Two-loop Bhabha Scattering at High Energy beyond Leading Power Approximation

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DESY Theory Workshop, Hamburg 2016

[Phys.Lett. B760 (2016) 816-822]

Overview

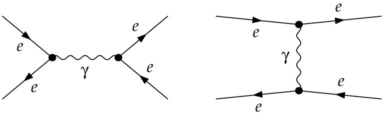
Motivation

Bhabha scattering alias $e^+e^- \to e^+e^-$

- Is “Standard Candle” process at e^+e^--colliders
- Is testing ground for new *HEP* calculation techniques
- Allows to explore the structure of QFT in classic QED using PT

$e^+e^- \to e^+e^-$ @ QED tree level [Bhabha'36]

- Amplitude $\mathcal{M}^{\text{tree}}$

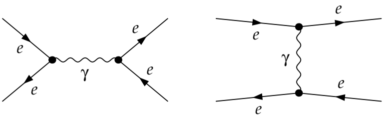


- Unpolarized amplitude square in terms of Mandelstam variables s, t $(u + s + t = 4m_e^2)$

$$\frac{1}{4}\sum_S |\mathcal{M}^{\text{tree}}|^2 = 4(4\pi\alpha)^2 \left[\frac{s^2}{t^2} + \frac{t^2}{s^2} + 2\left(\frac{s}{t} + \frac{t}{s}\right) + 3 - 4\left(\frac{t}{s^2} + \frac{s}{t^2}\right) m_e^2 + 4\left(\frac{1}{s^2} + \frac{1}{t^2} - \frac{1}{st}\right) m_e^4 \right].$$

$e^+e^- \to e^+e^-$ @ QED tree level [Bhabha'36]

- Amplitude $\mathcal{M}^{\text{tree}}$

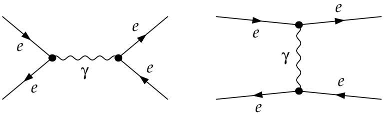


- Differential cross section at tree level

$$\frac{d\sigma^{\text{tree}}}{d\Omega} \sim \frac{1}{4} \sum_S |\mathcal{M}^{\text{tree}}|^2$$

$e^+e^- \to e^+e^-$ @ QED tree level [Bhabha'36]

- Amplitude $\mathcal{M}^{\text{tree}}$



- Differential cross section at tree level $\rho = m_e^2/s$

$$\frac{d\sigma^{\text{tree}}}{d\Omega} = \frac{\alpha^2}{s}\sum_{n=0}^{2}\rho^n \frac{d\sigma_n^{(0)}}{d\Omega}$$

$\sigma(e^+e^- \to e^+e^-)$ in QED at High Energy

- Differential cross section to fixed order in α and $\rho = m_e^2/s$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s} \sum_{n=0}^{2} \sum_{l=0}^{\infty} \rho^n \frac{d\sigma_n^{(l)}}{d\Omega} ,$$

$$\frac{d\sigma_n^{(l)}}{d\Omega} \sim \alpha^l .$$

- Logarithmic structure $M \in \{s, t, u, m_\gamma^2\}$

$$\frac{d\sigma_n^{(l)}}{d\Omega} = \sum_{m=0}^{m_{\max}=2l} \frac{\alpha^l}{(4\pi)^l} C_{m,n}^{(l)} \log^m \left(\frac{M}{m_e^2} \right) .$$

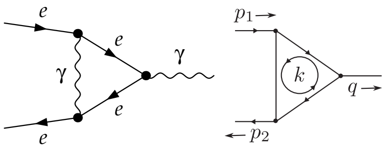
$\to m_{\max} = 2 \times (\#\text{loops}) \to$ double-logarithmic order
HERE: We present $C_{2l,1}^{(l)}$ up to including $l = 2$ (2-loops)

