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Vacuum Selection on Axionic Landscapes

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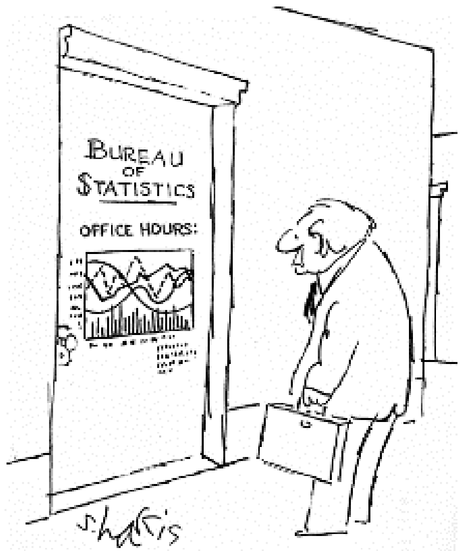
Motivation: Cosmology on Landscapes

To reconcile apparently fine tuned measured quantities (value of the CC etc.) and predict other observables (scalar spectral index, tensor to scalar ratio etc.) it is tempting to use statistical arguments, noting that:

- Multi-field models with complicated potentials are common in String Theory. Many different histories of the early universe are possible.

- Eternal inflation samples different histories.



Some Concerns

Landscape in ST: plethora of metastable vacua – eternal inflation.

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Caveat I: Vacua may not be long lived

- Resonant tunneling (Tye; Saranki Shiu, Shlaer 07)
- Disorder (Podolsky et.al. 08)
- Giant Leaps (Brown, Dhalen, 10,11)
- Instabilities (Greene at.al.13)
- Classical unstable directions (D.Battefeld, T.B. 12)

Working hypothesis:

eternal inflation is present (can check In concrete setups, see e.g. Masoumi, Vilenkin 16 for axionic landscapes)

Some Concerns

Landscape in ST: plethora of metastable vacua – eternal inflation.

Caveat I: Vacua may not be long lived

Caveat II: Measure Problem

no mathematically rigorous way to make predictions in the multiverse (dependence on choice of measure)

See e.g. Schiffrin and Wald 12, Freivogel 11 for reviews.

Approach:

Pragmatism: choose a sensible one and try to make predictions, Guth 12, Vanchurin 15, …

Some Concerns

Landscape in ST: plethora of metastable vacua – eternal inflation.

Caveat I: Vacua may not be long lived
Caveat II: Measure Problem
Caveat III: Potential on Landscape relevant for our universe is unknown - Constructions of concrete toy-models (e.g. Denev-Douglas, Random N=1 SUGRA, see i.e. Marsh, McAllister, Wrase 11; Pedro, Westphal 13, …) - Use of random potentials (Easther, Aazami 05; Agarwal et.al.11; Frazer, Liddle 11; T.B., D.B., Schulz 12, …) Approach: Phenomenology: pick framwork, use rand. potentials + rand. matrix theory.

Some Concerns

Landscape in ST: plethora of metastable vacua – eternal inflation.

Caveat I: Vacua may not be long lived
Caveat II: Measure Problem
Caveat III: Potential on Landscape relevant for our universe is unknown

Let's assume that the basic idea is right.

To make any progress, we need to understand how to

- model concrete landscapes via random potentials
- efficiently sample different histories (numerically challenging for many fields even at the background level) and compute observables (work in progress, not this talk.)
- make analytic predictions (this talk): use e.g. random Matrix theory.

Some Known Results based on Random Matrix theory:

Super exponential suppression for non-diagonal Hessians with zero mean:
For large D, almost all critical points are saddle points.

Aazami, Easther 05;
Dean, Majumdar 06; Vivo, Majumdar, Bohigas 07;
Marsh, McAllister, Wrase 11;
Chen, Shiu, Sumitomo, Tye 11

If Inflation takes place, it will most likely occur at a single saddle point.

Aazami, Easther 05

If Inflation takes place, it is more likely to be of short duration.

