Generic dijet soft functions to two-loop order

Guido Bell || **Rudi Rahn** || Jim Talbert

22 March 2016 || SCET2016, Hamburg

UNIVERSITÄT SIEGEN
Oxford Physics
UNIVERSITY OF OXFORD

Outline

1. ***Automating SCET***
   (a) Motivation and starting point
   (b) Generic dijet soft functions
   (c) Measurement functions
   (d) SecDec

2. ***Advances and new features***
   (e) SCET_{II}
   (f) Improved numerics
   (g) Universal analytic structures

3. ***Results***

Motivation

- Resummation in full QCD automated in *CAESAR/ARES (NLL / e^+e^- NNLL)*

  [Banfi, Salam, Zanderighi, '04]
  [Banfi, McAslan, Monni, Zanderighi, '14]

- Resummation accuracy in SCET overtaking full QCD

- Yet we proceed observable by observable individually:

  - Thrust
    [Becher, Schwartz, '08]
  - C-Parameter
    [Hoang, Kolodrubetz, Mateu, Stewart, '14]
  - Angularities
    [Bell, Hornig, Lee, Talbert, in 30 min]
  - …
  - Threshold Drell-Yan
    [Becher, Neubert, Xu, '07]
  - W/Z/H @ large p_T
    [Becher, Bell, Lorentzen, Marti, '13,'14]
  - Jet veto
    [Becher et al. '13, Stewart et al., '13]
  - …

Resummation in SCET$_\text{i}$

SCET$_\text{I}$

- Collinear and soft scales *different*
- Jet and soft functions *factorisable*

$$\sigma \sim H(Q,\mu)\, J(Q\lambda,\mu) \otimes S(Q\lambda^2,\mu)$$

- Solve RG equations for hard, jet and soft functions *individually*, e.g.

$$H(Q,\mu) = U_H(\mu,\mu_H) H(Q,\mu_H)$$

- Resummation requires *anomalous dimensions, matching corrections*

$$\Longrightarrow \quad \Gamma_{Cusp},\ \gamma_{H,J,S},\ c_{H,J,S}$$

SCET$_\text{II}$

- Collinear and soft scales *equal*
- Jet and soft function factorisable only with *additional rapidity regulator*

$$\sigma \sim H(Q,\mu)\, J(Q\lambda,\nu,\mu) \otimes S(Q\lambda,\nu,\mu)$$

- ν*-independence* enforces exponentiation of rapidity logarithms

$$JS \sim \underbrace{\left(Q^2 x_T^2\right)}_{\lambda^{-2}}{}^{-F(x_T,\mu)} W(x_T,\mu)$$

- Resummation requires *anomaly exponent, remainder function*

$$\Longrightarrow \quad \Gamma_{Cusp}, \gamma_H, F, c_H, W$$

Resummation ingredients

To achieve NNLL resummation, we need the soft anomalous dimension or anomaly exponent to two-loop accuracy

Logarithmic accuracy	$\boldsymbol{\Gamma_{Cusp}}$	$\boldsymbol{\gamma_H, \left\{ \begin{array}{c} \gamma_J, \gamma_S \\ F \end{array} \right.}$	$\boldsymbol{c_H, \left\{ \begin{array}{c} c_J, c_S \\ W \end{array} \right.}$
LL	1-loop	tree	tree
NLL	2-loop	1-loop	tree
NNLL	3-loop	2-loop	1-loop
$\mathrm{N^3LL}$	4-loop	3-loop	2-loop

- The missing 2-loop ingredients for *NNLL* resummation of *hadronic* event shapes can be obtained from *dijet* soft functions: [Becher, Garcia, Piclum, '15]

[Piclum, in 30h 25']

$$S(\tau,\mu) = \frac{1}{N_c}\sum_X \mathcal{M}(\tau,k_i)\ \mathrm{Tr}\ |\langle 0|S^\dagger_{\bar{n}}(0)S_n(0)|X\rangle|^2 \qquad S_n(x) = Pexp(ig_s\int_{-\infty}^0 n\cdot A_s(x+sn)ds)$$

Generic dijet soft functions

- We consider soft functions of the form:

$$S(\tau,\mu) = \frac{1}{N_c}\sum_X \mathcal{M}(\tau, k_i)\ \mathrm{Tr}\ |\langle 0|S_{\bar{n}}^{\dagger}(0)S_n(0)|X\rangle|^2 \qquad S_n(x) = Pexp(ig_s\int_{-\infty}^{0} n\cdot A_s(x+sn)ds)$$

- The **matrix element** of soft wilson lines is *independent of the observable*. It contains the universal (implicit) UV/IR-divergences of the function.