History of $e^+e^- \to e^+e^-$

- Tree[Bhabha'36]
- One-loop EW correcions [Böhm,Denner,Hollik'88]
- Small angle e^+e^- scattering amplitude at high-energy [Fadin,Kuraev,Trentadue,Lipatov,Merenkov'93]
- Two-loop virtual QED corrections [Bern,Dixon,Ghinculov'01]
- $\mathcal{O}(\alpha^2 \log s/m_e^2)$ [Glover,Tausk,Van der Bij'01]
- Two-loop $\alpha^4(N_F = 1)$ QED differential cross section + soft emission [Bonciani,Ferroglia,Mastrolia,Remiddi,vanderBij'04/05]
- Two-loop Vertex and one-loop by one-loop contributions $\sim \alpha^4$ (analytic/full m_e dep.) [Bonciani,Ferroglia 05]
- Two-loop $\sim \alpha^4$ (leading m_e dep.) [Penin'05/06]
- Two-loop fermionic corrections to massive Bhabha scattering ($m_e \ll m_f \ll s, t, u$) [Actis,Czakon,Gluza,Riemann'07]
- $\mathcal{M}(m_e \ll s, t, u) \leftrightarrow \mathcal{M}(m_e = 0 \ll s, t, u)$ [Becher,Melnikov'07]
- Two-loop Heavy-Flavor Contribution to Bhabha Scattering ($m_e \ll m_f, s, t, u$) [Bonciani,Ferroglia,Penin'08]
- Virtual Hadronic and Leptonic Contributions [Actis,Czakon,Gluza,Riemann'08,Kuhn,Uccirati'09]
- Summing up subleading Sudakov logarithms [Kühn,Penin,V.A.Smirnov'00]
- NNL logarithms in four-fermion electroweak processes at high energy [Kühn,Moch,Penin,V.A.Smirnov'01]
- Two-loop Sudakov Form Factor in a Theory with a Mass Gap [Feucht,Kühn,Penin,Smirnov'04]
- Two-loop electroweak logarithms in four-fermion processes at high energy [Jantzen,Kühn,Penin,Smirnov'05/06]
- Two-loop electroweak corrections to high energy large-angle Bhabha scattering [Penin,Ryan'11]
- Review [Actis,etal.'10]

Double Logs @ 1-Loop Sudakov Form Factor

- 1-loop form factor in Sudakov limit ($s \gg m_e^2$)



- Scalar Integral $p_i^2 = m_e^2$, $s = q^2 = (p_1 - p_2)^2$ evaluated for $s \gg m_e^2$

$$\mathcal{I}_S = \int \mathrm{d}^4k \frac{1}{k^2} \frac{1}{(k^2 - 2k \cdot p_1)} \frac{1}{(k^2 - 2k \cdot p_2)} \,.$$

Double Logs @ 1-Loop Sudakov Form Factor

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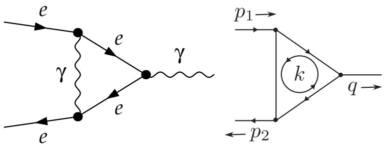

- Scalar Integral $p_i^2 = m_e^2$, $s = q^2 = (p_1 - p_2)^2$ evaluated for $s \gg m_e^2$
  Dim REG $\rightarrow d = 4 - 2\epsilon, \mu$

$$\mathcal{I}_S = \int \mathrm{d}^d k \frac{1}{k^2} \frac{1}{(k^2 - 2k \cdot p_1)} \frac{1}{(k^2 - 2k \cdot p_2)} \, .$$

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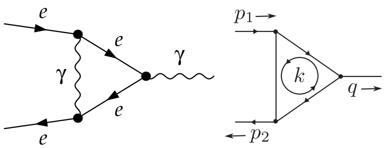
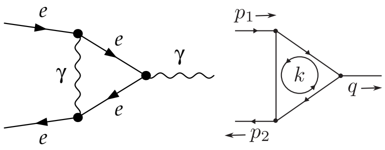


