

Three-loop cusp anomalous dimension

A. Grozin, J. Henn, G. Korchemsky, P. Marquard

Plan

- ▶ Result for Γ up to 3 loops in QCD and its supersymmetric extensions
- ▶ Limiting case $\varphi_M \rightarrow \infty$: light-like anomalous dimension
- ▶ Universal function Ω
- ▶ Limiting case $\varphi \rightarrow \pi$: Relation to potential
- ▶ Abelian large- n_f color structures to all orders in α_s

History

1 loop

$$\Gamma(\alpha_s, \varphi) = C_F \frac{\alpha_s}{\pi} (\varphi \coth \varphi - 1)$$

Follows from the soft radiation function
in classical electrodynamics

The Guinness Book of Records The anomalous
dimension known for a longest time
(> 100 years)

2 loops Korchemsky, Radyushkin (1987)
Kidonakis (2009)

3 loops here

History

1 loop

$$\Gamma(\alpha_s, \varphi) = C_F \frac{\alpha_s}{\pi} (\varphi \coth \varphi - 1)$$

Follows from the soft radiation function
in classical electrodynamics

The Guinness Book of Records The anomalous
dimension known for a longest time
(> 100 years)

2 loops Korchemsky, Radyushkin (1987)
Kidonakis (2009)

3 loops here

γ_h at 3 loops — Melnikov, van Ritbergen (2000)
Chetyrkin, Grozin (2003)

Cusp on a Wilson line



Euclidean

$$x = e^{i\varphi}$$

Cusp on a Wilson line



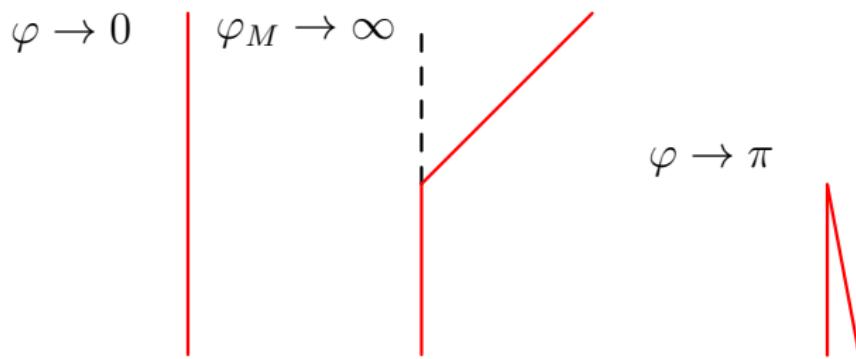
Euclidean

$$x = e^{i\varphi}$$

Minkowski

$$\varphi = i\varphi_M \quad x = e^{-\varphi_M}$$

Limiting cases



n_f quarks in fundamental representation

Wilson line in representation R

$$\begin{aligned}\Gamma = & C_R \frac{\alpha_s}{\pi} \left\{ \tilde{A}_1 \right. \\ & + \left[\frac{1}{2} C_A \left(\tilde{A}_2 + \tilde{A}_3 \right) + \left(\frac{67}{36} C_A - \frac{5}{9} T_F n_f \right) \tilde{A}_1 \right] \frac{\alpha_s}{\pi} \\ & \left. + \left[C_A^2 \gamma_{AA} + C_A T_F n_f \gamma_{Af} + C_F T_F n_f \gamma_{Ff} + (T_F n_f)^2 \gamma_{ff} \right] \left(\frac{\alpha_s}{\pi} \right)^2 \right\}\end{aligned}$$

n_f quarks in fundamental representation

Wilson line in representation R

$$\begin{aligned}\Gamma = & C_R \frac{\alpha_s}{\pi} \left\{ \tilde{A}_1 \right. \\ & + \left[\frac{1}{2} C_A \left(\tilde{A}_2 + \tilde{A}_3 \right) + \left(\frac{67}{36} C_A - \frac{5}{9} T_F n_f \right) \tilde{A}_1 \right] \frac{\alpha_s}{\pi} \\ & \left. + \left[C_A^2 \gamma_{AA} + C_A T_F n_f \gamma_{Af} + C_F T_F n_f \gamma_{Ff} + (T_F n_f)^2 \gamma_{ff} \right] \left(\frac{\alpha_s}{\pi} \right)^2 \right\}\end{aligned}$$

Casimir scaling breaks at 4 loops

n_s scalars and n_f fermions
in adjoint representation
with some specific interactions

$$\begin{aligned}\Gamma = C_R \frac{\alpha_s}{\pi} & \left\{ \tilde{A}_1 \right. \\ & + \left[\frac{1}{2} \left(\tilde{A}_2 + \tilde{A}_3 \right) + \left(\frac{67}{36} - \frac{5}{18} n_f - \frac{1}{9} n_s \right) \tilde{A}_1 \right] C_A \frac{\alpha_s}{\pi} \\ & + \left[\gamma_{AA} + \frac{1}{2} n_f (\gamma_{Af} + \gamma_{Ff}) + n_s \gamma_s \right. \\ & \quad \left. + \frac{1}{4} n_f^2 \gamma_{ff} + n_s^2 \gamma_{ss} + \frac{1}{2} n_s n_f \gamma_{sf} \right] C_A^2 \left(\frac{\alpha_s}{\pi} \right)^2 \end{aligned}\right\}$$

n_s scalars and n_f fermions
in adjoint representation
with some specific interactions

