Soft Theorems in Scattering Amplitudes

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New Postdoc in Uni HH

H.L., P. Mastrolia and W.J. Torres Bobadilla, Phys.Rev. D91(2015) 065018
H.L. & Y. Du, JHEP 1301(2013)129
H.L. & Y. Du, working in progress
Y. Huang., H. L., C. Wen, in preparation

@DESY, Nov 10th 2015

Why Scattering Amplitudes?

- Gauge invariant on-shell scattering amplitudes as input for computing the (differential) cross-section

- No. of particles in scattering increasing
  → No. of Feynman diagrams increasing exponentially

n=	4	5	6	7	8	9	10
	4	25	220	2485	34300	559,405	10,525,900

However, the on-shell scattering amplitudes can be written compactly and simply, e.g. n-gluon scattering amplitude

$$A_n[1^+ \ldots i^- \ldots j^- \ldots n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

- Feynman Diagrams (starting from Lagrangian) depend on
  # gauge choice
  # field redefinitions

- Work only with on-shell invariant input, avoiding complications in Feynman diagrams

- On-shell method: on-shell recursion relations
  - BCFW [Britto, Cachazo, Feng & Witten, 05']
  - CSW
  - multi-line shifts …

On-Shell Recursion Relation

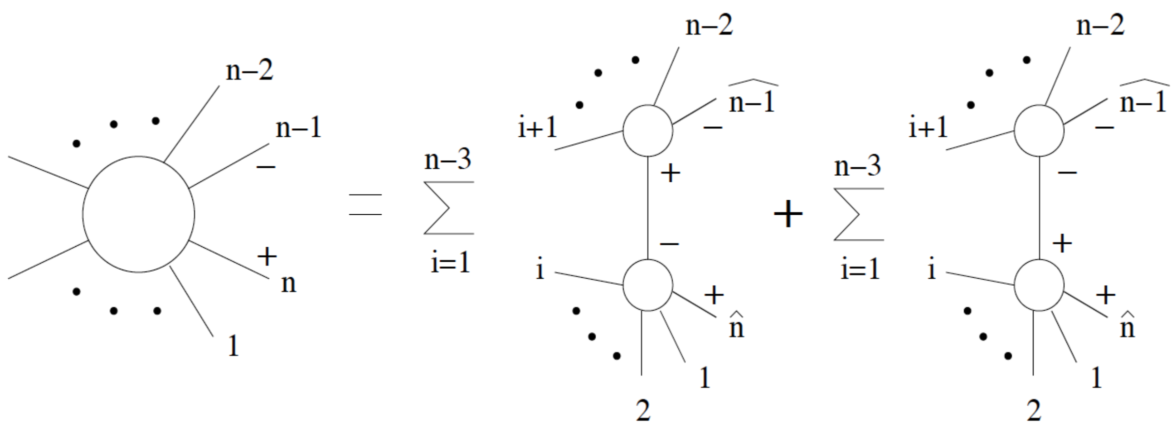
- Idea: Build up n-point amplitudes from lower-point on-shell amplitudes
- Scattering amplitudes: determined by poles through complex deformation of external momenta: $\frac{1}{(p+p_i(z))^2} = \frac{1}{(p+p_i)^2 + z(2q\cdot(p+p_i))}$

$$I = \oint \frac{dz}{z} A(z) \qquad I = A(z=0) + \sum_{z_\alpha} \text{Res}\left(\frac{A(z)}{z}\right)_{z_\alpha}$$

- If no boundary contribution: $\left(\frac{A(z)}{z}\right)_{z_\alpha} = -\sum_{h=\pm} A_L(p_i(z_\alpha), p^h(z_\alpha)) \frac{1}{P_\alpha^2} A_R(-p^{-h}(z_\alpha), p_j(z_\alpha))$

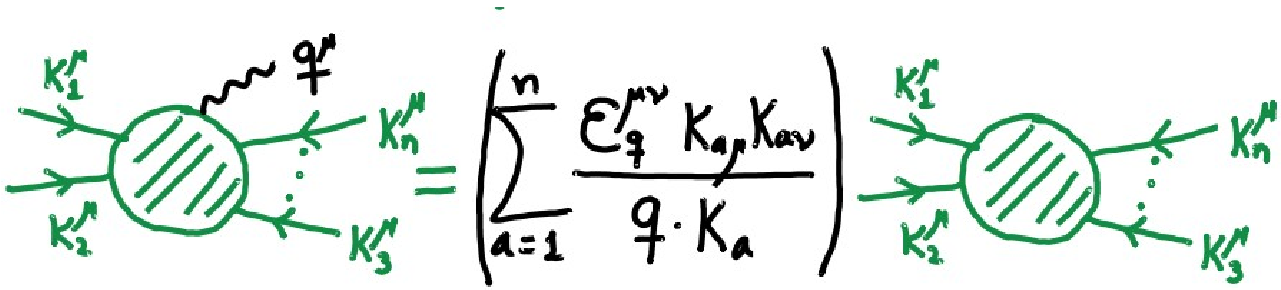
$z \to \infty$

$A(z) \to 0$



Why soft theorems?