- Scalar Integral $p_i^2 = m_e^2$, $s = q^2 = (p_1 - p_2)^2$ evaluated for $s \gg m_e^2$
  Light Cone Coordinates $s = Q^2$

$$\mathcal{I}_S = \int \mathrm{d}^d k \frac{1}{k^2} \frac{1}{(k^2 - k_- Q - (k_+ + k_-) Q \rho^2 + \dots)} \times \frac{1}{(k^2 - k_+ Q - (k_+ + k_-) Q \rho^2 + \dots)} .$$

Double Logs @ 1-Loop Sudakov Form Factor

- 1-loop form factor in Sudakov limit ($s \gg m_e^2$)



- Scalar Integral $p_i^2 = m_e^2$, $s = q^2 = (p_1 - p_2)^2$ evaluated for $s \gg m_e^2$
  Expansion by Region

$$\mathcal{I}_S = \mathcal{I}_V^{\text{hard}} + \mathcal{I}_V^{\text{c}_1} + \mathcal{I}_V^{\text{c}_2} \; .$$

Double Logs in 1-Loop Sudakov Form Factor

Contributing regions (using light cone coordiantes & $s = Q^2$)

- hard $(k_+, k_-, k_\perp) \sim Q \cdot (1,1,1)^T$ $\qquad (L_Q = \log Q^2/\mu^2)$

$$\mathcal{I}_S^{\text{hard}} = \int \mathrm{d}^d k \frac{1}{k^2} \frac{1}{(k^2 - k_- Q)} \frac{1}{(k^2 - k_+ Q)} + \mathcal{O}(\rho^2)\,.$$

- collinear I $(k_+, k_-, k_\perp) \sim Q \cdot (1, \rho^2, \rho)^T$ $\qquad (L_m = \log m_e^2/\mu^2)$

$$\mathcal{I}_S^{c_1} = \int \mathrm{d}^d k \frac{1}{k^2} \frac{1}{(k^2 - k_- Q - k_+ Q \rho^2)} \frac{1}{(-k_+ Q)} + \mathcal{O}(\rho^2)\,.$$

soft photon & eikonal $e^\pm$ propagator

- collinear II $(k_+, k_-, k_\perp) \sim Q \cdot (\rho^2, 1, \rho)^T$ $\qquad (\rho = m_e^2/Q^2)$

$$\mathcal{I}_S^{c_2} = \int \mathrm{d}^d k \frac{1}{k^2} \frac{1}{(-k_- Q)} \frac{1}{(k^2 - k_+ Q - k_- Q \rho^2)} + \mathcal{O}(\rho^2)\,.$$

soft photon & eikonal $e^\mp$ propagator

Double Logs in 1-Loop Sudakov Form Factor

Contributing regions (using light cone coordiantes & $s = Q^2$)

- hard $(k_+, k_-, k_\perp) \sim Q \cdot (1, 1, 1)^T$ $\qquad (L_Q = \log Q^2/\mu^2)$

$$Q^2 \cdot \tilde{\mathcal{I}}_S^{\text{hard}} = -\frac{1}{\epsilon^2} + \frac{1}{\epsilon} L_Q - \frac{1}{2} L_Q^2 + \frac{1}{12}\pi^2 + \rho\left(\frac{2}{\epsilon^2} + \frac{1}{\epsilon}(4 - 2L_Q) + L_Q^2 - 4L_Q - \frac{1}{6}\pi^2\right) + \ldots .$$

- collinear I $(k_+, k_-, k_\perp) \sim Q \cdot (1, \rho^2, \rho)^T$ $\qquad (L_m = \log m_e^2/\mu^2)$

$$Q^2 \cdot \tilde{\mathcal{I}}_S^{\mathrm{c}_1} = +\frac{1}{2\epsilon^2} - \frac{1}{2\epsilon} L_m + \frac{1}{4} L_m^2 + \frac{1}{24}\pi^2 + \rho\left(-\frac{1}{\epsilon^2} + \frac{1}{\epsilon}(-1 + L_m) - \frac{1}{2} L_m^2 + L_m - 1 - \frac{1}{12}\pi^2\right) + \ldots .$$

- collinear II $(k_+, k_-, k_\perp) \sim Q \cdot (\rho^2, 1, \rho)^T$ $\qquad (\rho = m_e^2/Q^2)$

$$\tilde{\mathcal{I}}_S^{\mathrm{c}_2} = \tilde{\mathcal{I}}_S^{\mathrm{c}_1} .$$

due to $\pm$-symmetry...

