Automated NLO QCD calculations for the LHC

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Introduction: **Jet physics**

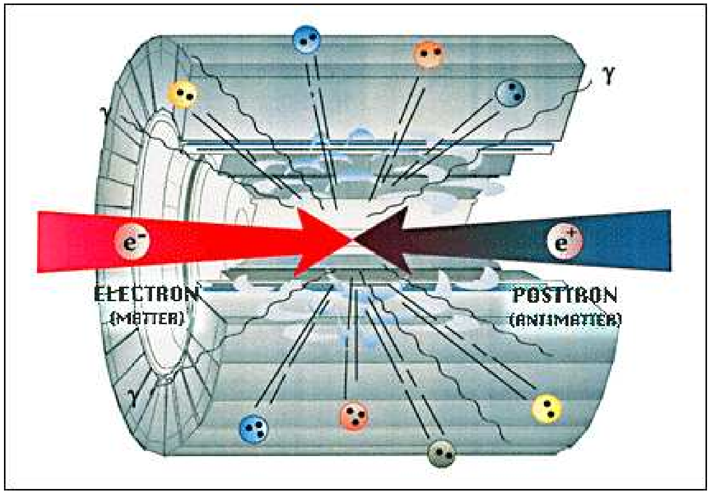
I.: **Recursive calculation of amplitudes**

II: **Organization of colour structures**

III: **Automated generation of subtraction terms**

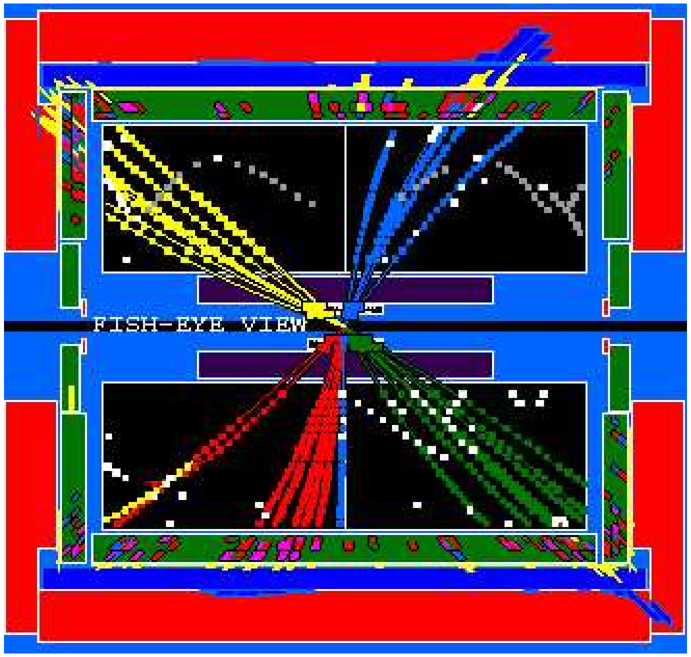
IV.: **Loop integrals**

Jet physics



A schematic view of electron-positron annihilation.

A four-jet event from the Aleph experiment at LEP:

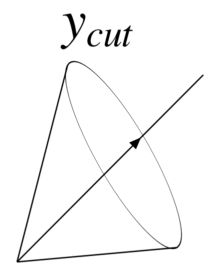


Jets: A bunch of particles moving in the same direction

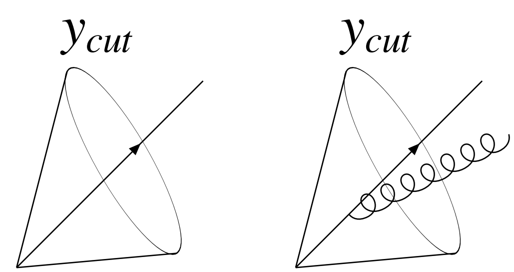
Modeling of jets:

In a perturbative calculation jets are modeled by only a few partons. This improves with the order to which the calculation is done.