- The **measurement function** (M) encodes all of the information of the particular observable at hand. It is *independent of the singularity structure*.

- **Idea**: isolate singularities at each order and calculate the associated coefficient numerically:

$$\boxed{\mathcal{S}(\tau,\mu) \sim 1 + \alpha_s\left(\frac{c_2}{\varepsilon^2} + \frac{c_1}{\varepsilon} + c_0\right) + \mathcal{O}(\alpha_s^2)}$$

- To avoid distribution valued measurement functions and facilitate renormalisation, we work in *Laplace space.*

Generic dijet soft functions: NLO and NNLO

$$\mathcal{S}^{(n)}(\tau,\mu) = \frac{\mu^{2n\varepsilon}}{(2\pi)^{n(d-1)}} \left(\prod_{i=1}^{n} \int \mathrm{d}^d k_i \, \delta(k_i^2)\, \theta(k_i^0)\, \underbrace{\mathcal{R}_\alpha(\nu, k_i)}_{\text{analytic regulator}} \right) \underbrace{|\mathcal{A}^{(n)}(\{k_i\},\mu)|^2}_{\text{Matrix element}} \, \underbrace{\mathcal{M}(\tau, \{k_i\})}_{\text{measurement function}}$$

- The above structure is generic, use n=1 for *NLO*, n=1,2 for *NNLO*, and the appropriate matrix elements, analytic regulator (for $\mathrm{SCET_{II}}$) and measurement function

- Parameterising in terms of total transverse momentum p_T and rapidity y of the radiated system, and for *NNLO* a measure for rapidity differences a , the ratio of transverse momenta b, and the angle θ between emissions, assume:

$$\mathcal{M}^{(1)}(\tau,k) = \exp\left(-\tau\, p_T\, y^{\frac{n}{2}}\; f\left(y, \vartheta_k\right)\right)$$
$$\mathcal{M}^{(2)}(\tau,k,l) = \exp\left(-\tau\, p_T\, y^{\frac{n}{2}}\; F\left(y, a, b, \theta, \vartheta_k, \vartheta_l\right)\right)$$

- We assume the observable is *measured with respect to an arbitrary axis:* $\vartheta_k = \angle(v_\perp, k_\perp),\ \vartheta_l = \angle(v_\perp, l_\perp)$

- This form is both generic enough for a wide range of observables, and can be easily generalised.

Measurement functions: NLO examples

$$\mathcal{M}^{(1)}(\tau, k) = \exp\left(-\tau\, p_T\, y^{\frac{n}{2}}\, f\,(y, \vartheta)\right)$$

Observable	n	$f(y, \vartheta)$
Thrust	1	1
Angularities	$1-A$	1
Recoil-free broadening	0	$1/2$
C-Parameter	1	$1/(1+y)$
Threshold Drell-Yan	-1	$1+y$
W @ large p_T	-1	$1+y-2\sqrt{y}\,\cos\theta$
e^+e^- transverse thrust	1	$\frac{1}{s\sqrt{y}}\left(\sqrt{\left(c\cos\theta+\left(\frac{1}{\sqrt{y}}-\sqrt{y}\right)\frac{s}{2}\right)^2+1-\cos^2\theta}-\left\lvert c\cos\theta+\left(\frac{1}{\sqrt{y}}-\sqrt{y}\right)\frac{s}{2}\right\rvert\right)$