$$\begin{aligned}\Gamma = C_R \frac{\alpha_s}{\pi} & \left\{ \tilde{A}_1 \right. \\ & + \left[\frac{1}{2} \left(\tilde{A}_2 + \tilde{A}_3 \right) + \left(\frac{67}{36} - \frac{5}{18} n_f - \frac{1}{9} n_s \right) \tilde{A}_1 \right] C_A \frac{\alpha_s}{\pi} \\ & + \left[\gamma_{AA} + \frac{1}{2} n_f (\gamma_{Af} + \gamma_{Ff}) + n_s \gamma_s \right. \\ & \quad \left. + \frac{1}{4} n_f^2 \gamma_{ff} + n_s^2 \gamma_{ss} + \frac{1}{2} n_s n_f \gamma_{sf} \right] C_A^2 \left(\frac{\alpha_s}{\pi} \right)^2 \end{aligned}\right\}$$

$$\Gamma_{N=1} = \Gamma(n_f = 1, n_s = 0)$$

$$\Gamma_{N=2} = \Gamma(n_f = 2, n_s = 2)$$

$$\Gamma_{N=4} = \Gamma(n_f = 4, n_s = 6)$$

Functions

$$\begin{aligned}\gamma_{AA} &= \frac{1}{4} \left(\tilde{A}_5 + \tilde{A}_4 + \tilde{B}_5 + \tilde{B}_3 \right) + \frac{67}{36} \tilde{A}_3 + \frac{29}{18} \tilde{A}_2 \\ &\quad + \left(\frac{11}{24} \zeta_3 + \frac{245}{96} \right) \tilde{A}_1 \\ \gamma_{Af} &= -\frac{5}{9} \left(\tilde{A}_3 + \tilde{A}_2 \right) - \frac{1}{6} \left(7\zeta_3 + \frac{209}{36} \right) \tilde{A}_1 \\ \gamma_{Ff} &= \left(\zeta_3 - \frac{55}{48} \right) \tilde{A}_1 \\ \gamma_s &= -\frac{1}{9} \left(\tilde{A}_3 + \tilde{A}_2 \right) - \left(\frac{1}{48} \zeta_3 + \frac{1039}{1728} \right) \tilde{A}_1 \\ \gamma_{ff} &= -\frac{1}{27} \tilde{A}_1 \qquad \gamma_{ss} = \frac{1}{432} \tilde{A}_1 \qquad \gamma_{sf} = \frac{7}{16} \tilde{A}_1\end{aligned}$$

Functions

$$\gamma_{AA} = \frac{1}{4} \left(\tilde{A}_5 + \tilde{A}_4 + \tilde{B}_5 + \tilde{B}_3 \right) + \frac{67}{36} \tilde{A}_3 + \frac{29}{18} \tilde{A}_2$$

$$+ \left(\frac{11}{24} \zeta_3 + \frac{245}{96} \right) \tilde{A}_1$$

$$\gamma_{Af} = -\frac{5}{9} \left(\tilde{A}_3 + \tilde{A}_2 \right) - \frac{1}{6} \left(7\zeta_3 + \frac{209}{36} \right) \tilde{A}_1$$

$$\gamma_{Ff} = \left(\zeta_3 - \frac{55}{48} \right) \tilde{A}_1$$

$$\gamma_s = -\frac{1}{9} \left(\tilde{A}_3 + \tilde{A}_2 \right) - \left(\frac{1}{48} \zeta_3 + \frac{1039}{1728} \right) \tilde{A}_1$$

$$\gamma_{ff} = -\frac{1}{27} \tilde{A}_1 \quad \gamma_{ss} = \frac{1}{432} \tilde{A}_1 \quad \gamma_{sf} = \frac{7}{16} \tilde{A}_1$$

$$\tilde{A}_i(x) = A_i(x) - A_i(1) \quad \tilde{B}_i(x) = B_i(x) - B_i(1)$$

Functions

$$A_1(x) = \frac{\xi}{2} H_1(y)$$

$$A_2(x) = \frac{1}{2} H_{1,1}(y) + \frac{\pi^2}{3} - \xi \left[\frac{1}{2} H_{1,1}(y) - H_{1,0}(y) \right]$$

$$A_3(x) = -\xi \left[\frac{1}{4} H_{1,1,1}(y) + \frac{\pi^2}{6} H_1(y) \right]$$

$$+ \xi^2 \left[\frac{1}{4} H_{1,1,1}(y) + \frac{1}{2} H_{1,0,1}(y) \right]$$

...

$$y = 1 - x^2 \quad \xi = \frac{1 + x^2}{1 - x^2}$$

Uniform weight

Limit $\varphi_M \rightarrow \infty$

$$x \rightarrow 1 \quad \Gamma = K(\alpha_s) \log \frac{1}{x} + \mathcal{O}(1)$$

Limit $\varphi_M \rightarrow \infty$

$$x \rightarrow 1 \quad \Gamma = K(\alpha_s) \log \frac{1}{x} + \mathcal{O}(1)$$

$$P_{q \rightarrow q}(z) = K(\alpha_s) \left(\frac{1}{1-z} \right)_+ + \dots$$

Korchemsky (1989); Korchemsky, Marchesini (1993)

n_f quarks in fundamental representation

$$\begin{aligned} K(\alpha_s) = & C_R \frac{\alpha_s}{\pi} \left\{ 1 + \left[\frac{1}{12} \left(\pi^2 - \frac{67}{3} \right) C_A + \frac{5}{9} T_F n_f \right] \frac{\alpha_s}{\pi} \right. \\ & + \left[\frac{1}{24} \left(\frac{11}{30} \pi^4 + 11 \zeta_3 - \frac{67}{9} \pi^2 + \frac{245}{4} \right) C_A^2 \right. \\ & - \frac{1}{6} \left(7 \zeta_3 - \frac{5}{9} \pi^2 + \frac{209}{36} \right) C_A T_F n_f \\ & \left. + \left(\zeta_3 - \frac{55}{48} \right) C_F T_F n_f - \frac{1}{27} (T_F n_f)^2 \right] \left(\frac{\alpha_s}{\pi} \right)^2 \left\} \right. \end{aligned}$$

