Recursion relations from soft theorems

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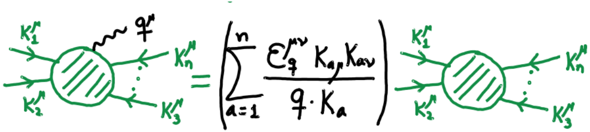
Overview

1. Soft theorems
2. Scattering Amplitudes and On-shell Construction
   - On Shell Recursion Relations at classical level
   - Boundary Contributions to On-shell Recursion Relations
3. Recursion Relations from Soft theorems for effective theories
   - Amplitudes of pure-Goldstones with vanishing soft limits
   - Amplitudes with non-vanishing soft limits
   - Amplitudes with non-Goldstone bosons

- Universal properties of low energy particle emissions, Weinberg's soft photon/graviton theorem [Weinberg, 1965]



$$= \left(\sum_{a=1}^{n} \frac{\epsilon_q^{\mu\nu} k_{a\mu} k_{a\nu}}{q \cdot k_a} \right)$$

- A powerful constraint to help to derive
  - electric charge conservation
  - universality of gravitational coupling [Weinberg, 1965; Weinberg's QFT Vol1]
  - no particles with helicities larger than 2 whose couplings cannot survive in the low energy limit
  - an evidence for Bondi-van der Burg-Matzner-Sachs (BMS) group, symmetries of asymptotic spacetime [Cachazo & Strominger,14']

Soft theorems

- Graviton Amplitudes obeying a soft identity [Cachazo & Strominger,14']

$$\mathcal{M}_{n+1} = (S^{(0)} + S^{(1)} + S^{(2)})\mathcal{M}_n + \mathcal{O}(q^2)$$

- Amplitudes in effective field theories (EFT) related to the symmetry breaking usually have nice properties in the soft limits $p_i \to 0$
  - Amplitudes of Goldstone bosons of a spontaneous global symmetry breaking vanish in the single soft limits ("Adler zero"): $A_n \sim p^\sigma,\ p \to 0$ e.g., NLSM, $\sigma = 1$ [Weinberg, QFT Vol. 2]
  - Alder zero valid for Goldstinos from spontaneous SUSY breaking e.g., Akulov-Volkov (A-V) theory [Chen, Huang & Wen, 14']
  - Soft theorems for broken conformal invariance

$$A_n\Big|_{p_1 \to \tau p_1} \to \left(\tau^{-1} S^{(-1)}_{M,1} + S^{(0)}_{M,1} + \tau S^{(1)}_{M,1} + S^{(0)}_1 + \tau S^{(1)}_1\right) A_{n-1,1}(p_2, \ldots, \bar{p}_n) + \mathcal{O}(\tau^2)$$

[Boels & Wormsbecher, 15'; Di Vecchia et al., 15']

Why scattering amplitudes?

- Gauge invariant on-shell scattering amplitudes as input for computing the (differential) cross-section
- Number of Feynman diagrams increases exponentially as number of particles in scattering increases

$n =$	**4**	**5**	**6**	**7**	**8**	**9**	**10**
	4	25	220	2485	34300	559,405	10,525,900

However, the on-shell scattering amplitudes can be written compactly and simply, e.g. MHV pure-gluon amplitudes, [Parke & Taylor, 80']

$$A_n[1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

- Feynman Diagrams (starting from Lagrangian) depend on
  - gauge choice
  - field redefinitions
- Work only with on-shell invariant inputs, avoiding complications in Feynman diagrams
- On-shell method:
  - On-shell recursion relations (BCFW) [Britto, Cachazo, Feng & Witten, 05']
  - "Sewing" MHV amplitudes (CSW) [Cachazo, Svrcek & Witten, 04']
  - ...
- Review the main idea of BCFW recursion relations

On-shell Recursion Relations at classical level

- Idea: Build up n-point amplitudes from lower-point amplitudes
  [See reviews: Feng & Luo, 11'; Elvang & Huang, 13']