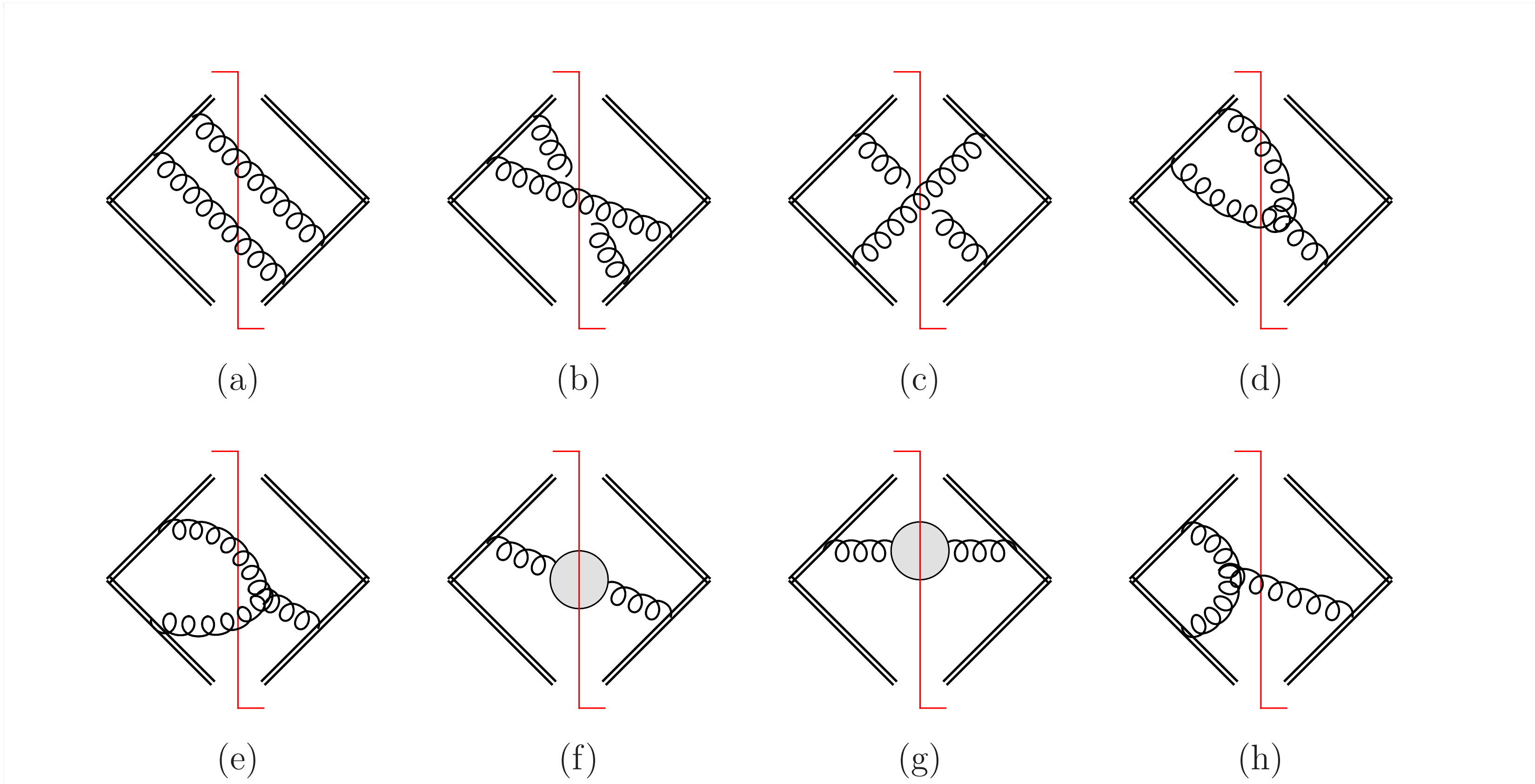
- For transverse thrust, $s = \sin\theta_B$, $c = \cos\theta_B$, with $\theta_B = \angle$beam axis, thrust axis

NLO vs. NNLO

- *NLO* is straightforward:

$$|\mathcal{A}(k)|^2 \sim \frac{\alpha_s C_F}{k_+ k_-}$$

- *NNLO* has more colour structures:



- and overlapping divergences, e.g. $C_F T_F\, n_f$ structure

$$|\mathcal{A}(k,l)|^2 = 128\pi^2 \alpha_s^2 C_F T_F n_f \frac{2k\cdot l(k_- + l_-)(k_+ + l_+) - (k_- l_+ - k_+ l_-)^2}{(k_- + l_-)^2 (k_+ + l_+)^2 (2k\cdot l)^2}$$

Numerical evaluation

- We perform the p_T and most angle integrations analytically, and hand the rest over to the program *SecDec*

- *SecDec* was developed for loop integrals with overlapping divergences, and provides interfaces to different integrators. We use the *general* branch of the program.

https://secdec.hepforge.org

Borowka, Heinrich, Jones, Kerner, Schlenk, Zirke

- *SecDec* resolves one overlapping divergence, and calls the *Cuba* library or *BASES* on a 6-dimensional integral

Output: Bare soft functions $\longrightarrow$ γ_s, finite terms

Limitations

Problems from last year's SCET workshop:

- No independent checks
  - ➞ three independent approaches are now available $\left\{\begin{array}{l}\textit{SecDec}\\ \textit{C++}\\ \text{Analytic}\end{array}\right.$
- SCET_{II} requires a second regulator, which SecDec doesn't provide
  - ➞ now possible in private code, to be implemented in *SecDec4*
- abysmal convergence for angle dependent observables: 1 week for 10^{-3} accuracy
  - ➞ analytic pre-treatment improves numerics drastically
- We focus on *NLO* and the $C_F\, C_A$ and $C_F\, T_F\, n_f$ colour structures for *NNLO*
  - ➞ no C_F^2 -Terms by choice, we assume non-abelian exponentiation (for now)

SCET$_{\mathrm{II}}$

- While support for a second regulator is being added to *SecDec4*, we didn't want to wait
  → C++ program developed specifically for SCET$_{\mathrm{II}}$

- We use a variation of the analytic regulator in [Becher, Bell, '12]:

$$\boxed{R_\alpha(\nu; k_i) = \frac{\nu}{k_i^+ + k_i^-}}$$

- This form is symmetric under $k_i^+ \leftrightarrow k_i^-$ exchange and parton relabelling, and easy to implement in a program.