Moch, Vermaseren, Vogt (2004)

n_s scalars and n_f fermions
in adjoint representation

$$\begin{aligned} K(\alpha_s) = & C_R \frac{\alpha_s}{\pi} \left\{ 1 + \left[-\frac{\pi^2}{12} + \frac{16}{9} - \frac{5}{18} n_f - \frac{n_s}{9} \right] C_A \frac{\alpha_s}{\pi} \right. \\ & + \left[\frac{11}{720} \pi^4 + \frac{11}{24} \zeta_3 - \frac{8}{27} \pi^2 + \frac{1817}{864} \right. \\ & + n_f \left(-\frac{1}{12} \zeta_3 + \frac{5}{108} \pi^2 - \frac{91}{96} \right) \\ & + n_s \left(-\frac{1}{48} \zeta_3 + \frac{1}{54} \pi^2 - \frac{1007}{1728} \right) \\ & \left. - \frac{n_f^2}{108} + \frac{n_s^2}{432} + \frac{7}{32} n_s n_f \right] C_A^2 \left(\frac{\alpha_s}{\pi} \right)^2 \left. \right\} \end{aligned}$$

Universal function

$$K(\alpha_s) = C_R \frac{a}{\pi} \quad \Omega(a, x) = \Gamma(\alpha_s, \varphi)$$

Universal function

$$K(\alpha_s) = C_R \frac{a}{\pi} \quad \Omega(a, x) = \Gamma(\alpha_s, \varphi)$$

$$\begin{aligned}\Omega(a, x) &= C_R \frac{a}{\pi} \left[\tilde{A}_1 + \frac{1}{2} \left(\tilde{A}_3 + \tilde{A}_2 + \frac{\pi^2}{6} \tilde{A}_1 \right) C_A \frac{a}{\pi} \right. \\ &\quad \left. + \frac{1}{4} \left(\tilde{A}_5 + \tilde{A}_4 - \tilde{A}_2 + \tilde{B}_5 + \tilde{B}_3 + \frac{\pi^2}{3} \tilde{A}_3 + \frac{\pi^2}{3} \tilde{A}_2 - \frac{\pi^4}{180} \tilde{A}_1 \right) C_A^2 \left(\frac{a}{\pi} \right)^2 \right]\end{aligned}$$

Limit $\varphi \rightarrow \pi$

$$\varphi = \pi - \delta \quad \Gamma = \frac{rV(r)}{\delta}$$

Kilian, Mannel, Ohl (1993)

Conformal symmetry

Euclidean space

$$ds^2 = dx_0^2 + d\vec{x}^2$$

Conformal symmetry

Euclidean space

$$ds^2 = dx_0^2 + d\vec{x}^2$$

Spherical coordinates

$$x_0 = r \cos \delta \quad \vec{x} = r \vec{n} \sin \delta$$

$$ds^2 = dr^2 + r^2(d\delta^2 + \sin^2 \delta d\vec{n}^2)$$

Conformal symmetry

Euclidean space

$$ds^2 = dx_0^2 + d\vec{x}^2$$

Spherical coordinates

$$x_0 = r \cos \delta \quad \vec{x} = r \vec{n} \sin \delta$$

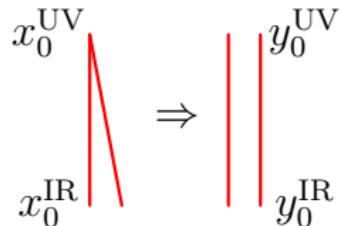
$$ds^2 = dr^2 + r^2(d\delta^2 + \sin^2 \delta d\vec{n}^2)$$

$$\delta \ll 1$$

$$r = e^{y_0} \quad \vec{y} = \delta \vec{n}$$

$$ds^2 = e^{2y_0} (dy_0^2 + d\vec{y}^2)$$

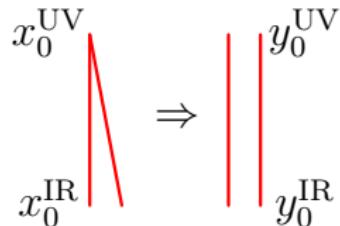
Conformal symmetry



$$\log W = \Gamma \log \frac{x_0^{\text{IR}}}{x_0^{\text{UV}}} = V(\vec{y}) (y_0^{\text{IR}} - y_0^{\text{UV}})$$

$$\Gamma = \frac{yV(y)}{\delta} = \frac{\vec{q}^2 V(\vec{q}; \alpha_s)}{4\pi\delta}$$

Conformal symmetry



$$\log W = \Gamma \log \frac{x_0^{\text{IR}}}{x_0^{\text{UV}}} = V(\vec{y}) (y_0^{\text{IR}} - y_0^{\text{UV}})$$

$$\Gamma = \frac{yV(y)}{\delta} = \frac{\vec{q}^2 V(\vec{q}; \alpha_s)}{4\pi\delta}$$

$\mathcal{N} = 4$: 2-loop $V(\vec{q})$ Prausa, Steinhauser (2013)

Conformal anomaly

In QCD conformal symmetry is anomalous
⇒ β function

Conformal anomaly

In QCD conformal symmetry is anomalous

$\Rightarrow \beta$ function

2 loops

$$\delta\Gamma(\alpha_s, \pi - \delta) - \frac{\vec{q}^2 V(\vec{q}; \alpha_s)}{4\pi} = 0 \quad \beta_0 = 0$$