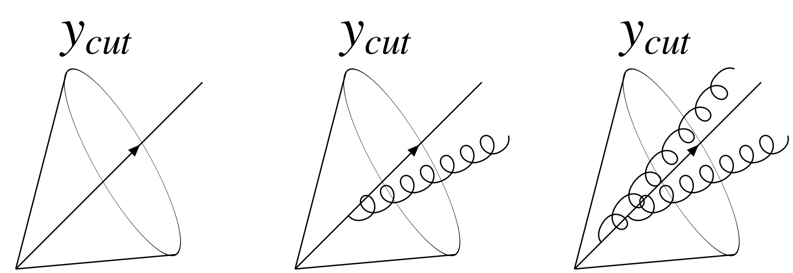
At leading order:



At next-to-leading order:



At next-to-next-to-leading order:



The master formula for the calculation of observables

$$\langle O \rangle = \underbrace{\frac{1}{2K(s)}}_{\text{flux factor}} \underbrace{\frac{1}{(2J_1+1)} \frac{1}{(2J_2+1)}}_{\text{average over initial spins}} \sum_n \underbrace{\int d\phi_{n-2}}_{\text{integral over phase space}} O(p_1, \ldots, p_n) \sum_{\text{helicity}} \underbrace{|\mathcal{A}_n|^2}_{\text{amplitude}}$$

Perturbative expansion of the amplitude (LO, NLO):

$$|\mathcal{A}_n|^2 = \underbrace{\mathcal{A}_n^{(0)\,*} \mathcal{A}_n^{(0)}}_{Born} + \underbrace{\left(\mathcal{A}_n^{(0)\,*} \mathcal{A}_n^{(1)} + \mathcal{A}_n^{(1)\,*} \mathcal{A}_n^{(0)} \right)}_{virtual},$$

$$|\mathcal{A}_{n+1}|^2 = \underbrace{\mathcal{A}_{n+1}^{(0)\;*} \mathcal{A}_{n+1}^{(0)}}_{real}.$$

Automated NLO calculations

Automated NLO calculations for $2 \to n$ ($n = 4..6, 7$) processes relevant for physics at the LHC and the ILC.

Technical challenges:

Automated numerical evaluation of one-loop amplitudes.

Automated subtraction procedure for the infrared divergences.

A. Ferroglia, M. Passera, G. Passarino, and S. Uccirati,

Z. Nagy and D. E. Soper,

W. Giele, E. W. N. Glover, and G. Zanderighi,

F. del Aguila and R. Pittau,

T. Binoth, G. Heinrich, and N. Kauer,

A. Denner and S. Dittmaier,

A. van Hameren, J. Vollinga and S.W.

Colour decomposition

Amplitudes in QCD may be decomposed into group-theoretical factors carrying the colour structures multiplied by kinematic functions called partial amplitudes.

The partial amplitudes do not contain any colour information and are gauge-invariant. Each partial amplitude has a fixed cyclic order of the external legs.

Examples: The n-gluon amplitude:

$$\mathcal{A}_n(1,2,...,n) \quad = \quad g^{n-2} \sum_{\sigma \in S_n / Z_n} \underbrace{2\,\mathrm{Tr}\left(T^{a_{\sigma(1)}} ... T^{a_{\sigma(n)}}\right)}_{\text{Chan Patton factors}} \; \underbrace{A_n\left(\sigma(1),...,\sigma(n)\right)}_{\text{partial amplitudes}}.$$

P. Cvitanovic, P. G. Lauwers, and P. N. Scharbach,

F. A. Berends and W. Giele,

M. L. Mangano, S. J. Parke, and Z. Xu,

D. Kosower, B.-H. Lee, and V. P. Nair,

Z. Bern and D. A. Kosower.

The spinor helicity method

- Basic objects: Massless two-component Weyl spinors

$$|p\pm\rangle, \qquad \langle p\pm|$$

- Gluon polarization vectors:

$$\varepsilon^{+}_{\mu}(k,q) = \frac{\langle k+|\gamma_\mu|q+\rangle}{\sqrt{2}\langle q-|k+\rangle}, \qquad \varepsilon^{-}_{\mu}(k,q) = \frac{\langle k-|\gamma_\mu|q-\rangle}{\sqrt{2}\langle k+|q-\rangle}$$

  q is an arbitrary null reference momentum. Dependency on q drops out in gauge invariant quantities.

