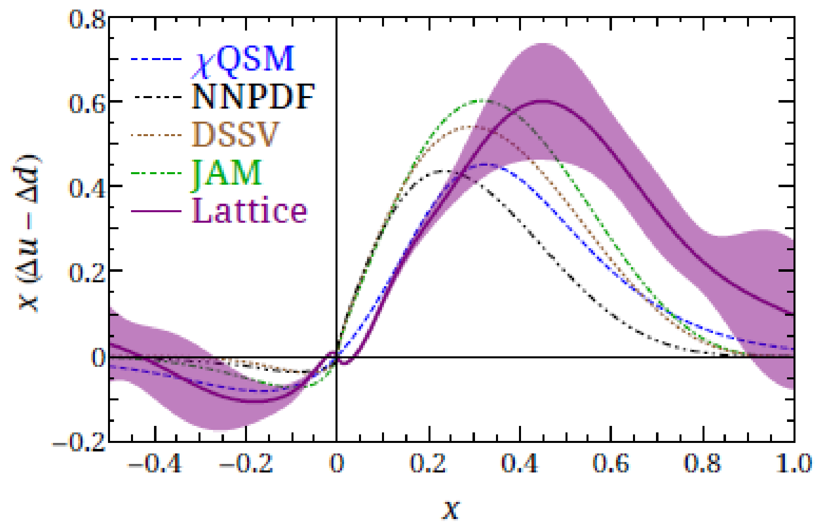


Helicity Distributions

Only other result



Huey-Wen Lin et al., Phys. Rev. D91 (2015) 054510



J.-W. Chen et al., arXiv:1603.066664

$24^3 \times 48$

$a \approx 0.12 fm \qquad N_f = 2 + 1 + 1$

$m_{PS} \approx 310\ MeV$

Uses highly improved staggered quarks and HYP smearing

Summary & Outline

- First attempts of a direct QCD calculation of quark distributions;
- Valuable information from intermediate to large x region;
- Asymmetric sea appears naturally. Imaginary part plays a fundamental role;
- Renormalization;
- Higher order correction;
- Implement Momentum Smearing: it will allow access to higher momentum;
- Compute at the physical mass – smaller number of configurations available at the moment;
- Go to the continuum;
- Singlet distributions, gluon distributions etc
- Much to be done!