Conformal anomaly

In QCD conformal symmetry is anomalous

$\Rightarrow \beta$ function

2 loops

$$\delta\Gamma(\alpha_s, \pi - \delta) - \frac{\vec{q}^2 V(\vec{q}; \alpha_s(\mu))}{4\pi} \sim \beta_0 C_R \left(\frac{\alpha_s}{\pi} \right)^2 \log \frac{\vec{q}^2}{\mu^2}$$

Conformal anomaly

In QCD conformal symmetry is anomalous

$\Rightarrow \beta$ function

2 loops

$$\delta\Gamma(\alpha_s, \pi - \delta) - \frac{\vec{q}^2 V(\vec{q}; \alpha_s(\mu))}{4\pi} = 0 \quad \mu^2 = \vec{q}^2$$

Conformal anomaly

In QCD conformal symmetry is anomalous

$\Rightarrow \beta$ function

2 loops

$$\delta\Gamma(\alpha_s, \pi - \delta) - \frac{\vec{q}^2 V(\vec{q}; \alpha_s(\mu))}{4\pi} = 0 \quad \mu^2 = \vec{q}^2$$

3 loops

$$\begin{aligned}\Delta &\equiv \delta\Gamma(\alpha_s, \pi - \delta) - \frac{\vec{q}^2 V(\vec{q}; \alpha_s(\mu))}{4\pi} \\ &= \frac{\pi}{108} \beta_0 C_R \left(\frac{\alpha_s}{\pi} \right)^3 (47C_A - 28T_F n_f)\end{aligned}$$

Conformal anomaly

In QCD conformal symmetry is anomalous

$\Rightarrow \beta$ function

2 loops

$$\delta\Gamma(\alpha_s, \pi - \delta) - \frac{\vec{q}^2 V(\vec{q}; \alpha_s(\mu))}{4\pi} = 0 \quad \mu^2 = \vec{q}^2$$

3 loops

$$\begin{aligned}\Delta &\equiv \delta\Gamma(\alpha_s, \pi - \delta) - \frac{\vec{q}^2 V(\vec{q}; \alpha_s(\mu))}{4\pi} \\ &= \frac{\pi}{108} \beta_0 C_R \left(\frac{\alpha_s}{\pi} \right)^3 (47C_A - 28T_F n_f)\end{aligned}$$

At 4 loops we expect ultrasoft

$$\Gamma(\alpha_s, \pi - \delta) \sim C_R C_A^3 \frac{\log \delta + \text{const}}{\delta} \left(\frac{\alpha_s}{\pi} \right)^4$$

Abelian large n_f structures

$$C_F(T_F n_f)^{L-1} \alpha_s^L \quad (L \geq 1)$$

$$C_F^2(T_F n_f)^{L-2} \alpha_s^L \quad (L \geq 3)$$

Abelian large n_f structures

$$C_F(T_F n_f)^{L-1} \alpha_s^L \quad (L \geq 1)$$

$$C_F^2(T_F n_f)^{L-2} \alpha_s^L \quad (L \geq 3)$$

QED with n_f flavors $C_F = 1$, $C_A = 0$, $T_F = 1$, $\beta_0 = -\frac{4}{3}n_f$

$$b = \beta_0 \frac{\alpha}{4\pi} \sim 1 \qquad \frac{1}{\beta_0} \ll 1 \text{ — expansion parameter}$$

Abelian large n_f structures

$$C_F(T_F n_f)^{L-1} \alpha_s^L \quad (L \geq 1)$$

$$C_F^2(T_F n_f)^{L-2} \alpha_s^L \quad (L \geq 3)$$

QED with n_f flavors $C_F = 1$, $C_A = 0$, $T_F = 1$, $\beta_0 = -\frac{4}{3}n_f$

$$b = \beta_0 \frac{\alpha}{4\pi} \sim 1 \quad \frac{1}{\beta_0} \ll 1 \text{ — expansion parameter}$$

Coordinate space, Wilson line of any shape, up to $\text{NL}\beta_0$

$$\log W = \text{[Diagram: A red horizontal line with blue wavy lines attached at both ends, forming a semi-circle above it.]}$$

First broken at $\text{NNL}\beta_0$: $n_f^{L-3} \alpha^L \quad (L \geq 4)$



Renormalization

$$\log W(t, t'; \varphi) - \log W(t, t'; 0)$$
$$= \begin{array}{c} \text{Diagram A: A triangle with red lines and blue wavy lines on the left leg.} \\ = \end{array} - \begin{array}{c} \text{Diagram B: A horizontal red line with a blue wavy loop attached to its center.} \end{array} = \log Z + \text{finite}$$

(external-leg corrections cancel)

Renormalization

$$\log W(t, t'; \varphi) - \log W(t, t'; 0)$$
$$= \begin{array}{c} \text{Diagram of } \log W(t, t'; \varphi) \\ \text{A red triangle with blue wavy lines on its sides.} \end{array} - \begin{array}{c} \text{Diagram of } \log W(t, t'; 0) \\ \text{A red rectangle with blue wavy lines on its top and bottom edges.} \end{array} = \log Z + \text{finite}$$

(external-leg corrections cancel)

Momentum space

$$\log V(\omega, \omega; \varphi) - \log V(\omega, \omega; 0)$$
$$= \begin{array}{c} \text{Diagram of } \log V(\omega, \omega; \varphi) \\ \text{A red triangle with blue wavy lines on its sides.} \end{array} - \begin{array}{c} \text{Diagram of } \log V(\omega, \omega; 0) \\ \text{A red rectangle with blue wavy lines on its top and bottom edges.} \end{array} = \log Z + \text{finite}$$