- Scattering amplitudes: determined by poles through complex deformation of external momenta

$$p_i(z) = p_i + zq,\ \ p_j(z) = p_j - zq; \quad q^2 = q \cdot p_i = q \cdot p_j = 0$$
$$\frac{1}{(P' + p_i(z))^2} = \frac{1}{(P' + p_i)^2 + z(2q \cdot (P' + p_i))}$$

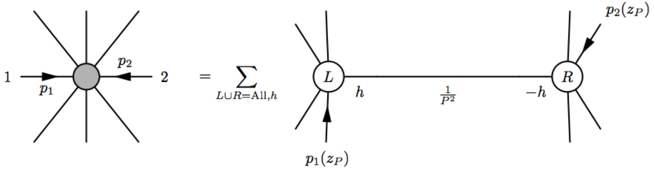
- Construct a contour integral and apply Cauchy's theorem

$$I = \oint \frac{dz}{z} A(z), \quad I = A(z=0) + \sum_{z_\alpha} \mathrm{Res}\left(\frac{A(z)}{z}\right)_{z_\alpha}$$

Boundary Contributions to On-shell Recursion Relations

- The on-shell recursion relations is constructible or not depending on the boundary contribution at $z = \infty$, i.e., if $A(z) \to 0$ while $z \to \infty$

$$A(0) = -\sum_{z_\alpha}\sum_{h=\pm} A_L(p_i(z_\alpha), p^h(z_\alpha))\frac{1}{P_\alpha^2}A_R(-p^h(z_\alpha), p_j(z_\alpha)).$$



[Arkani-Hamed et al.,10']

- In EFTs, high-order contact operators result in non-zero boundary terms at $z = \infty$; Additional information are required besides pole structures: Soft Theorems!

Amplitudes of pure-Goldstones with vanishing soft limits

- All-line "soft-BCFW" shifts: $\hat{p}_i = (1 - a_i z) p_i$, $\sum_{i=1}^{n} a_i p_i = 0$
  - It has non-trivial solutions of a_i when $n > d + 1$ [Cheung et al.,15']
  - When z approaches $1/a_i$, $\hat{p}_i \to 0$
- Construct contour integral of the complex parameter z

$$I = \oint \frac{dz}{z} \frac{A_n(z)}{F_n^{(\sigma)}(z)}, \quad F_n^{(\sigma)}(z) = \prod_{i=1}^{n} (1 - a_i z)^{\sigma}$$

  - $F_n^{(\sigma)}(z)$ improves the large-z behavior: no boundary contribution at $z = \infty$ if σ large enough;
  - $F_n^{(\sigma)}(z)$ might introduce additional singularities (poles or branch cut)
  - Such construction can be useful only if we know the residues of the additional poles introduced by $F_n^{(\sigma)}(z)$ for a positive integer σ

- A special and simple case: additional poles have vanishing residues, $A(z = 1/a_i) = 0$ (vanishing soft limits);
  - Soft limits $A_n(\tau p_i) \sim \tau^\sigma, \tau \to 0$ or $A_n(z) \sim (1 - a_i z)^\sigma, z \to 1/a_i$ large-z behavior $A_n(z) \sim z^m, z \to \infty$
  - $m - n\sigma < 0$ to ensure vanishing residue at $z = \infty$
- From Cauchy's theorem, recursion relations are

$$A_n = A_n(0) = \sum_I \frac{1}{P_I^2} \frac{A_L(z_I^-)A_R(z_I^-)}{(1 - z_I^-/z_I^+)F_n^{(\sigma)}(z_I^-)} + (z_I^- \leftrightarrow z_I^+)$$

summing over all simple poles from factorization diagrams of $A_n(z)$
  - $1/P_I^2$: the internal propagator with $P_I = \sum_{i \in I} p_i$
  - $z_I^\pm$: solutions to the condition $0 = \left(P_I - z \sum_{i \in I} a_i p_i\right)^2$
- Construct amplitudes in EFTs, e.g., NLSM, scalar-DBI and the Galileon theory; as A-V theory from SUSY breaking