- A clever choice of the reference momentum can reduce significantly the number of diagrams which need to be calculated.

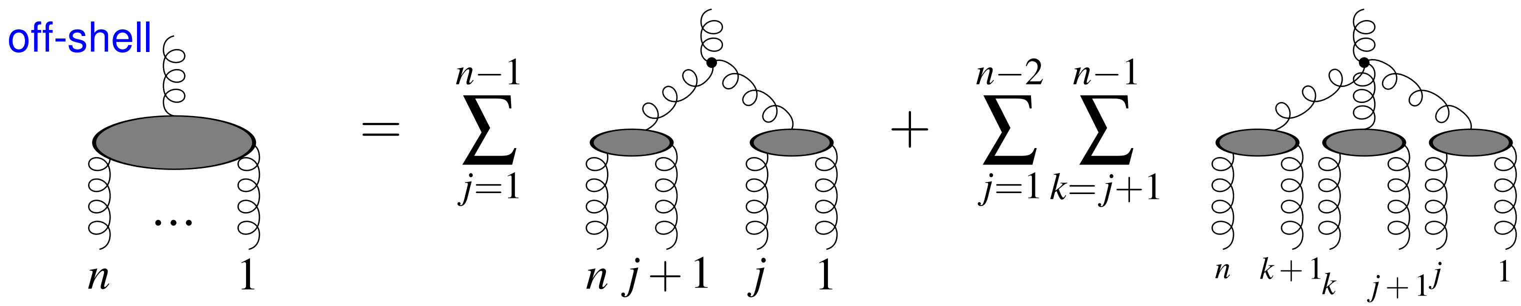
P. De Causmaecker, R. Gastmans, W. Troost and T. Wu;

R. Kleiss and W. Stirling; J. Gunion and Z. Kunszt;

Z. Xu, D.-H. Zhang, and L. Chang.

Recurrence relations

Off-shell currents provide an efficient way to calculate amplitudes:



No Feynman diagrams are calculated in this approach !

F. A. Berends and W. T. Giele,

D. A. Kosower.

The Parke-Taylor formulae

For specific helicity combinations the amplitudes have a remarkably simple analytic formula or vanish altogether:

$$\begin{aligned} A_n^{tree}(g_1^+, \ldots, g_n^+) &= 0, \\ A_n^{tree}(g_1^+, \ldots, g_j^-, \ldots, g_n^+) &= 0, \\ A_n^{tree}(g_1^+, \ldots, g_j^-, \ldots, g_k^-, \ldots, g_n^+) &= i\left(\sqrt{2}\right)^{n-2} \frac{\langle jk \rangle^4}{\langle 12 \rangle \ldots \langle n1 \rangle}. \end{aligned}$$

The n-gluon amplitude with $n-2$ gluons of positive helicity and 2 gluons of negative helicity is called a maximal-helicity violating amplitude (MHV amplitude).

F. A. Berends and W. T. Giele,

S. J. Parke and T. R. Taylor.

The CSW construction

Cachazo, Svrček and Witten proposed that the gluonic Born amplitude with an arbitrary helicity configuration can be calculated from diagrams with scalar propagators and new vertices, which are MHV-amplitudes continued off-shell.

$$A_n(1^+, ..., j^-, ..., k^-, ..., n^+) \quad = \quad i\left(\sqrt{2}\right)^{n-2} \frac{\langle jk\rangle^4}{\langle 12\rangle ... \langle n1\rangle}.$$