➢ Universal properties of low energy particle emissions, Weinberg's soft theorem



$$= \left(\sum_{a=1}^{n} \frac{\varepsilon_q^{\mu\nu} K_{a\mu} K_{a\nu}}{q \cdot K_a} \right)$$

[Cachazo, Amplitude 2015]

➢ A powerful constraint to help to derive

- universality of gravitational coupling
- electric charge conservation [Weinberg, 1965]
- no particles with helicities larger than 2
- an evidence for Bondi-van der Burg-Matzner-Sachs (BMS) group, symmetries of asymptotic spacetime [Cachazo & Strominger, 14']

Soft-Limit Behaviors at Tree-Level

- *Graviton Amplitudes obeying a soft identity* [Cachazo & Strominger, 14']

$$\mathcal{M}_{n+1}(k_1, k_2, ...k_n, q) = \left(S^{(0)} + S^{(1)} + S^{(2)}\right) \mathcal{M}_n(k_1, k_2, ...k_n) + \mathcal{O}(q^2)$$

$$S^{(0)} \equiv \sum_{a=1}^{n} \frac{E_{\mu\nu} k_a^\mu k_a^\nu}{q \cdot k_a} \qquad S^{(1)} \equiv -i \sum_{a=1}^{n} \frac{E_{\mu\nu} k_a^\mu (q_\rho J_a^{\rho\nu})}{q \cdot k_a} \qquad S^{(2)} \equiv -\frac{1}{2} \sum_{a=1}^{n} \frac{E_{\mu\nu} (q_\rho J_a^{\rho\mu})(q_\sigma J_a^{\sigma\nu})}{q \cdot k_a}$$

- *Extensions of the soft-limit topic*

➢ In Different Theoretical Frames:

Pure-YM, String theory, with/without SUSY…

➢ With Different Methods:

BCFW, Scattering Equation; Conformal Invariance; Gauge Invariance…

➢ In Different Dimensions:

4-dim; D-dim

➢ Involve Quantum Contributions:

Loop-correction; Soft-Collinear ET;…

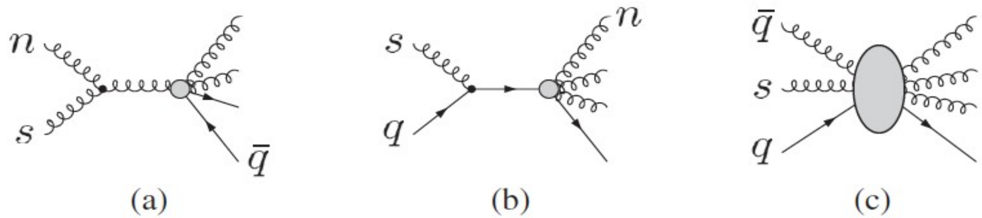
[Bern, Davies, Vecchia & Nohle;
Geyer, Lipstein, Mason;
Schwab & Volovich;
Larkoski;
E. Casali;
Broedel, Leeuw, Plafka & Rosso;
He, Huang & Wen;
Bonocore, Laenen, Magnean, Vernazza & White;
Afkhami-Jeddi;
…]

• *Soft-Gluon Limit in QCD Amplitude:* [HL, Mastrolia & Torres Bobadilla, 14']

$$A_n(k_s; k_1, \ldots, k_{n-1}) = \left[S_{\rm G}^{(0)} + S_{\rm G}^{(1)}\right] A_{n-1}(k_1, \ldots, k_{n-1}) + \mathcal{O}(k_s)$$