Leading β_0 order

Photon self energy

$$\Pi_0(k^2) = \text{Diagram} = \beta_0 \frac{e_0^2}{(4\pi)^{d/2}} e^{-\gamma\varepsilon} \frac{D(\varepsilon)}{\varepsilon} (-k^2)^{-\varepsilon}$$

$$D(\varepsilon) = e^{\gamma\varepsilon} \frac{(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{(1-2\varepsilon)(1-\frac{2}{3}\varepsilon)\Gamma(1-2\varepsilon)} = 1 + \frac{5}{3}\varepsilon + \dots$$

Leading β_0 order

Photon self energy

$$\Pi_0(k^2) = \text{Diagram} = \beta_0 \frac{e_0^2}{(4\pi)^{d/2}} e^{-\gamma\varepsilon} \frac{D(\varepsilon)}{\varepsilon} (-k^2)^{-\varepsilon}$$

$$D(\varepsilon) = e^{\gamma\varepsilon} \frac{(1-\varepsilon)\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{(1-2\varepsilon)(1-\frac{2}{3}\varepsilon)\Gamma(1-2\varepsilon)} = 1 + \frac{5}{3}\varepsilon + \dots$$

Charge renormalization

$$\beta_0 \frac{e_0^2}{(4\pi)^{d/2}} e^{-\gamma\varepsilon} = b Z_\alpha(b) \mu^{2\varepsilon}$$

$$\frac{d \log Z_\alpha}{d \log b} = -\frac{b}{\varepsilon + b} \quad \quad Z_\alpha = \frac{1}{1 + b/\varepsilon}$$

Vertex function

1 loop with $1/(1 - \Pi_0(k^2))$

$$V(\omega, \omega; \varphi) = \text{Diagram} = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{f(\varepsilon, L\varepsilon; \varphi)}{L} \Pi_0^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

Π_0 is taken at $-k^2 = (-2\omega)^2$

Vertex function

1 loop with $1/(1 - \Pi_0(k^2))$

$$V(\omega, \omega; \varphi) = \text{Diagram} = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{f(\varepsilon, L\varepsilon; \varphi)}{L} \Pi_0^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

Π_0 is taken at $-k^2 = (-2\omega)^2$

Choosing $\mu^2 = D(\varepsilon)^{-1/\varepsilon}(-2\omega)^2 \rightarrow e^{-5/3}(-2\omega)^2$

$$V(\omega, \omega; \varphi) - V(\omega, \omega; 0) = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{\bar{f}(\varepsilon, L\varepsilon; \varphi)}{L} \left(\frac{b}{\varepsilon + b}\right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\bar{f}(\varepsilon, u; \varphi) = f(\varepsilon, u; \varphi) - f(\varepsilon, u; 0)$$

Leading β_0 result

Expand in b , $\bar{f}(\varepsilon, u; \vartheta) = \sum_{n,m=0}^{\infty} \bar{f}_{n,m} \varepsilon^n u^m$, coefficient of ε^{-1} contains only \bar{f}_{n0}

$$Z_1(b; \varphi) = 2 \frac{\varphi \cot \varphi - 1}{\beta_0} \sum_{n=0}^{\infty} \frac{\hat{f}_n}{n+1} (-b)^{n+1} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\bar{f}(\varepsilon, 0; \varphi) = -2\hat{f}(\varepsilon)(\varphi \cot \varphi - 1) \quad \hat{f}(\varepsilon) = \sum_{n=0}^{\infty} \hat{f}_n \varepsilon^n$$

Leading β_0 result

Expand in b , $\bar{f}(\varepsilon, u; \vartheta) = \sum_{n,m=0}^{\infty} \bar{f}_{n,m} \varepsilon^n u^m$, coefficient of ε^{-1} contains only \bar{f}_{n0}

$$Z_1(b; \varphi) = 2 \frac{\varphi \cot \varphi - 1}{\beta_0} \sum_{n=0}^{\infty} \frac{\hat{f}_n}{n+1} (-b)^{n+1} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\bar{f}(\varepsilon, 0; \varphi) = -2\hat{f}(\varepsilon)(\varphi \cot \varphi - 1) \quad \hat{f}(\varepsilon) = \sum_{n=0}^{\infty} \hat{f}_n \varepsilon^n$$

$$\Gamma(b; \varphi) = 4 \frac{b}{\beta_0} \gamma_0(b) (\varphi \cot \varphi - 1) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\begin{aligned} \gamma_0(b) &= \hat{f}(-b) = \frac{(1 + \frac{2}{3}b)\Gamma(2 + 2b)}{(1 + b)\Gamma^3(1 + b)\Gamma(1 - b)} \\ &= 1 + \frac{5}{3}b - \frac{1}{3}b^2 - \left(2\zeta_3 - \frac{1}{3}\right)b^3 + \left(\frac{\pi^4}{30} - \frac{10}{3}\zeta_3 - \frac{1}{3}\right)b^4 + \dots \end{aligned}$$

Beneke, Braun (1995)

HQET field anomalous dimension

$\varphi = 0$ Ward identity

$$V(\omega, \omega'; 0) = \frac{S^{-1}(\omega') - S^{-1}(\omega)}{\omega' - \omega} \quad V(\omega, \omega; 0) = \frac{dS^{-1}(\omega)}{d\omega}$$
$$\log V(\omega, \omega'; 0) = -\log Z_h + \text{finite}$$

HQET field anomalous dimension

$\varphi = 0$ Ward identity

$$V(\omega, \omega'; 0) = \frac{S^{-1}(\omega') - S^{-1}(\omega)}{\omega' - \omega} \quad V(\omega, \omega; 0) = \frac{dS^{-1}(\omega)}{d\omega}$$
$$\log V(\omega, \omega'; 0) = -\log Z_h + \text{finite}$$