Off-shell continuation:

$$P \quad = \quad p^\flat + \frac{P^2}{2Pq} q.$$

Propagators are scalars:

$$\frac{-i}{P^2}$$

Cachazo, Svrček and Witten, JHEP 0409:006, (hep-th/0403047)

The BCF recursion relations

R. Britto, F. Cachazo and B. Feng gave a recursion relation for the calculation of the n-gluon amplitude:

$$A_n\left(p_1, p_2, \ldots, p_{n-1}^-, p_n^+\right) = \sum_{i=1}^{n-3} \sum_{\lambda=+,-} A_{i+2}\left(\hat{p}_n, p_1, p_2, \ldots, p_i, -\hat{P}_{n,i}^{\lambda}\right) \left(\frac{-i}{P_{n,i}^2}\right) A_{n-i}\left(\hat{P}_{n,i}^{-\lambda}, p_{i+1}, \ldots, p_{n-2}, \hat{p}_{n-1}\right).$$

No off-shell continuation needed. The amplitudes on the r.h.s. are evaluated with shifted momenta.

Britto, Cachazo and Feng, Nucl. Phys. B715, (2005), 499, (hep-th/0412308);

Britto, Cachazo, Feng and Witten, Phys. Rev. Lett. 94:181602, (2005), (hep-th/0501052)

Axial gauge

Polarisation sum, continued off-shell:

$$\sum_{\lambda=+/-} \varepsilon_\mu^{\lambda}(k^\flat, q)\varepsilon_\nu^{-\lambda}(k^\flat, q) \quad = \quad -g_{\mu\nu} + 2\frac{k^\flat_\mu q_\nu + q_\mu k^\flat_\nu}{2kq}.$$

The gluon propagator in the axial gauge is given by

$$\frac{i}{k^2} d_{\mu\nu} \quad = \quad \frac{i}{k^2}\left(-g_{\mu\nu} + 2\frac{k_\mu q_\nu + q_\mu k_\nu}{2kq}\right) = \frac{i}{k^2}\left(\varepsilon_\mu^+ \varepsilon_\nu^- + \varepsilon_\mu^- \varepsilon_\nu^+ + \varepsilon_\mu^0 \varepsilon_\nu^0\right),$$

where we introduced an unphysical polarisation

$$\varepsilon_\mu^0(k, q) \quad = \quad 2\frac{\sqrt{k^2}}{2kq} q_\mu.$$

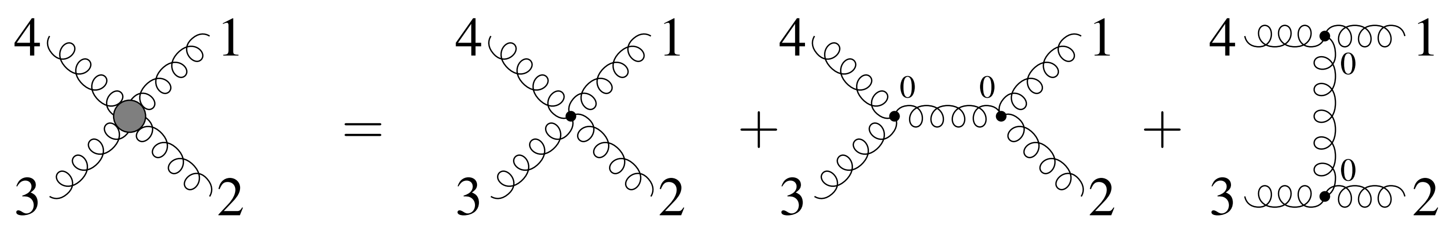
Ch. Schwinn and S.W., JHEP 0505:006, (hep-th/0503015)

Modified vertices

The only non-zero contribution containing ε^0 is obtained from a contraction of a single ε^0 into a three-gluon vertex.

In this case the other two helicities are necessarily ε^+ and ε^-.

The additional polarisation ε^0 can be absorbed into a redefinition of the four-gluon vertex.