Double Logs in 1-Loop Sudakov Form Factor

Combined scalar integral result

- $\mathcal{I}_S = \mathcal{I}_V^{\text{hard}} + \mathcal{I}_V^{c_1} + \mathcal{I}_V^{c_2}$ $\qquad (L_\rho = \log m_e^2/Q^2,\, L_\mu = \log Q^2/\mu^2)$

$$Q^2 \cdot \tilde{\mathcal{I}}_S = -\frac{L_\rho}{\epsilon} + \frac{1}{2}L_\rho^2 + L_\rho L_\mu + \frac{\pi^2}{6}$$
$$+ \rho\left(\frac{2+2L_\rho}{\epsilon} - L_\rho^2 + L_\rho(2-2L_\mu) - 2L_\mu - 2 - \frac{\pi^2}{3}\right)$$
$$+ \mathcal{O}(\rho^2)\,.$$

- Integrals and method see text book [V.A.Smirnov'02]
- L^2 terms are generated by highest poles $\sim \epsilon^{-2}$
- Higher order terms in ρ exp $\rightarrow k$ in numerator $\rightarrow$ more IR finite/spoils L^2 scaling $\rightarrow$ no L^2
- Coefficient of identified L^2 is independent of REG

Double Logs in 1-Loop Sudakov Form Factor

Fermion form factor integral result

- On shell conditions for external fermions reduce the fermionic integral to

$$\mathcal{I}_F \sim (p_1 \cdot p_2)\mathcal{I}_S$$

- Expressing Q^2 via $p_1 \cdot p_2$ (or vice versa) one finds:

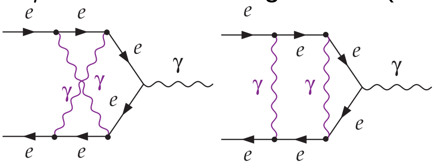
$$2\mathcal{I}_F \sim -\frac{L_\rho}{\epsilon} + \frac{1}{2}L_\rho^2 + L_\rho L_\mu + \dots \\ + \rho\left(0 \cdot L^2 + \dots\right) + \mathcal{O}(\rho^2)\,.$$

  - $\sim \rho^0 : L^2 =$ Sudakov logarithms generated by soft photon & eikonal $e^\pm$
  - Calculated + resummed by Sudakov [Sudakov'54] (using $d = 4$ and $m_\gamma \neq 0$)
  - $\sim \rho^1$: No L^2 terms in fermionic 1-loop form factor
  - Dirac structure does not allow m_e^2 and ϵ^{-2} in the same term

Double Logs in 2-Loop Sudakov Form Factor

Fermion form factor integral result

- $\sim \rho^0$: L^4 Sudakov logarithms (no $L^4 \sim \rho^1$!)



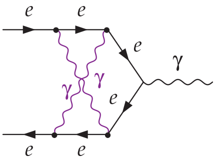
soft γ & eikonal $e^\pm$

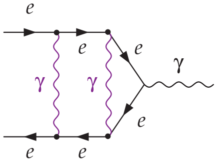
$$S \approx -\frac{\not{p}_j + m_e}{2p_j k_i}, \qquad D^{\mu\nu} \approx \frac{-g^{\mu\nu}}{k_i^2 - m_\gamma^2}.$$

Double Logs in 2-Loop Sudakov Form Factor

Fermion form factor integral result

- $\sim \rho^0$: L^4 Sudakov logarithms (no $L^4 \sim \rho^1$!)