Amplitudes with non-vanishing soft limits

- Amplitudes have non-vanishing and universal soft behavior

$$A_n(\tau p_n)\big|_{\tau\to 0} = \sum_{k=q_1}^{q_2} \tau^k \left(\mathcal{S}_n^{(k)} A_{n-1}\right) + \mathcal{O}(\tau^{q_2+1})$$

- Perform the same "soft-BCFW" shifts and contour integral as before

$$\hat{p}_i = p_i(1 - a_i z), \quad A_n(0) = \oint_{|z|=0} \frac{dz}{z} \frac{A_n(z)}{F_n^{(\sigma)}(z)}, \quad \sigma = q_2 + 1$$

all the residues from $F_n^{(\sigma)}(z)$ can be extracted from soft theorem

- Recursion relations are constructed as [Luo & Wen, 15']

$$A_n(0) = \sum_I{}' \frac{1}{P_I^2} \frac{A_L(z_I^-)A_R(z_I^-)}{(1 - z_I^-/z_I^+)F_n^{(q_2+1)}(z_I^-)} + (z_I^- \leftrightarrow z_I^+) - \sum_{i=1}^{n} \sum_{k=q_1}^{q_2} R_i^{(k)}$$

If $A_n(z) \sim z^m$ at $z \to \infty$, $m < n(q_2 + 1)$ to ensure the recursion relations

Amplitudes with non-vanishing soft limits

- Recall the recursion relations

$$A_n(0) = \sum_I{}' \frac{1}{P_I^2} \frac{A_L(z_I^-)A_R(z_I^-)}{(1 - z_I^-/z_I^+)F_n^{(q_2+1)}(z_I^-)} + (z_I^- \leftrightarrow z_I^+) - \sum_{i=1}^{n} \sum_{k=q_1}^{q_2} R_i^{(k)}$$

  - $\sum_I{}'$ excludes factorizations containing 3-pt amplitudes, which are included in the soft theorem contributions $R_i^{(k)}$
  - The additional term

$$R_i^{(k)} = \frac{1}{2\pi i} \oint_{z=\frac{1}{a_i}} dz \frac{\left(\mathcal{S}_i^{(k)} A_{n-1,i}\right)(z)}{z(1-a_i z)^{q_2+1-k} F_{n-1,i}^{(q_2+1)}(z)}, \quad F_{n-1,i}^{(q_2+1)}(z) = \frac{F_n^{(q_2+1)}(z)}{(z-a_i)^{(q_2+1)}}$$

- Example: amplitudes from dilaton effective action used to prove a-theorem can be constructed with $\sigma = q_2 + 1 = 2$ and explicitly $S_i^{(0)}$ and $S_i^{(1)}$ known in the soft theorem
[Komargodski & Schwimmer, 11'; Huang & Wen, 15'; Di Vecchia et al., 15']

Amplitudes with non-Goldstone bosons

- A more general case: Goldstone particles + other particles;
  Soft theorems still valid by taking a single Goldstone particle soft

- Soft-BCFW shifts only on (soft) Goldstone particles: $\hat{p}_i = (1 - a_i z)p_i$;
  Usual BCFW shifts on other particles: $\hat{p}_j = p_j + zq$, $\hat{p}_k = p_k - zq$

- Example: two-scalar model

$$\mathcal{L} = \frac{1}{2}\partial_\mu \xi \partial^\mu \xi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}\lambda^{\frac{4}{d-2}} \phi^2 (\xi + v)^{\frac{4}{d-2}}$$

  with soft theorem

$$A_n\big|_{p_1 \to \tau p_1} \to \left(\tau^{-1} \mathcal{S}^{(-1)}_{\mathrm{M},1} + \mathcal{S}^{(0)}_{\mathrm{M},1} + \tau \mathcal{S}^{(1)}_{\mathrm{M},1} + \mathcal{S}^{(0)}_{1} + \tau \mathcal{S}^{(1)}_{1}\right) A_{n-1,1}(p_2, \ldots, \bar{p}_n) + \mathcal{O}(\tau^2)$$