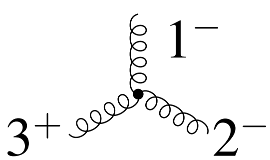
Scalar diagrammatic rules

Extension to massive and massless quarks: Born amplitudes in QCD can be computed from scalar propagators and a set of three- and four-valent vertices. Only vertices of degree zero and one occur.

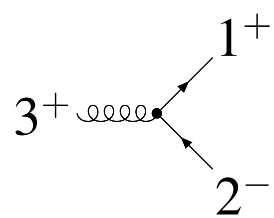
Propagators:

$$\frac{i}{p^2 - m^2}$$

Vertices:



$$= i\sqrt{2}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle},$$



$$= i\sqrt{2}\frac{[13]^2}{[12]}.$$

Ch. Schwinn and S.W., JHEP 0505:006, (hep-th/0503015)

Comparison

n	4	5	6	7	8	9	10	11	12
Berends-Giele	0.00005	0.00023	0.0009	0.003	0.011	0.030	0.09	0.27	0.7
Scalar	0.00008	0.00046	0.0018	0.006	0.019	0.057	0.16	0.4	1
MHV	0.00001	0.00040	0.0042	0.033	0.24	1.77	13	81	—
BCF	0.00001	0.00007	0.0003	0.001	0.006	0.037	0.19	0.97	5.5

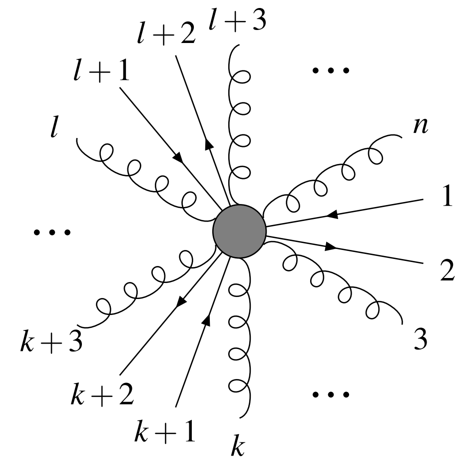
CPU time in seconds for the computation of the n gluon amplitude on a standard PC (2 GHz Pentium IV), summed over all helicities.

M. Dinsdale, M. Ternick and S.W., JHEP 0603:056, (hep-ph/0602204);

C. Duhr, S. Höche and F. Maltoni, hep-ph/0607057.

Colour structures and cyclic ordering

For Born graphs: cyclic order such that a quark follows immediately its corresponding antiquark in the clockwise orientation.
Treat a $(\bar{q}, q)$ pair as a pseudo-leg.
All possible cyclic orderings are obtained by summing over all permutations of the pseudo-legs and factoring out the cyclic permutations.



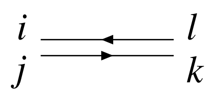
Colour cluster: part of an amplitude, which is connected to the rest of the amplitude only by an $U(1)$ gluon and which does not contain by itself any $U(1)$ gluon.

The double line notation

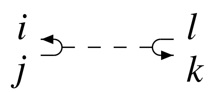
Replace a colour index in the adjoint representation by two indices in the fundamental representation:

$$V^a E^a \quad = \quad = \left(\sqrt{2} T^a_{ij} V^a\right) \left(\sqrt{2} T^b_{ji} E^b\right).$$

Then split a $SU(N)$ gluon into an $U(N)$-part and an $U(1)$-part:

$U(N):$ 

$= \delta_{il}\delta_{kj},$

$U(1):$ 

$= -\frac{1}{N}\delta_{ij}\delta_{kl}.$

One can show that the $U(1)$ gluon couples only to quarks.

The algorithm

Colour decomposition:

$$\mathcal{A} = \sum_i c_i A_i,$$

- Loop over all quark permutations.
- Loop over all cyclic permutations.
- Loop over all colour clusters.