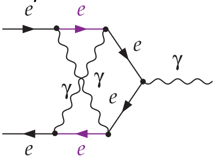




soft γ & eikonal $e^{\pm}$

$$S \approx -\frac{\not{p}_j + m_e}{2p_j k_i}, \qquad D^{\mu\nu} \approx \frac{-g^{\mu\nu}}{k_i^2 - m_\gamma^2}.$$

- $\sim \rho^1$: L^4 non-Sudakov logarithms are known [Penin'15]



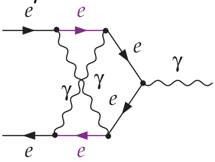
soft $e^{\pm}$ & eikonal γ (soft electron pair exchange)

$$S \approx \frac{m_e}{k_i^2 - m_e^2}, \qquad D^{\mu\nu} \approx \frac{g^{\mu\nu}}{2p_j k_i + m_\gamma^2 - m_e^2}.$$

Double Logs in 2-Loop Sudakov Form Factor

Fermion form factor integral result

- $\sim \rho^1 : L^4$ non-Sudakov logarithms are known [Penin'15]



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- In specific region double-logarithmic contributions extracted using Sudakov's method
- Double-logarithmic integrand scaling $+$ momentum conservation $+$ m_e^2 Dirac-structure $\neq$ easy to get

Double Logs in 2-Loop Sudakov Form Factor

Example for double[2] logarithmic integral extraction ala Sudakov
We use $m_e = m_\gamma \neq 0$ in $d = 4$[1]

$$\begin{aligned} I_1(p_i, p_j) &= \int d^4 l_1 d^4 l_2 D(l_1) D(l_2) D(p_i + l_1 + l_2) D(p_i + l_1) \\ &\quad \times D(p_j - l_1 - l_2) D(p_j - l_2)\,. \end{aligned}$$

Picking up only the $\sim L^4$ contribution of the integral
(see QED text book Landau-Lifshitz)

$$\begin{aligned} I_1(p_i, p_j) &\approx \left(\frac{i\pi^2}{s_{ij}}\right)^2 \int_\rho^1 \frac{\mathrm{d}v_1}{v_1} \int_{v_1}^1 \frac{\mathrm{d}v_2}{v_2} \int_{\rho/v_1}^1 \frac{\mathrm{d}u_1}{u_1} \int_{\rho/v_2}^{u_1} \frac{\mathrm{d}u_2}{u_2} \\ &\approx \frac{1}{12 s_{ij}^2} \left(i\pi^2 \ln^2 \rho\right)^2 \,. \end{aligned}$$

With $s_{ij} = (p_i + p_j)^2$.

[1]Soft photon $\log(m_e^2/m_\gamma^2)$ can be restored [Yennie,Frautschi,Suura'61]

Double Logs $\sim \rho^1$ in $e^+e^- \to e^+e^-$ up to 2-Loop

Amplitude Level

- NOT generated by Sudakov type soft γ corrections

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Amplitude Level

- NOT generated by Sudakov type soft γ corrections
- Are generated by diagrams containing
  $n\times$soft $e^\pm$ & $m\times$ soft γ ($n \geq 1$, $n+m$ =#loops)
  - soft $e^\pm/\gamma$ carry only loop momentum $S(k_i)/D(k_j)$
  - eikonal $e^\pm/\gamma$ carry single external momentum $S(p_i)/D(p_j)$
  - hard $e^\pm/\gamma$ carry two external momenta $S(p_i+p_j)/D(p_i+p_j)$