Freivogel, Kleban, Rodriguez Martinez, Susskind 05;
Agarwal, Bean, McAllister, Xu 11; Schulz, D.B., T.B. 12;
Marsh, McAllister, Pajer, Wrase 13

A concrete Landscape: Axions

- Common in String Theory.
- Potential protected by (softly broken) shift symmetry.
- Candidates for (multi-field) inflation: alignment effects (KNP mechanism, natural inflation, N-flation, …).
- Dark matter candidates (wide range of possible masses).
- A simple landscape to try probabilistic arguments and/or anthropics.

Potential:

mixing matrix

$$V = \sum_{i=1}^{N_{\rm source}} \Lambda_i^4 \left(1 - \cos\left(\sum_{j=1}^{N_{\rm axion}} n_{ij} \frac{\phi_j}{f_j} + \theta_i\right)\right) + C$$

strength of source terms

decay constant

e.g.: T. Higaki, F. Takahashi 15

A priori, we do not know the values of parameters.
Consider limit $N_{\rm source} \gg N_{\rm axion}$

Make educated guess about the distribution of parameters.

Expect: Since the potential becomes random, universal features should become apparent.

Question:

Given the relatively simple potential of multi-axion systems,
can we compute the distribution of vacua that are reached dynamically after fields evolve on such a landscape?

Steps:

1.Counting of all minima

- numerically (only viable for small n and D),

  T. Higaki, F. Takahashi 15
- analytic approximation of Probability Distribution Function (PDF)

  (application of random matrix theory (RMT), large n, D limit).

  G. Wang, T. Battefeld 15

2. Incorporate effect of classical evolution, G. Wang, T. Battefeld 15
   - numeric (only viable for small n and D),
   - analytic: we will see that RMT is not sufficient!

3. Incorporate stability of vacua w.r.t. tunneling, A. Masoumi, A. Vilenkin 16
   - numeric
   - analytic (reliable analytic approximations are possible)

Conventions:

$$V = \sum_{i=1}^{N_{\mathrm{source}}} \Lambda_i^4 \left(1 - \cos\left(\sum_{j=1}^{N_{\mathrm{axion}}} n_{ij}\frac{\phi_j}{f_j} + \theta_i\right)\right) + C$$

$$D \equiv N_{\mathrm{axion}} \qquad C \equiv 0$$

$$n \equiv N_{\mathrm{source}} \qquad \tilde{n} \equiv n^{1/D}$$

Mixing matrix: real valued, so we can set $f_j = 1$ without loss of generality, distributed over the interval $[-3, 3]$

Source terms: distributed over the unit interval, and rescaled such that

$$\sum_{i=1}^{n} \Lambda_i^4 = 1$$

Phases are chosen randomly.

Note: we varied the type of distributions: results become insensitive in the large n, D limit.

Find Minima numerically:

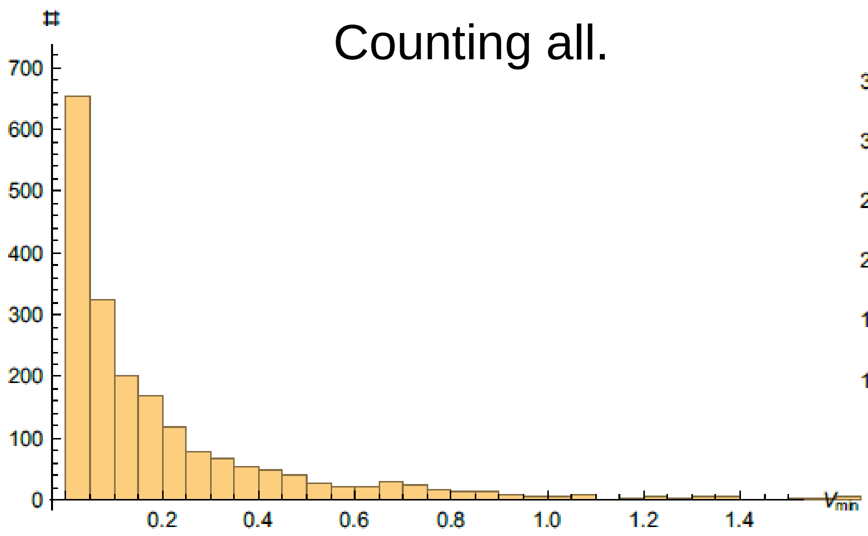
Choose 5000 random initial values, and solve