- The subtractions are performed manually, and we integrate everything using the *Cuba* library [Hahn, '04]

- We are a bit slower for SCET$_{\mathrm{I}}$ observables than *SecDec*, but have improved error estimates, and can compute SCET$_{\mathrm{II}}$ observables, and a few problematic SCET$_{\mathrm{I}}$ ones.

- The SCET$_{\mathrm{II}}$ branch computes all α, ε-divergent and -finite terms

Advances: precision

- **Problem**: Logarithmic and square root divergences at the integration boundaries slowed the integral convergence, and upset the error estimate.

- **Solution**: Substitute wisely

  - $$\int_0^1 \mathrm{d}y\, \log y \xrightarrow{y \to x^2} \int_0^1 \mathrm{d}x\, 4x \log x$$
  - $$\int_0^1 \mathrm{d}v\, \frac{\log v}{\sqrt{v}} \xrightarrow{v \to w^4} \int_0^1 \mathrm{d}w\, 16w \log w$$
  - $$\int_0^1 \mathrm{d}u\, \frac{1}{\sqrt{1-u}\sqrt{u}} \xrightarrow{u \to 1-(1-z^4)^4} \int_0^1 \mathrm{d}z\, \frac{16\, z\, (1-z^4)}{\sqrt{2-z^4}\sqrt{2-2z^4+z^8}}$$

- These also work for plus distributions of the form $\left[\frac{\log^n x}{x}\right]_+$

- With these substitutions, we get 10^{-7} precision for $\mathrm{SCET_I}$ observables in <1h on an 8 core desktop machine

Hitting machine precision

- Transverse Thrust cannot be computed using SecDec, due to its structure:

$$“\frac{1}{\sqrt{x}}\left(\sqrt{\frac{1}{x}+2\Delta}-\sqrt{\frac{1}{x}}\right)“$$

, but

$$\lim_{x\to 0}\frac{1}{\sqrt{x}}\left(\sqrt{\frac{1}{x}+2\Delta}-\sqrt{\frac{1}{x}}\right)=\Delta$$

- For the actual *NNLO* function, for all input variables at 10^{-8}, the two roots differ in the ***75th digit***
  ➡ *Fortran* and C++ *double* type variables evaluate to 0

- So any +-distribution of the form $\frac{\log F(x) - \log F(0)}{x}$ is seen as $\frac{\log 0 - \log \Delta}{x}$,

  rather than $\frac{\log(\Delta+\epsilon) - \log \Delta}{x}$, for small x.

- **Solution**: Add a branch to our program, using the *cpp_dec_float_100* variables defined in the *boost* library to provide enough digits for this type of calculation
- The program becomes slower (10^{-4} accuracy after a few hours), but at least we get results.

Universal analytic structures

- We found a phase space parameterisation that resolves *all* overlapping divergences
- The *NLO/NNLO* measurement functions are linked via infrared and collinear safety

$$\boxed{\begin{aligned} F(y,a,b,\theta,\vartheta_k,\vartheta_l) &\xrightarrow{a\to 1,\vartheta_k\to\vartheta_l} f(y,\vartheta_l) \\ F(y,a,b,\theta,\vartheta_k,\vartheta_l) &\xrightarrow{b\to 0} f(y,\vartheta_l) \end{aligned}}$$

- This allows us to derive analytic formulae for anomalous dimensions
- Example: Angularities

$$\gamma_1^{C_A}(A) = -\frac{808}{27} + \frac{11\pi^2}{9} + 28\zeta_3 - \int_0^1\int_0^1 \mathrm{d}a\,\mathrm{d}b\,\frac{32a^2\left(1+ab+b^2\right)\left(a\left(1+b^2\right)+(a+b)(1+ab)\right)}{b(1-a)(1+a)(a+b)^2(1+ab)^2}\ln\left(\frac{\left(a^A+ab\right)\left(a+ba^A\right)}{a^A(1+ab)(a+b)}\right)$$

$$\gamma_1^{n_f}(A) = \frac{224}{27} - \frac{4\pi^2}{9} - \int_0^1\int_0^1 \mathrm{d}a\,\mathrm{d}b\,\frac{64a^2\left(1+b^2\right)}{(1-a)(1+a)(a+b)^2(1+ab)^2}\ln\left(\frac{\left(a^A+ab\right)\left(a+ba^A\right)}{a^A(1+ab)(a+b)}\right)$$