Double Logs $\sim \rho^1$ in $e^+e^- \to e^+e^-$ up to 2-Loop

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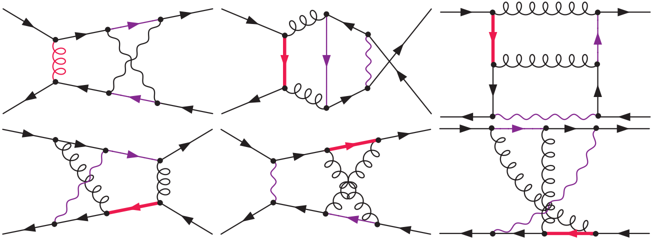
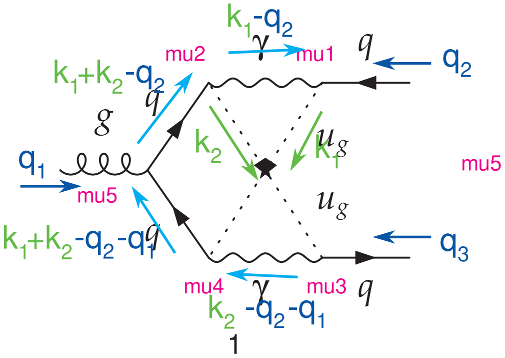
- example 1-PI diagrams

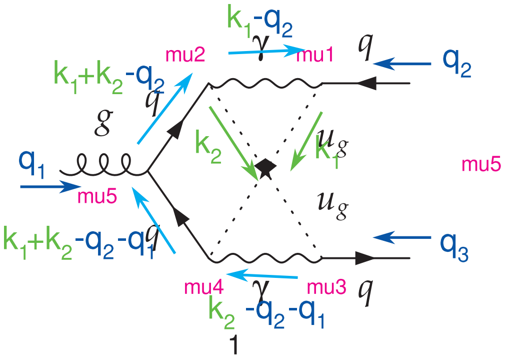


By Hand?



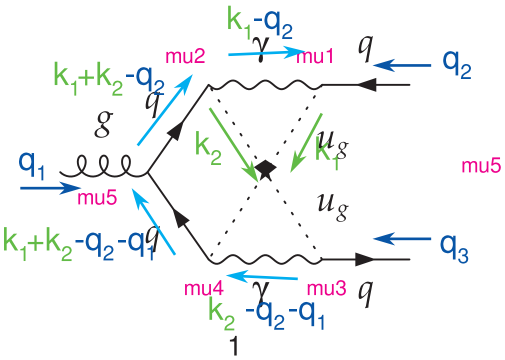
k1-q2
γ
mu2
mu1
q
q2
k1+k2-q2
q
g
q1
mu5
k2
k1
ug
mu5
ug
k1+k2-q2-q1
q
q3
mu4
γ
mu3
q
k2 -q2-q1
1

By Hand?



just vertex...

By Hand?


k1-q2
γ
mu2
mu1
q
q2
k1+k2-q2
q
g
q1
k2
ug
k1
mu5
mu5
ug
k1+k2-q2-q1
q
q3
mu4
γ
mu3
q
k2 -q2-q1
1

NO!

Technical Setup

- Diagram generation *QGRAF* [Nogueira'91]
- Drawing diagrams *FeynArts* [Hahn']
- Own *QGRAF* $\to$ *FeynArts* converter *Q2FGraphs*
- Own amplitude generator *amper* (controllable momentum routing) in *Mathematica*
- Amplitude calculation
  - Spin trace evaluation and LT contraction in *FORM* [Kuipers,Ueda,Vermaseren,Vollinga'12]
  - Own (very basic) master integral mapper in *Mathematica*
  - Lorentz decomposition [...] for $1l \times 1l$ box amplitudes in *FORM*

Double Logs $\sim \rho^1$ in $e^+e^- \to e^+e^-$ up to 2-Loop

Cross-Section $\sim$ |Amplitude|^2

- $1l \times 1l$ & $0l \times 2l$.
- 1-PR & 1-PI.
- Known Sudakov logs contribute through $\times$ with tree amp terms $\sim \rho^1$.
- $1 -$ loop $\sim L^2 \rho^1$: $\qquad x = -t/s$

$$\left.\frac{d\sigma_1^{(1)}}{d\Omega}\right|_{L^2} \sim C_{2,1}^{(1)} = -4\frac{d\sigma_1^{(0)}}{d\Omega} + \frac{6 - 20x + 24x^2 - 20x^3 + 6x^4}{(1-x)x^2} \,.$$