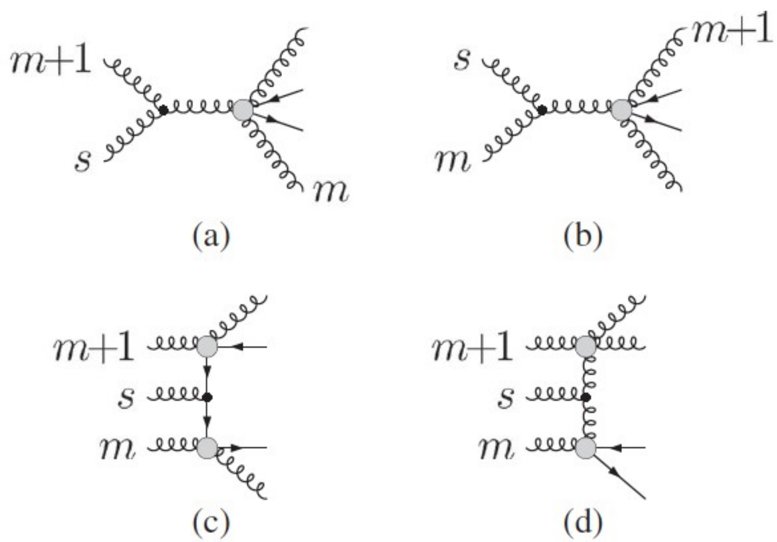
BCFW ⟺ **Spinorial Notation** ⟺ Gauge-Invariance Derivation

➢ Case 1: Soft gluon adjacent to one quark and one gluon



Combine results of pure-YM and soft-photon from fermionic emitter

➢ Case 2: Soft gluon adjacent to two gluons

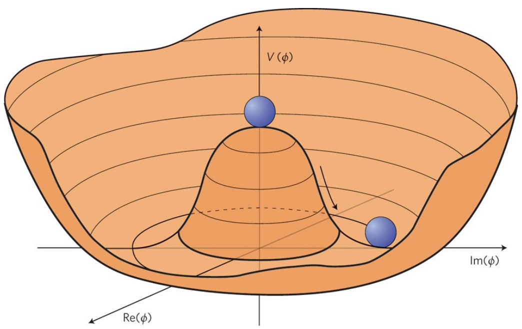


Although, results are similar to pure-YM, However, physics insight is different according to diagram (c) and (d).

Soft Behaviors and Symmetries

- Weinberg soft theorem as Ward identity of BMS group
- Soft structures might connect to some hidden symmetries, e.g. underlying patterns of spontaneously symm. breaking

- *Soft limits for massless Goldstone bosons of spontaneously broken symmetry can be studied via Amplitude*



- Single soft emission: Adler zero
- Double soft emission:

$$\lim_{\delta\to 0} \mathcal{A}_{n+2}(\phi^i(\delta q_1), \phi^j(\delta q_2), 3, \ldots n+2) = \sum_{a=3}^{n+2} \frac{p_a\cdot(q_1-q_2)}{p_a\cdot(q_1+q_2)} f^{ijk}\hat{T}_k \mathcal{A}_n(3, \ldots n+2)$$

[Plefka, Amplitude 2015]

One can read out symmetry algebra from double soft limit (rotation in the vaccum)! [Arkani-Hamed, Cachazo, Kaplan, 08']

Examples: Soft pions, Hidden E7(7) symmetry in N=8 SUGRA

[Kampf, Novotny & Trnka, 13'; Cachazo, He & Ye, 15'; Du & H.L.,15']

Non-linear sigma model $SU(N) \times SU(N) \to SU(N)$

- *Lagrangian for NLSM with Cayley parameterization*

$$\mathcal{L} = \frac{F^2}{4} \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) \qquad U = 1 + 2\sum_{n=1}^{\infty} \left(\frac{1}{2F}\phi\right)^n$$

- Vertices: $V_{2n+1} = 0$, **Odd-point amplitude vanishes in NLSM**

$$V_{2n+2} = \left(-\frac{1}{2F^2}\right)^n \left(\sum_{i=0}^{n} p_{2i+1}\right)^2 = \left(-\frac{1}{2F^2}\right)^n \left(\sum_{i=0}^{n} p_{2i+2}\right)^2$$

- *Color-like (Flavor) Decomposition:* [Kampf, Novotny & Trnka, 2013]

$$M(1^{a_1}, \ldots, n^{a_n}) = \sum_{\sigma \in S_{n-1}} \mathrm{Tr}(T^{a_1} T^{a_{\sigma_2}} \ldots T^{a_{\sigma_n}}) A(1, \sigma)$$

- *Berends-Giele recursion for NLSM with*

$$J(2, ..., 2n)$$

$$= \frac{i}{P_{2,2n}^2} \sum_{m=2}^{n} \sum_{\text{Divisions}} iV_{2m}(p_1 = -P_{2,2n}, P_{A_1}, \cdots, P_{A_{2m-1}}) \times \prod_{k=1}^{2m-1} J(A_k)$$