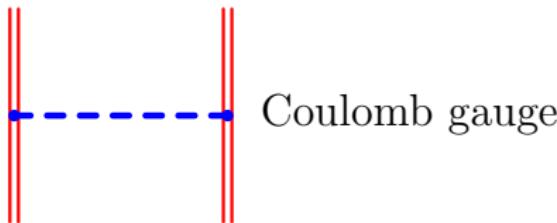
$$\gamma_h(b) = 2 \frac{b}{\beta_0} \gamma_{h0}(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\gamma_{h0}(b) = f(-b, 0; 0) = \frac{\left(1 + \frac{2}{3}b\right)^2 \Gamma(2 + 2b)}{(1 + b)^2 \Gamma^3(1 + b) \Gamma(1 - b)}$$
$$= 1 + \frac{4}{3}b - \frac{5}{9}b^2 - \left(2\zeta_3 - \frac{2}{3}\right)b^3 + \left(\frac{\pi^4}{30} - \frac{8}{3}\zeta_3 - \frac{7}{9}\right)b^4 + \dots$$

Broadhurst, Grozin (1995)

Quark-antiquark potential

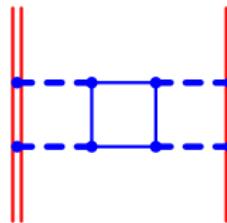
Up to $NL\beta_0$



$$V(\vec{q}) = -\frac{e_0^2}{\vec{q}^2} \frac{1}{1 - \Pi(-\vec{q}^2)}$$

$\Pi(q^2)$ gauge invariant

First broken at $NNL\beta_0$



Quark-antiquark potential

$$\mu^2 = \vec{q}^2$$

$$V(\vec{q}) = -\frac{(4\pi)^{d/2} e^{\gamma\varepsilon}}{\beta_0 D(\varepsilon) (\vec{q}^2)^{1-\varepsilon}} \varepsilon \sum_{L=1}^{\infty} \left(D(\varepsilon) \frac{b}{\varepsilon + b} \right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

Quark-antiquark potential

$$\mu^2 = \vec{q}^2$$

$$V(\vec{q}) = -\frac{(4\pi)^{d/2} e^{\gamma\varepsilon}}{\beta_0 D(\varepsilon) (\vec{q}^2)^{1-\varepsilon}} \varepsilon \sum_{L=1}^{\infty} \left(D(\varepsilon) \frac{b}{\varepsilon + b} \right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

The sum can be written as

$$\sum_{L=1}^{\infty} f(\varepsilon, L\varepsilon) \left(\frac{b}{\varepsilon + b} \right)^L = \frac{b}{\varepsilon} \sum_{n=0}^{\infty} n! f_{0n} b^n + \mathcal{O}(\varepsilon^0)$$

$$f(0, u) = e^{\frac{5}{3}u} \quad f_{0n} = \frac{1}{n!} \left(\frac{5}{3} \right)^n$$

Quark-antiquark potential

$$\mu^2 = \vec{q}^2$$

$$V(\vec{q}) = -\frac{(4\pi)^{d/2} e^{\gamma\varepsilon}}{\beta_0 D(\varepsilon) (\vec{q}^2)^{1-\varepsilon}} \varepsilon \sum_{L=1}^{\infty} \left(D(\varepsilon) \frac{b}{\varepsilon + b} \right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

The sum can be written as

$$\sum_{L=1}^{\infty} f(\varepsilon, L\varepsilon) \left(\frac{b}{\varepsilon + b} \right)^L = \frac{b}{\varepsilon} \sum_{n=0}^{\infty} n! f_{0n} b^n + \mathcal{O}(\varepsilon^0)$$

$$f(0, u) = e^{\frac{5}{3}u} \quad f_{0n} = \frac{1}{n!} \left(\frac{5}{3} \right)^n$$

$$V(\vec{q}) = -\frac{(4\pi)^2}{\vec{q}^2} \frac{b}{\beta_0} \frac{1}{1 - \frac{5}{3}b}.$$

Conformal anomaly

$$\begin{aligned}\Delta &= 4\pi \frac{b}{\beta_0} \delta_0(b) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \\ \delta_0(b) &= \frac{1}{1 - \frac{5}{3}b} - \frac{(1 + \frac{2}{3}b)\Gamma(2 + 2b)}{(1 + b)\Gamma^3(1 + b)\Gamma(1 - b)} \\ &= \frac{28}{9} + 2\left(\zeta_3 + \frac{58}{27}\right)b - \frac{1}{3}\left(\frac{\pi^4}{10} - 10\zeta_3 - \frac{652}{27}\right)b^2 + \dots\end{aligned}$$

Next to leading β_0 order

Photon self energy

$$\begin{aligned}\Pi(k^2) &= \text{Diagram: a simple circle} + 2 \cdot \text{Diagram: a circle with a wavy line loop attached to one point} + \text{Diagram: a circle with a wavy line loop attached to two points} \\ &= \Pi_0(k^2) + \frac{\Pi_1(k^2)}{\beta_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\end{aligned}$$

 at the $L\beta_0$ order

$$\Pi_1 = 3\varepsilon \sum_{L=2}^{\infty} \frac{F(\varepsilon, L\varepsilon)}{L} \Pi_0^L$$

Palanques-Mestre, Pascual (1984); Broadhurst (1993)

Charge renormalization

$$Z_\alpha(b) = \frac{1}{1 + b/\varepsilon} \left[1 + \frac{Z_1(b)}{\beta_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \right]$$
$$\log(1 - \Pi) = \log Z_\alpha + \text{finite}$$