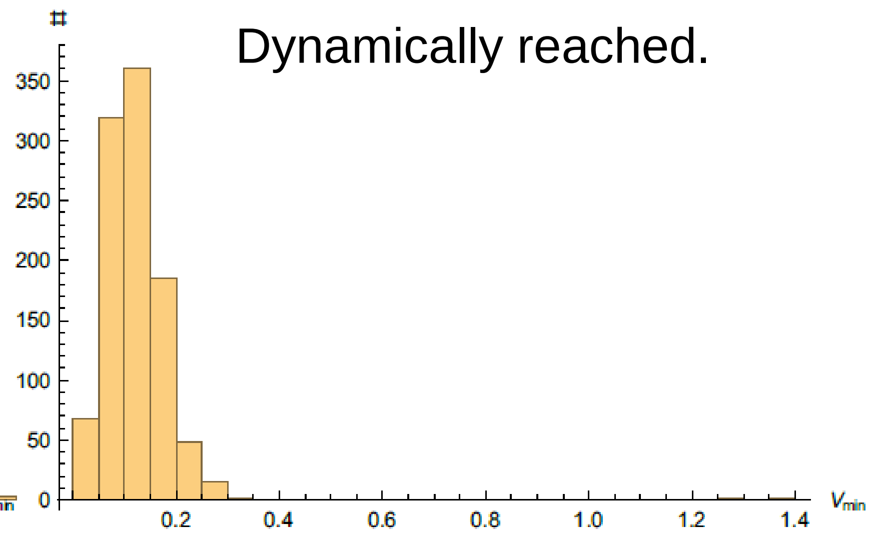
$$\ddot{\phi}_i + 3H\dot{\phi}_i = -\frac{\partial V}{\partial \phi_i}$$

$$3H^2 = V + \sum_{i=1}^{D} \frac{\dot{\phi}_i^2}{2}$$

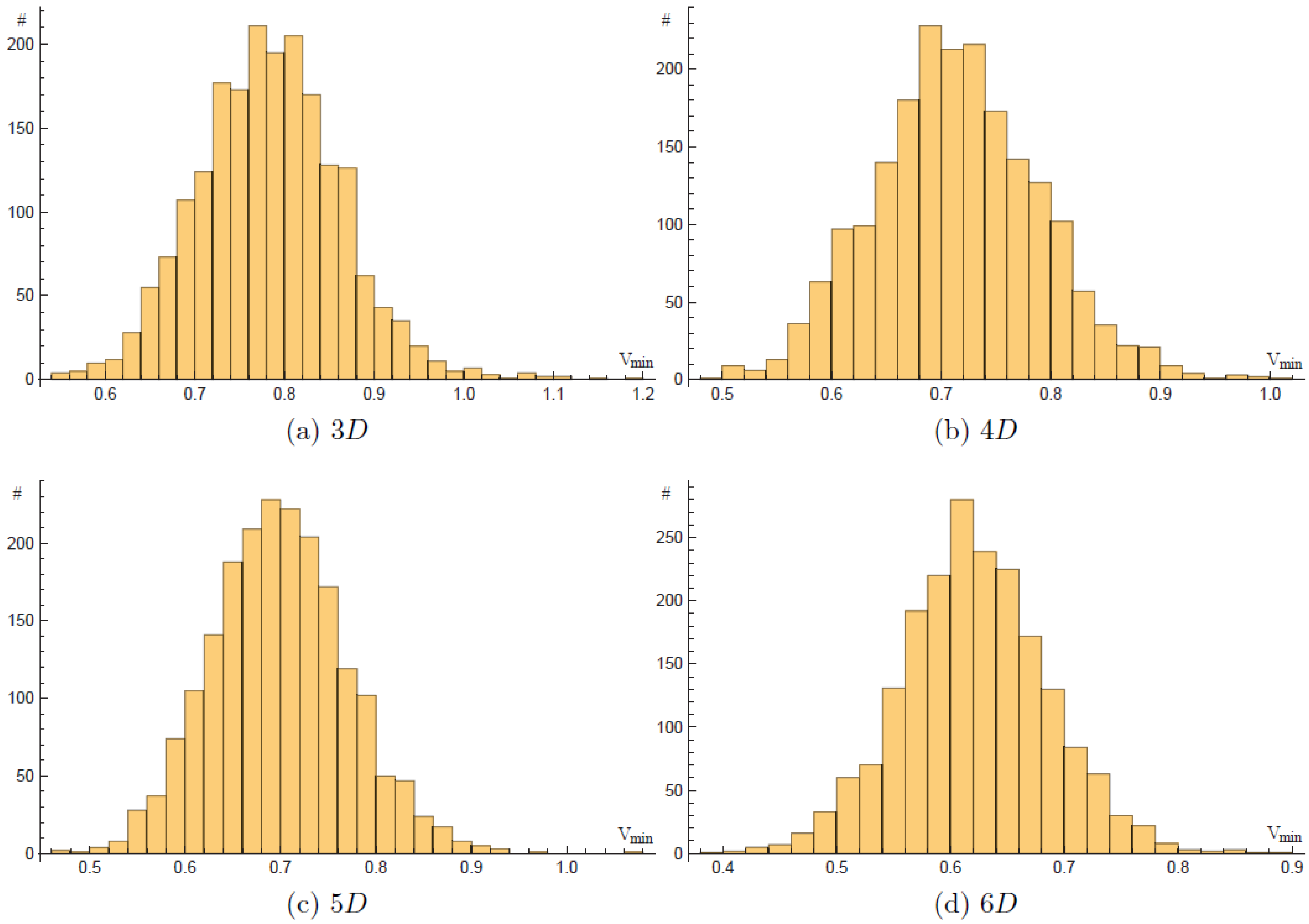
until a minimum is found.