Results - Systematics

- For SCET$_\mathrm{I}$ the soft function fulfils the RG equation

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\ln\mu}S(\tau,\mu) = -\frac{1}{n}\Big[4\,\Gamma_{Cusp}(\alpha_s)\ln(\mu\bar{\tau}) - 2\,\gamma^S(\alpha_s)\Big]S(\tau,\mu)} \qquad \bar{\tau} = \tau e^{\gamma_E}$$

Anomalous dimension $\gamma^S(\alpha_s) = \sum_{n=0}^{\infty}\gamma_n^S\left(\frac{\alpha_s}{4\pi}\right)^{n+1}$

Log independent part $c^S(\alpha_s) = \sum_{n=0}^{\infty}c_n^S\left(\frac{\alpha_s}{4\pi}\right)^{n}$

$\Rightarrow$ $\boxed{\gamma_1^S,\ c_2^S}$

- For SCET$_\mathrm{II}$, the anomaly exponent fulfils the equation

$$\boxed{\frac{\mathrm{d}F(\tau,\mu)}{\mathrm{d}\ln\mu} = 2\,\Gamma_{Cusp}(\alpha_s)}$$

Log independent part $d = \sum_{i=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^{n} d_n$ $\Rightarrow$ $\boxed{d_2}$

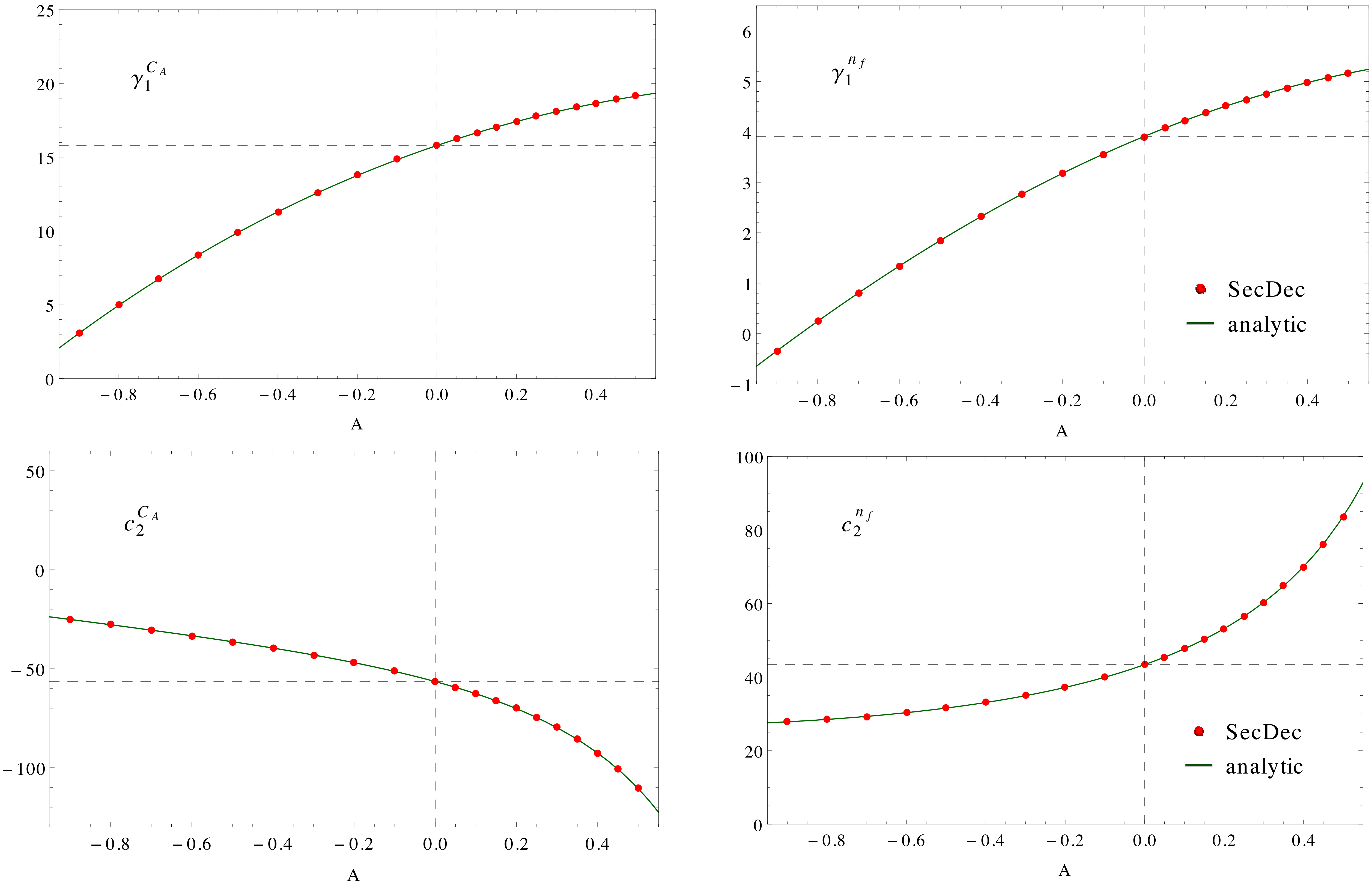
Results: SCET$_{\mathrm{I}}$