(✓ analytic result [Bonciani,Ferroglia 05])

- $2 -$ loop $\sim L^4 \rho^1$:

$$\left.\frac{d\sigma_1^{(2)}}{d\Omega}\right|_{L^4} \sim C_{4,1}^{(2)} = 8\frac{d\sigma_1^{(0)}}{d\Omega} - \frac{4 + 70x - 227x^2 + 266x^3 - 227x^4 + 70x^5 + 4x^6}{3(1-x)x^3}$$

$$= -\frac{4 + 166x - 323x^2 + 266x^3 - 323x^4 + 166x^5 + 4x^6}{3(1-x)x^3} \,.$$

Summary

1. In the high energy limit we determined on amplitude level the m_e^2 suppressed (process dependent) double-logarithmic contribution up to including 2-loops
2. Terms $\sim \rho L^{2l}$ originate from regions with at least one soft $e^{\pm}$
3. We calculated the m_e^2 suppressed $\sim \rho L^{2l}$ contributions for the Bhabha cross-section at the 2-loop order

Summary

Thanks for your attention!

Numerical effect

At $\theta = 30^\circ$

- maximal correction @ $\sqrt{s} \approx 8\, m_e \approx 4$ MeV:

$$\rho \frac{d\sigma_1^{(2)}}{d\Omega}\Big|_{L^4} \approx -24.4 \left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_0^{(0)}}{d\Omega}\,.$$

- $\downarrow$ when $s \uparrow$
- @ $\sqrt{s} = 300 m_e \approx 150$ MeV:

$$\rho \frac{d\sigma_1^{(2)}}{d\Omega}\Big|_{L^4} \approx -1 \left(\frac{\alpha}{\pi}\right)^2 \frac{d\sigma_0^{(0)}}{d\Omega}\,.$$

$\sim$ size of non-log 2-loop corrections $\sim \rho^0$

Restoring $m_\gamma \neq m_e$

Following [Yennie,Frautschi,Suura'61]:

$$\sigma_n = \exp\left[-\frac{2\alpha}{\pi}\left(\ln\rho + \mathcal{O}(1)\right)\ln\left(\lambda^2/m_e^2\right)\right]\sum_{m=0}^{\infty}\sigma_n^{(m)}\Big|_{m_\gamma=m_e}.$$

Factorization & Exponentiation

Known factorization and exponentiation of Sudakov double-logs at leading power $\sim \rho^0$

$$\frac{d\sigma_0}{d\Omega} = e^{-4\tau} \frac{d\sigma_0^{(0)}}{d\Omega}\,, \qquad \tau = \frac{\alpha}{4\pi} \log^2 \rho\,.$$

Light Cone Coordinates

$$n_{\pm}^2 = 0\,, \qquad n_+ \cdot n_- = 2\,.$$

Special frame choice

$$n_+ = (1,0,0,1)^T\,, \qquad n_- = (1,0,0,-1)^T\,,$$

$$p_1 = \left(\sqrt{\frac{Q^2}{4}+m^2}, 0, 0, \frac{Q}{2}\right)^T\,, \qquad p_2 = \left(\sqrt{\frac{Q^2}{4}+m^2}, 0, 0, -\frac{Q}{2}\right)^T\,.$$

For any 4-vector we use the shorthand:

$$\begin{aligned} p &= p_+ \tfrac{1}{2} n_+ + p_- \tfrac{1}{2} n_- + p_\perp \\ &= [p_+, p_-, p_\perp]^T\,. \end{aligned}$$

We have

$$p_1 \approx Q \begin{bmatrix} 1+\rho^2+\mathcal{O}(\rho^4) \\ 0+\rho^2+\mathcal{O}(\rho^4) \\ 0 \end{bmatrix}\,, \qquad p_2 \approx Q \begin{bmatrix} 0+\rho^2+\mathcal{O}(\rho^4) \\ 1+\rho^2+\mathcal{O}(\rho^4) \\ 0 \end{bmatrix}\,.$$