Colour factor:

$$c_i = \left(-\frac{1}{N}\right)^{(n_{cluster}-1)} \times \prod_{j=1}^{n_{cluster}} c_{i,j}.$$

The colour factors $c_{i,j}$ of an individual colour cluster consist only of Kronecker delta's:

$$g_1, g_2, ..., g_n : \qquad c_{i,j} = \delta_{i_1 j_2} \delta_{i_2 j_3} ... \delta_{i_{n-1} j_n} \delta_{i_n j_1}.$$

Mixing of symbolical and numerical code

Colour decomposition:

$$\mathcal{A} = \sum_i c_i A_i,$$

In squaring the amplitude we obtain

$$|\mathcal{A}|^2 = \sum_{i,j} A_i \left(c_i P c_j^\dagger \right) A_j^*.$$

The matrix

$$M_{ij} = \left(c_i P c_j^\dagger \right)$$

is independet of any kinematical information and can be computed at initialization time. The program uses the C++ library GiNaC for the symbolic computation of this matrix.

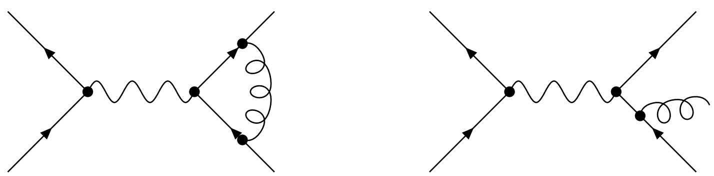
Checks: Comparison with Madgraph

Process	this work	Madgraph
$gg \to gg$	56203.4	56203.2
$g\bar{d} \to \bar{d}g$	8436.64	8436.62
$\bar{u}\bar{d} \to \bar{d}\bar{u}$	1374.01	1374.01
$\bar{d}\bar{d} \to \bar{d}\bar{d}$	1287.74	1287.74
$gg \to ggg$	21269.2	21269.3
$g\bar{d} \to \bar{d}gg$	3222.01	3222.02
$\bar{u}\bar{d} \to \bar{d}\bar{u}g$	56.459	56.4591
$\bar{d}\bar{d} \to \bar{d}\bar{d}g$	53.2424	53.2425
$gg \to gggg$	1354.24	1354.22
$g\bar{d} \to \bar{d}ggg$	138.691	138.689
$\bar{u}\bar{d} \to \bar{d}\bar{u}gg$	0.975563	0.975546
$\bar{d}\bar{d} \to \bar{d}\bar{d}gg$	0.902231	0.902215
$\bar{u}\bar{d} \to \bar{d}\bar{u}\bar{s}s$	0.0116469	0.0116467
$\bar{u}\bar{d} \to \bar{d}\bar{u}\bar{u}u$	0.0524928	0.0524927
$\bar{d}\bar{d} \to \bar{d}\bar{d}\bar{d}d$	0.0583822	0.0583821
$\bar{u}\bar{d} \to \bar{d}\bar{u}\bar{s}gs$	0.000453678	0.000453671
$\bar{u}\bar{d} \to \bar{d}\bar{u}\bar{u}gu$	0.00202449	0.00202446

Infrared divergences and the Kinoshita-Lee-Nauenberg theorem

In addition to ultraviolet divergences, loop integrals can have infrared divergences.

For each IR divergence there is a corresponding divergence with the opposite sign in the real emission amplitude, when particles becomes soft or collinear (e.g. unresolved).



The Kinoshita-Lee-Nauenberg theorem: Any observable, summed over all states degenerate according to some resolution criteria, will be finite.

General methods at NLO

Fully differential NLO Monte Carlo programs need a general method to handle the cancelation of infrared divergencies.