Soft function	$\gamma_1^{C_A}$	$\gamma_1^{n_f}$	$c_2^{C_A}$	$c_2^{n_f}$
Thrust [Kelley et al, '11] [Monni et al, '11]	15.7945 (15.7945)	3.90981 (3.90981)	−56.4992 (−56.4990)	43.3902 (43.3905)
C-Parameter [Hoang et al, '14]	15.7947 (15.7945)	3.90980 (3.90981)	−57.9754 [-58.16 ± 0.26]	43.8179 [43.74 ± 0.06]
Threshold Drell-Yan [Belitsky, '98]	15.7946 (15.7945)	3.90982 (3.90981)	6.81281 (6.81287)	−10.6857 (−10.6857)
W @ large p_T [Becher et al, '12]	15.7947 (15.7945)	3.90981 (3.90981)	−2.65034 (−2.65010)	−25.3073 (−25.3073)
Transverse Thrust [Becher, Garcia, Piclum, '15]	−158.278 [$-148\pm^{20}_{30}$]	19.3955 [$18\pm^{2}_{3}$]	*parameter dependent*	*parameter dependent*

$$\gamma_1 = \gamma_1^{C_A} C_F C_A + \gamma_1^{N_f} C_F T_F n_f \qquad c_2 = c_2^{C_A} C_F C_A + c_2^{N_f} C_F T_F n_f + \frac{1}{2}(c_1)^2$$

- Derived in few hours on an 8 core desktop machine
- Deviations from analytic results compatible with 1σ error estimate