  [Boels & Wormsbecher, 15'; Di Vecchia et al., 15']

  - Perform shifts $p_{\hat{1}} = (1 - z)p_1$, $p_{\hat{2}} = p_2 + zq_2$, $p_{\hat{3}} = p_3 + zq_3$ with $q_2^2 = q_3^2 = 0$, $q_2 \cdot p_2 = q_3 \cdot p_3 = 0$, $q_2 + q_3 = k_1$
  - Contour integral $A_n(0) = \oint_{|z|=0} dz[A_n(z)/(z(1 - z))]$, $F_1^{(1)}(z) = (1 - z)$;
  - Amplitudes from recursion relations have been checked up to 5-pt

Thank you!

- Study of classical gravitational waves: Expected Poincaré symmetry enlarged by BMS_4 group
- Acts at null infinity ($\mathcal{I}^{\pm}$) for asympt. flat space-times
- Coordinates: u (retarded time), r (radius), $x^A = \{\Theta, \phi\} \in S^2$ at $\mathcal{I}^{\pm}$

$$\boxed{ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}\,du\,dr + g_{AB}(dx^A + U^A du)(dx^B + U^B du)}$$

Metric functions β, V, U^A, g_{AB} have fall-off conditions in r:

$$g_{AB} = r^2(d\Theta^2 + \sin^2\Theta\, d\phi^2) + \mathcal{O}(r),\ \ \beta = \mathcal{O}(r^{-2}),\ \ \frac{V}{r} = \mathcal{O}(r),\ \ U^A = \mathcal{O}(r^{-2})$$

- BMS_4 group: Maps asymptotically flat space-times onto themselves

$$\boxed{\Theta' = \Theta'(\Theta,\phi) \qquad \phi' = \phi'(\Theta,\phi) \qquad u' = K(\Theta,\phi)\,(u - \alpha(\Theta,\phi))}$$

Where $(\Theta,\phi) \to (\Theta',\phi')$ is conformal transformation on S^2:

$$d\Theta'^2 + \sin^2\Theta' d\phi'^2 = K(\Theta,\phi)^2(d\Theta^2 + \sin^2\Theta\, d\phi^2)$$

- For $\Theta' = \Theta$ & $\phi' = \phi$ one has "supertranslations": $u' = u - \alpha(\Theta,\phi)$ with a general function $\alpha(\Theta,\phi)$.

In standard complex coordinates $z = e^{i\phi} \cot(\Theta/2)$ conformal symmetry generated by Virasoro generators ("superrotations")

$$l_n = -z^{n+1}\,\partial_z \qquad \bar{l}_n = -\bar{z}^{n+1}\,\partial_{\bar{z}}$$

Supertranslations generated by $T_{m,n} = z^m\,\bar{z}^n\,\partial_u$

Extended $\mathfrak{bms}_4$ algebra [Barnich, Troessart]

$$[l_n, l_m] = (m-n)\,l_{m+n} \qquad [\bar{l}_n, \bar{l}_m] = (m-n)\,\bar{l}_{m+n}$$

$$[l_l, T_{m,n}] = -m\,T_{m+l,n} \qquad [\bar{l}_l, T_{m,n}] = -n\,\bar{T}_{m,n+l}$$

Poincaré subalgebra spanned by $\underbrace{l_{-1}, l_0, l_1; \bar{l}_{-1}, \bar{l}_0, \bar{l}_1}_{\text{Lorentz}} \quad \underbrace{T_{0,0}, T_{0,1}, T_{1,0}, T_{1,1}}_{\text{Translation}}$

BMS_4 group maps gravitational wave solutions onto each other.

Claim: | Supertranslations $\hat{=}\ S_{\mathsf{G}}^{(0)}$ | Superrotations $\hat{=}\ S_{\mathsf{G}}^{(1)}$ | [Cachazo, Strominger]