- Divisions: all possible divisions of on-shell particle $\{2, \ldots, 2n\} \to \{A_1\}, \ldots, \{A_{2m-1}\}$

Soft Behaviors of the off-shell currents & on-shell limits:

- *Single soft behaviors* (τ parameterizes the soft momentum)

$$J(2,\ldots,i-1,\widetilde{i},i+1,\ldots,2n)=\left\{\begin{array}{cc} 0 & (i \text{ is even}) \\ \left(\frac{1}{2F^2}\right) J(2,\ldots,i-1)J(i+2,\ldots,2n) & (i \text{ is odd}) \end{array}\right\}+\mathcal{O}(\tau),$$

Taking on-shell limit $P^2_{2,2n}\to 0.$ → " *Adler Zero* "

Soft Limit and on-shell limit can be exchanged

- *Double soft behaviors*

$$J(2,\ldots,i-1,\widetilde{i},\widetilde{i+1},i+2,\ldots,2n)$$

$$=\tau^0 S^{(0)}_{i,i+1}J(2,\ldots,i-1,i+2,\ldots,2n)+\tau^1\Bigg[S^{(1)}_{i,i+1}J(2,\ldots,i-1,i+2,\ldots,2n)$$

$$+\left\{\begin{array}{cc} \left(\frac{1}{2F^2}\right) J^{(1)}(2,\ldots,i-1,\widetilde{i})J(i+2,\ldots,2n) & (i \text{ is even }) \\ \left(\frac{1}{2F^2}\right) J(2,\ldots,i-1)J^{(1)}(\widetilde{i+1},i+2,\ldots,2n) & (i \text{ is odd}) \end{array}\right\}\Bigg]+\mathcal{O}(\tau^2)$$

Taking on-shell limit $P^2_{2,2n}\to 0.$ →

Soft Limit and on-shell limit can be exchanged

$$A(1,\ldots,\widetilde{i},\widetilde{i}+1,\ldots,2n)=\left(\tau^0\mathbb{S}^{(0)}_{i,i+1}+\tau^1\mathbb{S}^{(1)}_{i,i+1}\right)A(1,\ldots,i-1,i+2,\ldots,2n)+\mathcal{O}(\tau^2)$$

• Possible implementations of soft theorems

- Single gluon soft limit in QCD amplitude and the double soft Goldstone bosons structures in NLSM to sub-leading order. It's quite interesting to discover the hidden (if exists) symmetry which makes the sub-(sub-)leading soft behaviors universal

- Soft limits of Goldstone-boson amplitudes encode underlying patterns of symm. breaking (e.g. in NLSM), which can also be implemented in N=8 SUGRA, where the classical theory has global continuous E7(7) symm. broken to SU(8)

- These scalar limits can be used to test the candidate counter terms for high-loop orders in N=8 SUGRA, in principle they should be E7(7)-compatible and match the scalar soft limits factorization. Only one 7-loop counter term $D^8 R^4$ pass the test of single and double scalar limits up to 6-point, can multi-scalar limits be further constraints to test its E7(7) compatibility [Huang, H.L. & Wen, in preparation]

[Beisert, Elvang, Freedman, Kiermaier, Morales & Stieberger, 10']

Thanks!

Back-Up:

Bondi-van der Burg-Metzner-Sachs(BMS) symm.:

- Study of classical gravitational waves: Expected Poincaré symmetry enlarged by BMS_4 group
- Acts at null infinity ($\mathcal{I}^{\pm}$) for asympt. flat space-times
- Coordinates: u (retarded time), r (radius), $x^A = \{\Theta, \phi\} \in S^2$ at $\mathcal{I}^{\pm}$

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}\,du\,dr + g_{AB}(dx^A + U^A du)(dx^B + U^B du)$$

Metric functions β, V, U^A, g_{AB} have fall-off conditions in r:

$$g_{AB} = r^2(d\Theta^2 + \sin^2\Theta\,d\phi^2) + \mathcal{O}(r), \quad \beta = \mathcal{O}(r^{-2}), \quad \frac{V}{r} = \mathcal{O}(r), \quad U^A = \mathcal{O}(r^{-2})$$

- BMS_4 group: Maps asymptotically flat space-times onto themselves

$$\Theta' = \Theta'(\Theta, \phi) \qquad \phi' = \phi'(\Theta, \phi) \qquad u' = K(\Theta, \phi)\,(u - \alpha(\Theta, \phi))$$

Where $(\Theta, \phi) \to (\Theta', \phi')$ is conformal transformation on S^2:

$$d\Theta'^2 + \sin^2\Theta' d\phi'^2 = K(\Theta, \phi)^2(d\Theta^2 + \sin^2\Theta\,d\phi^2)$$

- For $\Theta' = \Theta$ & $\phi' = \phi$ one has "supertranslations": $u' = u - \alpha(\Theta, \phi)$ with a general function $\alpha(\Theta, \phi)$.