Charge renormalization

$$Z_\alpha(b) = \frac{1}{1 + b/\varepsilon} \left[1 + \frac{Z_1(b)}{\beta_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \right]$$

$$\log(1 - \Pi) = \log Z_\alpha + \text{finite}$$

$1/\beta_0$ term: coefficient of ε^{-1} in $-(1 + b/\varepsilon)\Pi_1$

$$Z_{11} = -3 \sum_{n=0}^{\infty} \frac{F_{n0}(-b)^{n+2}}{(n+1)(n+2)}$$

Charge renormalization

$$Z_\alpha(b) = \frac{1}{1 + b/\varepsilon} \left[1 + \frac{Z_1(b)}{\beta_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \right]$$

$$\log(1 - \Pi) = \log Z_\alpha + \text{finite}$$

$1/\beta_0$ term: coefficient of ε^{-1} in $-(1 + b/\varepsilon)\Pi_1$

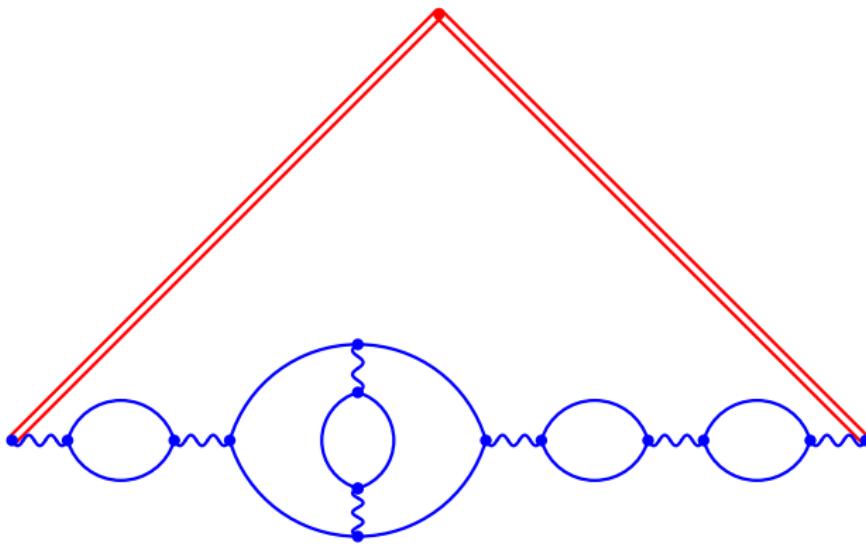
$$Z_{11} = -3 \sum_{n=0}^{\infty} \frac{F_{n0}(-b)^{n+2}}{(n+1)(n+2)}$$

$$\beta(b) = b + \frac{\beta_1(b)}{\beta_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)$$

$$\beta_1(b) = -\frac{dZ_{11}(b)}{d \log b} = 3 \sum_{n=0}^{\infty} \frac{F_{n0}(-b)^{n+2}}{n+1}$$

$$= 3b^2 + \frac{11}{4}b^3 - \frac{77}{36}b^4 - \frac{1}{2} \left(3\zeta_3 + \frac{107}{48} \right) b^5 + \dots$$

Next to leading β_0 order



$$V(\omega, \omega; \varphi) - V(\omega, \omega; 0) = \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{\bar{f}(\varepsilon, L\varepsilon; \varphi)}{L} \left(\frac{b}{\varepsilon + b} \right)^L$$

$$\times \left[1 + L \frac{Z_1}{\beta_0} + \frac{3\varepsilon}{\beta_0} \sum_{L'=2}^{L-1} \frac{L - L'}{L'} F(\varepsilon, L'\varepsilon) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)$$

Next to leading β_0 order

- ▶ Photon propagator $(1 - \Pi_0 - \Pi_1/\beta_0)^{-1}$ up to $1/\beta_0$
- ▶ Charge renormalization up to $1/\beta_0$ (Z_1/β_0)

$$\begin{aligned}\Gamma(b; \varphi) = & 4 \frac{b}{\beta_0} \hat{f}(-b) (\varphi \cot \varphi - 1) + 4 \frac{b^3}{\beta_0^2} (\varphi \cot \varphi - 1) \\ & \times \left\{ \frac{3}{2} [F_{10} + 2F_{01} - 2\hat{f}_1] \right. \\ & \left. - [2F_{20} + 3(F_{11} + F_{02}) + 3F_{01}\hat{f}_1 - 6\hat{f}_2] b + \dots \right\}\end{aligned}$$

Next to leading β_0 result

$$\Gamma(b; \varphi) = 4 \left[\frac{b}{\beta_0} \gamma_0(b) - \frac{b^3}{\beta_0^2} \gamma_1(b) \right] (\varphi \cot \varphi - 1) + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)$$

$$\begin{aligned}\gamma_1(b) &= 12\zeta_3 - \frac{55}{4} + \left[-\frac{\pi^4}{5} + 40\zeta_3 - \frac{299}{18} \right] b \\ &\quad + \left[24\zeta_5 - \frac{2}{3}\pi^4 + \frac{233}{6}\zeta_3 + \frac{15211}{864} \right] b^2 \\ &\quad + \left[-48\zeta_3^2 - \frac{2}{63}\pi^6 + 80\zeta_5 - \frac{167}{225}\pi^4 + \frac{1168}{15}\zeta_3 - \frac{971}{240} \right] b^3 + \dots\end{aligned}$$