Observation: the PDF of dynamically reached minima differs considerably from the one of all minima. Example: $n = 13$, $D = 8$





Goal: explain PDF of dynamically reached minima



(a) $3D$ (b) $4D$ (c) $5D$ (d) $6D$

Often used approach:

Outline of the (tedious) computation:

1. Based the definition of the potential, compute the PDF of V and its first and second derivative.

2. Compute the number of critical points that are reached if a given potential distance is traversed.

3. Based on the PDF of the Hessian and RMT, compute the probability that a given critical point is a minimum.

4. compute PDF of minima.

1. PDF of V

Take Gaussian distribution for Λ_i^4 with mean and variance,

$$\mu_\Lambda \equiv \frac{1}{\tilde{n}^D} = \frac{1}{n}$$
$$\sigma_\Lambda \equiv \frac{a}{\tilde{n}^D}\,, \quad a \equiv 0.1$$

The PDF of V becomes approximately Gaussian, with

$$\mu_V = 1\,,$$
$$\sigma_V = \frac{a}{\sqrt{n}}\,.$$

Note: ``a'' controls the scatter among different sources. As long as a << 1, our final results are independent of it.

1. PDF of the gradient's absolute magnitude

It obeys a chi^2-distribution:

$$f(V') = \frac{V'^{D-1}}{\sigma^D 2^{\frac{D-2}{2}} \Gamma\left(\frac{D}{2}\right)} \exp\left(-\frac{V'^2}{2\sigma^2}\right)$$

$$\sigma_{V_k}^2 \approx \frac{9C + 4a^2}{n} \approx \frac{1}{n}$$

$C \approx 0.11$, due to sum over random variables on unit interval, so $\sigma = \sigma_{V_k} \approx \frac{1}{\sqrt{n}}$

The expectation value becomes

$$\overline{V'} = \int_0^\infty V' f(V')\,\mathrm{d}V' = \sigma\sqrt{2}\frac{\Gamma\left(\frac{D+1}{2}\right)}{\Gamma\left(\frac{D}{2}\right)} \approx \sqrt{D/n}$$

Note: the last expression is valid in the large n, D limit, but analytic expressions are available for all values of n,D.