● BMS4 *Algebra:*

In standard complex coordinates $z = e^{i\phi}\cot(\Theta/2)$ conformal symmetry generated by Virasoro generators ("superrotations")

$$l_n = -z^{n+1}\,\partial_z \qquad \bar{l}_n = -\bar{z}^{n+1}\,\partial_{\bar{z}}$$

Supertranslations generated by $T_{m,n} = z^m\,\bar{z}^n\,\partial_u$

Extended $\mathfrak{bms}_4$ algebra [Barnich, Troessart]

$$[l_n, l_m] = (m-n)\,l_{m+n} \qquad [\bar{l}_n, \bar{l}_m] = (m-n)\,\bar{l}_{m+n}$$

$$[l_l, T_{m,n}] = -m\,T_{m+l,n} \qquad [\bar{l}_l, T_{m,n}] = -n\,\bar{T}_{m,n+l}$$

Poincaré subalgebra spanned by $\underbrace{l_{-1}, l_0, l_1; \bar{l}_{-1}, \bar{l}_0, \bar{l}_1}_{\text{Lorentz}} \quad \underbrace{T_{0,0}, T_{0,1}, T_{1,0}, T_{1,1}}_{\text{Translation}}$

BMS_4 group maps gravitational wave solutions onto each other.

Claim: | Supertranslations $\hat{=}\ S_{\mathsf{G}}^{(0)}$ | Superrotations $\hat{=}\ S_{\mathsf{G}}^{(1)}$ | [Cachazo, Strominger]

On-shell Limits of the Adjacent Double Soft Behaviors

- *<u>Boundary case:</u> Notice the orders in taking soft and on-shell limits:*

➢ While taking the on-shell limit of the off-shell leg $P^2_{2,2n} \to 0$ after deriving the soft limits, there is a 0/0 illed form

➢ In the boundary case, the on-shell limit should be imposed first, then the soft limits

- *<u>Other cases</u>: soft and on-shell limits can be exchanged*

- *With a careful treatment, the double soft behaviors of the amplitudes in the NLSM can be achieved as*

$$A(1,\ldots,\widetilde{i},\widetilde{i}+1,\ldots,2n) = \left(\tau^0 \mathbb{S}^{(0)}_{i,i+1} + \tau^1 \mathbb{S}^{(1)}_{i,i+1}\right) A(1,\ldots,i-1,i+2,\ldots,2n) + \mathcal{O}(\tau^2)$$

$$\mathbb{S}^{(0)}_{i,i+1} = \left(-\frac{1}{2F^2}\right)\frac{1}{2}\left[\frac{k_{i-1}\cdot(p-q)}{k_{i-1}\cdot(p+q)} + \frac{k_{i+2}\cdot(q-p)}{k_{i+2}\cdot(q+p)}\right]$$

$$S^{(0)}_{i,i+1} = \mathbb{S}^{(0)}_{i,i+1}$$

$$\mathbb{S}^{(1)}_{i,i+1} = \left(-\frac{1}{2F^2}\right)(p\cdot q)\left[\frac{k_{i-1}\cdot q}{(k_{i-1}\cdot(p+q))^2} + \frac{k_{i+2}\cdot p}{(k_{i+2}\cdot(p+q))^2}\right] + \left(-\frac{1}{2F^2}\right)\left[\frac{p_\mu q_\nu}{k_{i-1}\cdot(p+q)}\mathcal{J}^{\mu\nu}_{i-1} + \frac{q_\mu p_\nu}{k_{i+2}\cdot(p+q)}\mathcal{J}^{\mu\nu}_{i+2}\right]$$

$$S^{(1)}_{i,i+1} = \mathbb{S}^{(1)}_{i,i+1}$$

$$\mathcal{J}^{\mu\nu}_a \equiv k^\mu_a \frac{\partial}{\partial k_{a,\nu}} - k^\nu_a \frac{\partial}{\partial k_{a,\mu}}$$