Can be continued

F_{nm} at weight 6 contain $\zeta_{5,3}$ but it has cancelled

HQET field anomalous dimension

$$\begin{aligned}\gamma_h(b) &= -6 \left[\frac{b}{\beta_0} \gamma_{h0}(b) - \frac{b^3}{\beta_0^2} \gamma_{h1}(b) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right) \\ \gamma_{h1}(b) &= 3 \left(4\zeta_3 - \frac{17}{4} \right) + \left(-\frac{\pi^4}{5} + 36\zeta_3 - \frac{103}{9} \right) b \\ &\quad + \left(24\zeta_5 - \frac{3}{5}\pi^4 + \frac{59}{2}\zeta_3 + \frac{14579}{864} \right) b^2 \\ &\quad + \left(-48\zeta_3^3 - \frac{2}{63}\pi^6 + 72\zeta_5 - \frac{44}{75}\pi^4 + \frac{3229}{45}\zeta_3 - \frac{5191}{540} \right) b^3 + \dots\end{aligned}$$

$\zeta_{5,3}$ has cancelled again

Quark-antiquark potential

$$\begin{aligned} V(\vec{q}) &= -\frac{(4\pi)^2}{\beta_0 \vec{q}^2} \epsilon \sum_{L=1}^{\infty} f(\epsilon, L\epsilon) \left(\frac{b}{\epsilon + b} \right)^L \\ &\quad \times \left[1 + L \frac{Z_1}{\beta_0} + \frac{3\epsilon}{\beta_0} \sum_{L'=2}^{L-1} \frac{L-L'}{L'} F(\epsilon, L'\epsilon) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right) \\ &= -\frac{(4\pi)^2}{\vec{q}^2} \left[\frac{b}{\beta_0} v_0(b) - \frac{b^3}{\beta_0^2} v_1(b) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right) \\ v_0(b) &= \frac{1}{1 - \frac{5}{3}b} \\ v_1(b) &= -\frac{3}{2} [F_{10} + 2F_{01} + 2f_{01}] \\ &\quad + \frac{1}{2} [F_{20} - 6F_{02} - 6(F_{10} + 3F_{01})f_{01} - 30f_{02}] b + \dots \end{aligned}$$

Quark-antiquark potential

$$\begin{aligned}v_1(b) = & 12\zeta_3 - \frac{55}{4} + \left(78\zeta_3 - \frac{7001}{72}\right)b \\& + \left(60\zeta_5 + \frac{723}{2}\zeta_3 - \frac{147851}{288}\right)b^2 \\& + \left(770\zeta_5 + \frac{\pi^4}{200} + \frac{276901}{180}\zeta_3 - \frac{70418923}{25920}\right)b^3 + \dots\end{aligned}$$

Conformal anomaly

$$\Delta = \pi \left[\frac{b}{\beta_0} \delta_0(b) - \frac{b^4}{\beta_0^2} \delta_1(b) \right] + \mathcal{O}\left(\frac{1}{\beta_0^3}\right)$$
$$\delta_1(b) = \frac{\pi^4}{5} + 38\zeta_3 - \frac{645}{8} + \left(36\zeta_5 + \frac{2}{3}\pi^4 + \frac{968}{3}\zeta_3 - \frac{114691}{216} \right)b$$
$$+ \left(48\zeta_3^2 + \frac{2}{63}\pi^6 + 690\zeta_5 + \frac{269}{360}\pi^4 + \frac{52577}{36}\zeta_3 - \frac{14062811}{5184} \right)b^2 + \dots$$

Conclusion

- ▶ $\Gamma(\alpha_s, \varphi)$ at 3 loops has been calculated via harmonic polylogarithms up to weight 5 in QCD and some theories with scalars, including $\mathcal{N} = 1, 2, 4$ supersymmetric QCD

Conclusion

- ▶ $\Gamma(\alpha_s, \varphi)$ at 3 loops has been calculated via harmonic polylogarithms up to weight 5 in QCD and some theories with scalars, including $\mathcal{N} = 1, 2, 4$ supersymmetric QCD
- ▶ $\varphi_M \rightarrow \infty$: the known result is reproduced

Conclusion

- ▶ $\Gamma(\alpha_s, \varphi)$ at 3 loops has been calculated via harmonic polylogarithms up to weight 5 in QCD and some theories with scalars, including $\mathcal{N} = 1, 2, 4$ supersymmetric QCD
- ▶ $\varphi_M \rightarrow \infty$: the known result is reproduced
- ▶ $\Omega(a, x)$ does not contain n_f, n_s : only 1 gluonic color structure at each L

Conclusion

- ▶ $\Gamma(\alpha_s, \varphi)$ at 3 loops has been calculated via harmonic polylogarithms up to weight 5 in QCD and some theories with scalars, including $\mathcal{N} = 1, 2, 4$ supersymmetric QCD
- ▶ $\varphi_M \rightarrow \infty$: the known result is reproduced
- ▶ $\Omega(a, x)$ does not contain n_f, n_s : only 1 gluonic color structure at each L
- ▶ $\varphi = \pi - \delta$: the relation to $V(\vec{q})$ which follows from conformal invariance is violated at 3 loops by a term $\sim \beta_0 \alpha_s^3$

Conclusion

- ▶ $\Gamma(\alpha_s, \varphi)$ at 3 loops has been calculated via harmonic polylogarithms up to weight 5 in QCD and some theories with scalars, including $\mathcal{N} = 1, 2, 4$ supersymmetric QCD
- ▶ $\varphi_M \rightarrow \infty$: the known result is reproduced
- ▶ $\Omega(a, x)$ does not contain n_f, n_s : only 1 gluonic color structure at each L
- ▶ $\varphi = \pi - \delta$: the relation to $V(\vec{q})$ which follows from conformal invariance is violated at 3 loops by a term $\sim \beta_0 \alpha_s^3$
- ▶ $C_F(T_F n_f)^{L-1}$ and $C_F^2(T_F n_f)^{L-2}$ ($L \geq 3$) in Γ , γ_h and V are known to all loops