1. PDF of the Hessian:

The Hessian at a given height has a deterministic component and a random one,

$$
\begin{aligned}
H_{kl}\Big|_{V_c} &= \frac{\partial^2 V}{\partial \phi_k \phi_l}\Big|_{V_c} \\
&= -\frac{9}{\tilde{n}^2} V_c + \tilde{H}_{kl},
\end{aligned}
$$

which is approximately Gaussian distributed. The Hessian has therefore a mean and variance of:

$$
\mu_{kl} \approx \frac{9}{\tilde{n}^2}(1 - V_c)
$$

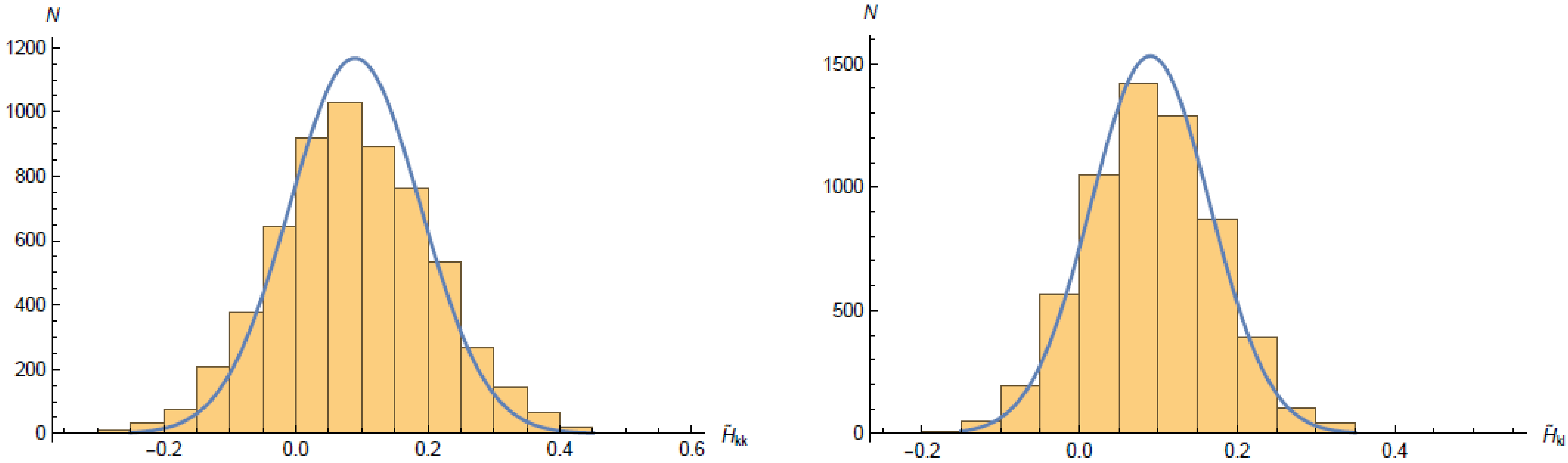
$$
\sigma_{kl} = \begin{cases} \sigma_{\rm dia} \approx \sqrt{\frac{3}{2}}\sigma_3 \approx \frac{3.98}{\sqrt{n}} & \text{for } k = l \\ \sigma_{\rm offdia} \approx \sqrt{\frac{3}{2}}\tilde{\sigma}_3 \approx \frac{2.2}{\sqrt{n}} & \text{for } k \neq l\,, \end{cases}
$$

where

$$
\tilde{\sigma}_3 = \sqrt{\frac{2C}{12}\frac{(1+\tilde{n})(1+2\tilde{n})(\tilde{n}^2+\tilde{n})}{\tilde{n}^D}\frac{9}{\tilde{n}^2}} \qquad \sigma_3 = \sqrt{\frac{2C}{6}\frac{(1+\tilde{n})(1+2\tilde{n})\tilde{n}^2}{\tilde{n}^D}\frac{9}{\tilde{n}^2}}
$$

1. Check: PDF of the Hessian

Compare diagonal and off-diagonal entries of the Hessian at
2000 random points (Histogram) with the analytic approximation (solid)



(a) Diagonal

(b) Off-Diagonal

$\tilde{n} = 10,\ D = 3$

2. Number of critical points:

Ansatz: potential difference to nearest critical point at a given random point:

$$\Delta V(V') = \frac{V'}{\beta\sqrt{D}}$$

Numerically, beta is insensitive to changes in n, D and of order one. With the PDF of the gradient, we get

$$\overline{\Delta V} = \int_0^{+\infty} \Delta V(V') f(V')\, \mathrm{d}V' \approx \sqrt{\frac{1}{n\beta}}$$

So that the average number of critical points when traversing δV becomes:

$$n_c(\delta V) = \frac{\delta V}{\overline{\Delta V}} = \beta\, \delta V \sqrt{\frac{D}{2n}} \frac{\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D+1}{2}\right)}$$

$$\boxed{n_c(\delta V) \approx \delta V \sqrt{\beta n}\,.}$$

3. Probability that a critical point is a minimum:

The Hessian has a deterministic component give by the height of the potential. The probability that all eigenvalues are positive can be approximated by (RMT)

$$P_{\min}(V_c) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{9(1-V_c)}{2\sigma_{\mathrm{offdia}}\tilde{n}^2}\right)\right) \cdot \exp\left(-\frac{\ln(3)}{4}(D-1)^2\right)$$

Probability that the largest eigenvalue is positive.