- Phase space slicing
  - e^+e^-: W. Giele and N. Glover, (1992)
  - initial hadrons: W. Giele, N. Glover and D.A. Kosower, (1993)
  - massive partons, fragmentation: S. Keller and E. Laenen, (1999)
- Subtraction method
  - residue approach: S. Frixione, Z. Kunzst and A. Signer, (1995)
  - dipole formalism: S. Catani and M. Seymour, (1996)
  - massive partons: L. Phaf and S.W. (2001), S. Catani, S. Dittmaier, M. Seymour and Z. Trócsányi, (2002)

The dipole formalism

The dipole formalism is based on the subtraction method. The NLO cross section is rewritten as

$$\begin{aligned}\sigma^{NLO} &= \int_{n+1} d\sigma^R + \int_n d\sigma^V \\ &= \int_{n+1} \left(d\sigma^R - d\sigma^A\right) + \int_n \left(d\sigma^V + \int_1 d\sigma^A\right)\end{aligned}$$

The approximation $d\sigma^A$ has to fulfill the following requirements:

- $d\sigma^A$ must be a proper approximation of $d\sigma^R$ such as to have the same pointwise singular behaviour in D dimensions as $d\sigma^R$ itself. Thus, $d\sigma^A$ acts as a local counterterm for $d\sigma^R$ and one can safely perform the limit $\varepsilon \to 0$.

- Analytic integrability in D dimensions over the one-parton subspace leading to soft and collinear divergences.

The subtraction terms

The approximation term $d\sigma^A$ is given as a sum over dipoles:

$$d\sigma^A = \sum_{pairs\, i,j} \sum_{k \neq i,j} \mathcal{D}_{ij,k}.$$

Each dipole contribution has the following form:

$$\mathcal{D}_{ij,k} = -\frac{1}{2p_i \cdot p_j} \mathcal{A}_n^{(0)\,*}\left(p_1, ..., \tilde{p}_{(ij)}, ..., \tilde{p}_k, ...\right) \frac{\mathbf{T}_k \cdot \mathbf{T}_{ij}}{\mathbf{T}_{ij}^2} V_{ij,k} \mathcal{A}_n^{(0)}\left(p_1, ..., \tilde{p}_{(ij)}, ..., \tilde{p}_k, ...\right).$$

- Colour correlations through $\mathbf{T}_k \cdot \mathbf{T}_{ij}$.
- Spin correlations through $V_{ij,k}$.

The physical origin of the correlations

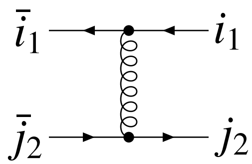
- In the soft limit, amplitudes factorize completely in spin space, but colour correlations remain.

- In the collinear limit, amplitudes factorize completely in colour space, but spin correlations remain.

Solution: Use colour decomposition and calculate helicity amplitudes.

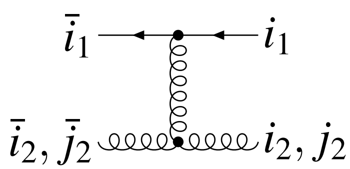
Colour correlations

Quark-antiquark:



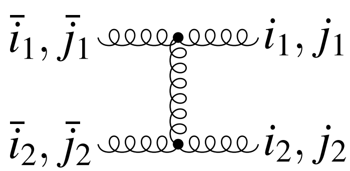
$$= -\frac{1}{2}\left(\delta_{\bar{i}_1\bar{j}_2}\delta_{j_2 i_1} - \frac{1}{N}\delta_{\bar{i}_1 i_1}\delta_{j_2\bar{j}_2}\right)$$

Quark-gluon:



$$= \frac{1}{2}\left(\delta_{\bar{i}_1 i_2}\delta_{\bar{i}_2 i_1}\delta_{j_2\bar{j}_2} - \delta_{\bar{i}_1\bar{j}_2}\delta_{j_2 i_1}\delta_{\bar{i}_2 i_2}\right)$$

Gluon-gluon:



$$= \frac{1}{2}\left(\delta_{\bar{i}_1 i_1}\delta_{\bar{i}_2 i_2}\delta_{j_1\bar{j}_2}\delta_{j_2\bar{j}_1} - \delta_{\bar{i}_1 i_1}\delta_{j_2\bar{j}_2}\delta_{j_1 i_2}\delta_{\bar{i}_2\bar{j}_1}\right.$$
$$\left. -\delta_{j_1\bar{j}_1}\delta_{\bar{i}_2 i_2}\delta_{\bar{i}_1\bar{j}_2}\delta_{j_2 i_1} + \delta_{j_1\bar{j}_1}\delta_{j_2\bar{j}_2}\delta_{\bar{i}_1 i_2}\delta_{\bar{i}_2 i_1}\right)$$