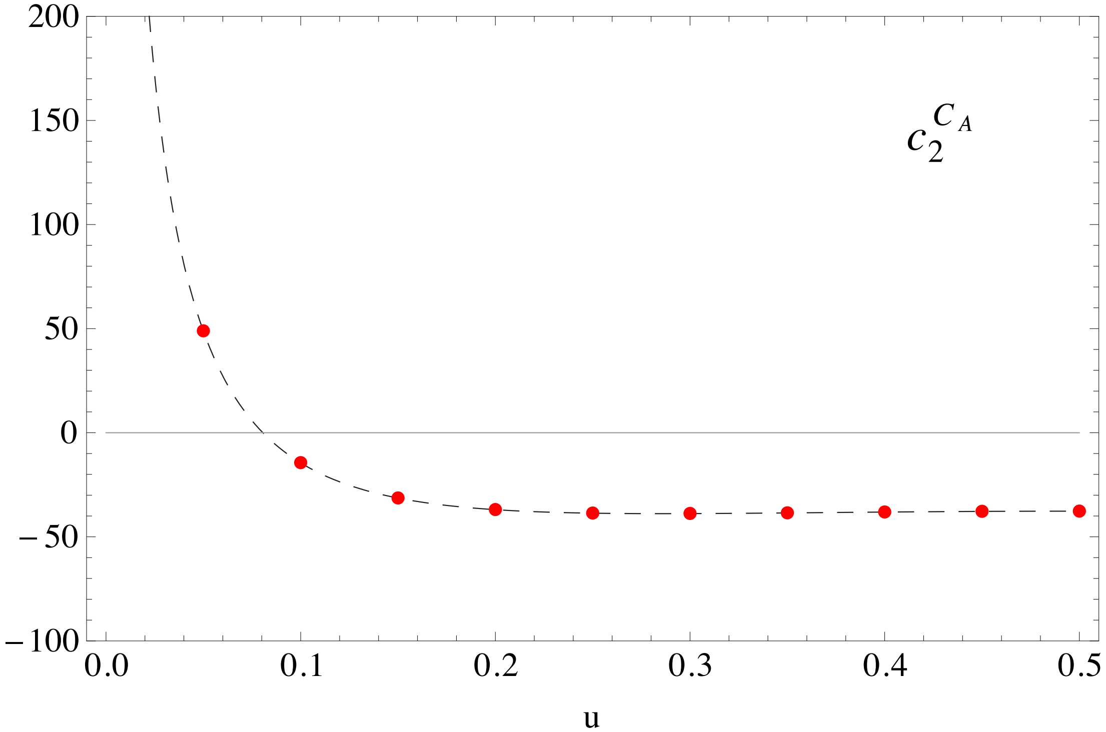
Results: Angularities

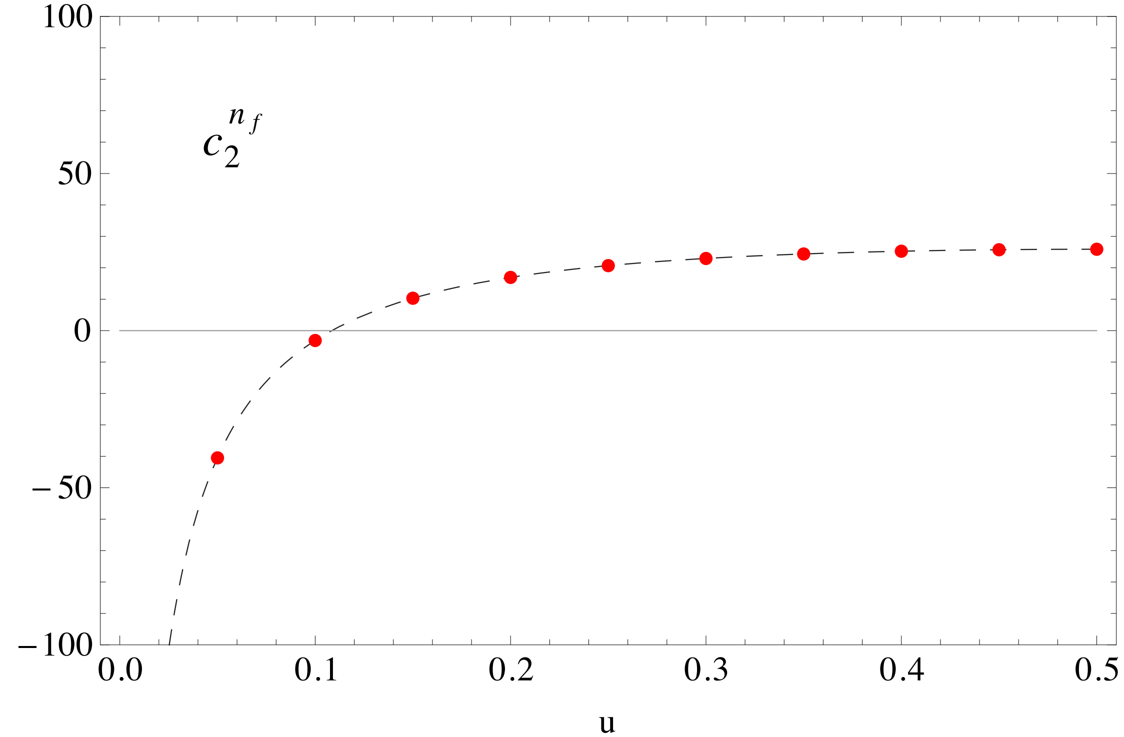


Results: Hemisphere masses

- Multi-differential functions can be computed by keeping the ratio of Laplace variables fixed, *e.g. here*

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}M_L\,\mathrm{d}M_R} \to \frac{\mathrm{d}^2\sigma}{\mathrm{d}\tau_L \mathrm{d}\tau_R} \qquad \tau = \tau_L + \tau_R \qquad u = \frac{\tau_L}{\tau_L + \tau_R}$$





- Dots are numerical, lines analytic ([Kelley, Schwartz, Schabinger, Zhu, ’11])

Results: SCET$_{\text{II}}$

preliminary

Observable	$d_2^{C_A}$	$d_2^{n_f}$
p_T-Resummation [Becher, Neubert, '07]	-3.73389 (-3.73167)	-8.29610 (-8.29630)
Recoil free broadening [Becher, Bell, '12]	7.03595 (7.03605)	-11.5393 (-11.5393)
E_T-Resummation [Grazzini et al. '14]	15.9804 [*below*]	-18.7370 [*below*]

Analytic result for E_T Resummation in Higgs production:

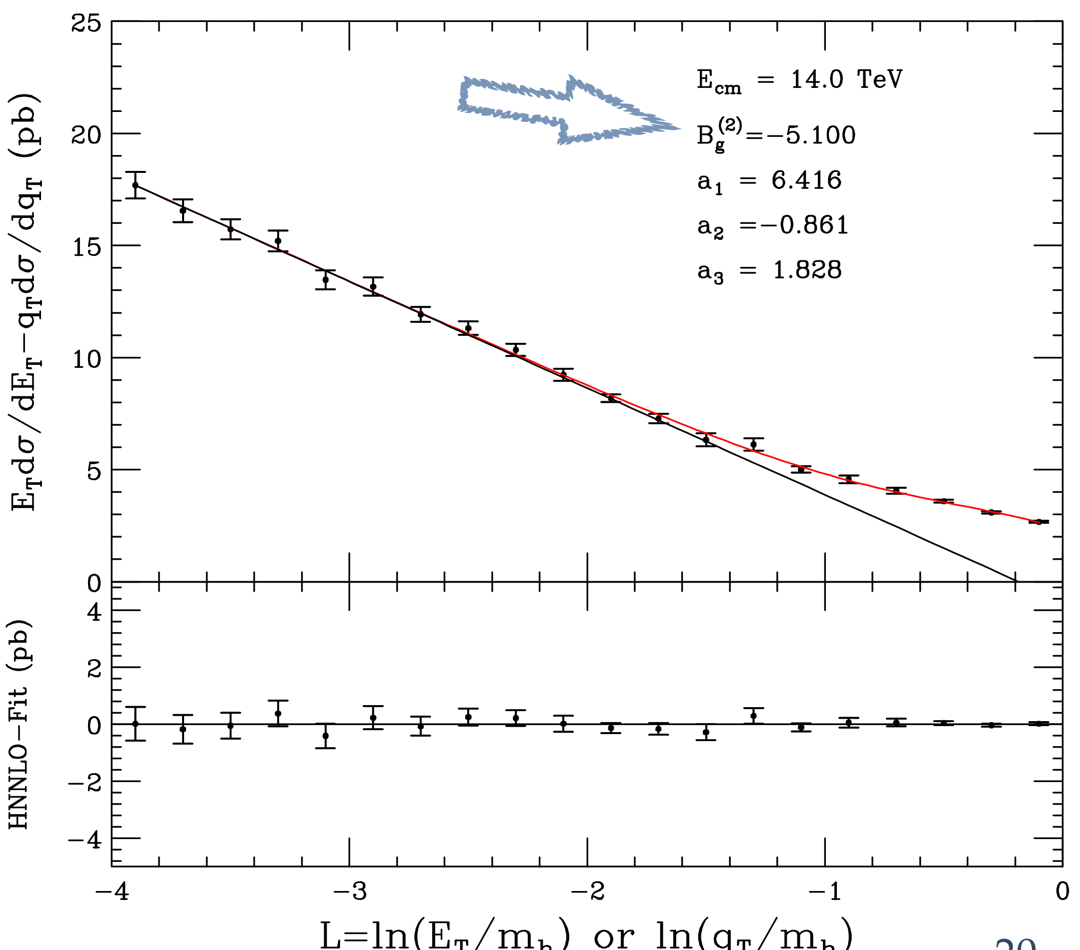
$$d_2^{C_A} = \frac{760}{27} + \frac{22\pi^2}{3} + 8\zeta_3 - \left(\frac{512}{9} + 8\pi^2\right)\log 2$$

$$d_2^{n_f} = -\frac{128}{27} - \frac{8\pi^2}{3} + \frac{160}{9}\log 2$$

- E_T-Resummation

  [Grazzini, Papaefstathiou, Smillie, Webber, '14]:

$$B_g^{(2)} = -5.1 \pm 1.6$$



- Our result:

$$B_g^{(2)} = 33.0$$

$$\left(B_g^{(2)} = 2\gamma_1^g + d_2^g + \beta_0 e_1^g\right)$$

Conclusion

- We have developed a framework to systematically compute generic *NNLO* dijet soft functions for $\mathrm{SCET_I}$ and $\mathrm{SCET_{II}}$ observables
- These are the missing puzzle pieces for *NNLL* resummation of (hadronic) dijet observables
- We have multiple independent methods to derive soft anomalous dimensions and anomaly exponents, both numerical and analytic
- The numerical code is now usable on non-cosmological time scales
- Our setup facilitates the computation of the missing *NNLO* ingredients needed for *NNLL* and *NNLL′* resummation

That's all folks!

Thank you!