The usual result due to Wiegners semicircle law for a Gaussian random matrix.

For large n and D, this probability becomes

$$P_{\min} \approx \begin{cases} \exp\left(-\frac{\ln(3)}{4}(D-1)^2\right) & \text{for } V_c < 1\ , \\ 0 & \text{for } V_c > 1\ . \end{cases}$$

4. Computing the Historgrams

The probability that no minimum is reached down to a given value of V, marginalized over initial values, becomes

$$P_{\rm nomin}(V) = \int_V^2 \prod_{j=0}^{n_c} (1 - P_{\rm min}(V_{\rm ini} - j \cdot \Delta V)) \frac{1}{V_{\rm ini} - V} dV_{\rm ini}$$

Starting only at V=1 and working in the large n, D limit, yields

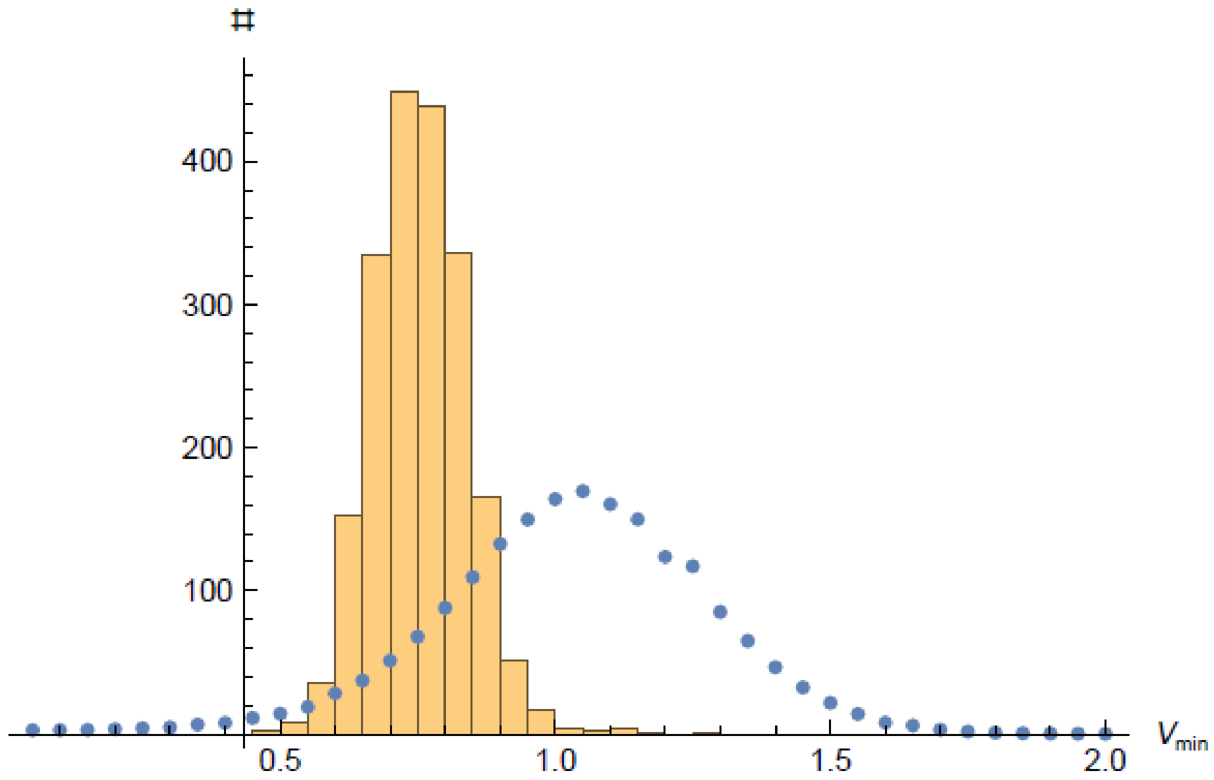
$$\begin{aligned} P_{\rm nomin}(V) &\approx (1 - P_{\rm min})^{n_c} \\ &\approx 1 - \sqrt{\beta n}(1 - V) \exp\left(-\frac{\ln(3)}{4} (D-1)^2 \right) \end{aligned}$$