Reduction of tensor integrals

Loop momentum in the numerator:

$$I_n^r = \langle a_1 - |\gamma_{\mu_1}| b_1 - \rangle \dots \langle a_r - |\gamma_{\mu_r}| b_r - \rangle \int \frac{d^D k}{i\pi^{\frac{D}{2}}} \frac{k^{\mu_1} \dots k^{\mu_r}}{k^2 (k-p_1)^2 \dots (k - p_1 - \dots p_{n-1})^2}.$$

Use spinor methods to decompose k^μ into

$$k^\mu = c_1\, l_1^\mu + c_2\, l_2^\mu + c_3\, \langle l_1 - |\gamma^\mu| l_2 - \rangle + c_4\, \langle l_2 - |\gamma^\mu| l_1 - \rangle.$$

F. del Aguila and R. Pittau,

A. van Hameren, J. Vollinga and S.W.

Reduction of scalar integrals

Reduction of pentagons (W. van Neerven and J. Vermaseren; Z. Bern, L. Dixon, and D. Kosower):

$$I_5 = \sum_{i=1}^{5} b_i I_4^{(i)} + O(\varepsilon)\,.$$

Reduction of hexagons (T. Binoth, J. P. Guillet, and G. Heinrich):

$$I_6 = \sum_{i=1}^{6} b_i I_5^{(i)}.$$

Reduction of scalar integrals with more than six propagators (G. Duplancic and B. Nizic):

$$I_n = \sum_{i=1}^{n} r_i I_{n-1}^{(i)}.$$

Here, the decomposition is no longer unique.

Performance of the tensor reduction

r / n	0	1	2	3	4	5	6	7	8	9	10
2	$1\cdot10^{-6}$	$5\cdot10^{-6}$	$2\cdot10^{-5}$								
3	$1\cdot10^{-6}$	$2\cdot10^{-5}$	$2\cdot10^{-4}$	$1\cdot10^{-3}$							
4	$2\cdot10^{-6}$	$5\cdot10^{-5}$	$4\cdot10^{-4}$	$2\cdot10^{-3}$	$6\cdot10^{-3}$						
5	$3\cdot10^{-5}$	$1\cdot10^{-4}$	$6\cdot10^{-4}$	$3\cdot10^{-3}$	$9\cdot10^{-3}$	0.03					
6	$2\cdot10^{-4}$	$3\cdot10^{-4}$	$9\cdot10^{-4}$	$4\cdot10^{-3}$	0.02	0.04	0.1				
7	$7\cdot10^{-4}$	$7\cdot10^{-4}$	$1\cdot10^{-3}$	$5\cdot10^{-3}$	0.02	0.06	0.1	0.4			
8	$3\cdot10^{-3}$	$3\cdot10^{-3}$	$4\cdot10^{-3}$	$8\cdot10^{-3}$	0.02	0.07	0.2	0.6	1.8		
9	0.01	0.01	0.01	0.02	0.03	0.08	0.3	0.9	2.6	7	
10	0.05	0.05	0.05	0.06	0.06	0.2	0.4	1.1	3.5	8	25

CPU time in seconds for a tensor integral with n external legs and rank r for $r \leq n$ on a standard PC (Pentium IV with 2 GHz).

Summary and outlook

Automated NLO calculations:

- Matrix elements: Colour decomposition, spinor methods and recurrence relations.

- New developments: Twistor space, MHV vertices and BCF recursion relations.

- Subtraction terms: Dipole formalism, spin- and colour-correlations.

- Loop integrals: Tensor reduction, reduction of higher-point scalar integrals.

To be done:

- Automated colour decomposition of one-loop amplitudes