with $n_c = (1 - V)/\overline{\Delta V} \approx (1 - V)\sqrt{\beta n}$

Note the exponential dependence on D.

4. Computing the Historgrams

Comparison to observed histograms shows large discrepancies, e.g: for D=3 and n=125



Causes for discrepancies:

- Two competing dynamical selection effects:

  1. Minima in the vicinity of the trajectory act as local attractors.

  2. Dynamical evolution tends to follow the steepest direction.

- Not all constraints are imposed (lower bound at V=0, gradient is zero at critical pt.)

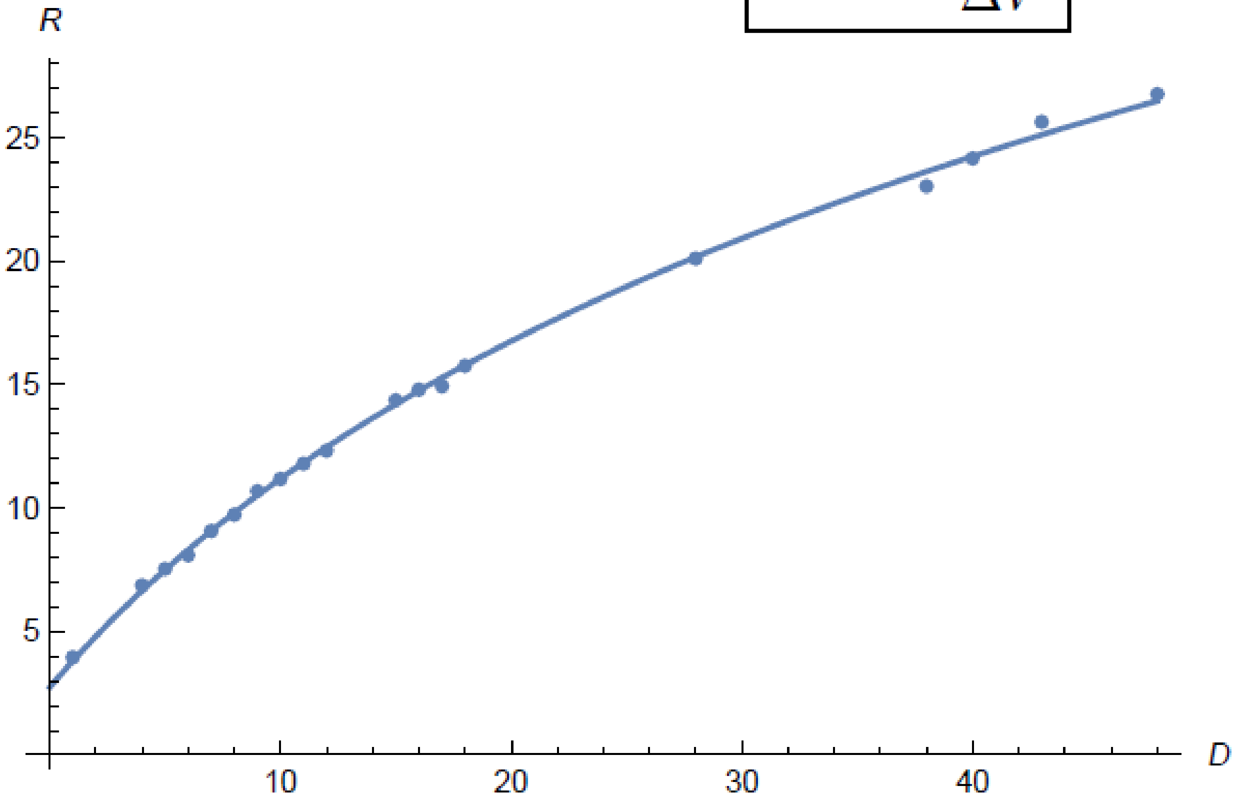
- entries in the Hessian are correlated

Empirical Result:

To bypass RMT, we measure 1/ $P_{\min}$ directly, that is, the ratio of

the average potential difference between encountered critical points to the mean potential difference to the next minimum.

$$R(D) \equiv \frac{dV}{\overline{\Delta V}}$$



R is well fitted by:

$$R(D) = a\ln(15 + D) + b$$

$$a \approx 16.38 \pm 0.14\,,$$

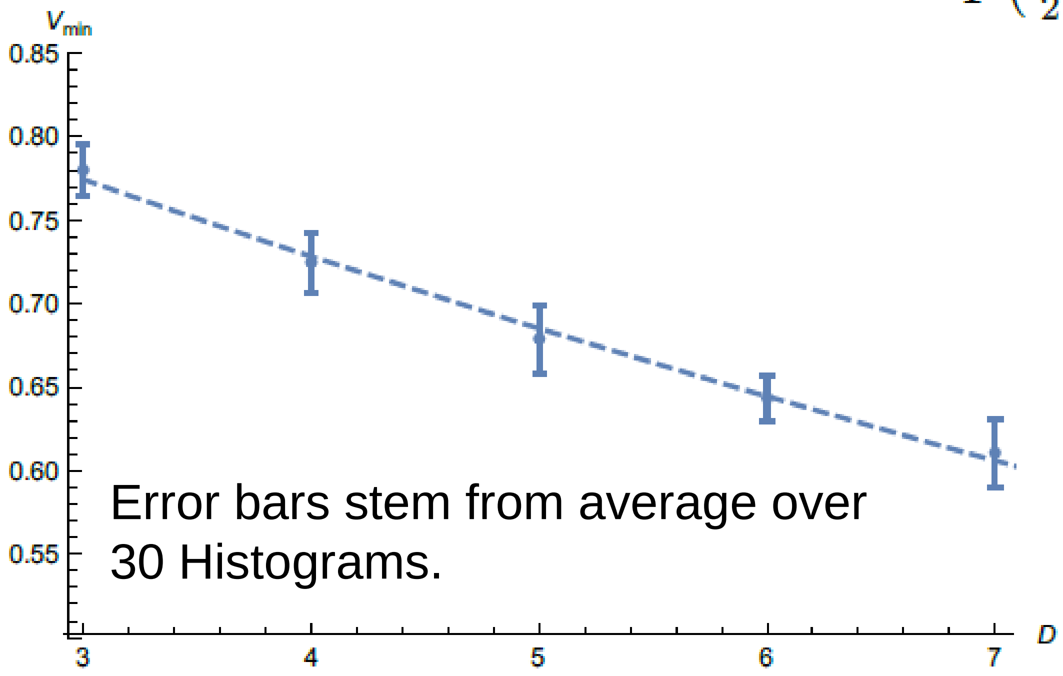
$$b \approx -41.56 \pm 0.48\,,$$

and insensitive to changes in n and only logarithmically dependent on D.

The mean of the Histograms

Given this simple empirical result, we get for the mean of the peaks position in the Histrograms:

$$V_{\rm min}(D,n) = V_{\rm ini} - \; R(D)\cdot\overline{\Delta V}$$
$$= V_{\rm ini} - (a\ln(15+D)+b)\,\frac{\Gamma\left(\frac{D+1}{2}\right)}{\Gamma\left(\frac{D}{2}\right)}\,\frac{\sqrt{2}}{\beta\sqrt{Dn}}$$



(a) $\bar{V}_{\rm min}$

One can also compute the width of the historgrams.

Summary

The peaks position in the PDF of dynamically reached minima depends only logarithmically on the number of fields:

$$V_{\min}(D,n) \propto \left(1 - \text{const.}\,\frac{\ln(15+D)}{\sqrt{n}}\right)$$

in contrast to the naïve expectation. A theoretical derivation is lacking so far.

Take home message: Counting all vacua is not sufficient for quantitative arguments.
RMT is not sufficient either.

Note: The PDF including dynamical selection effects needs to be multiplied with the one describing the stability of vacua with respect to tunneling computed in Masoumi, Vilenkin 16 to make quantitative predictions.

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