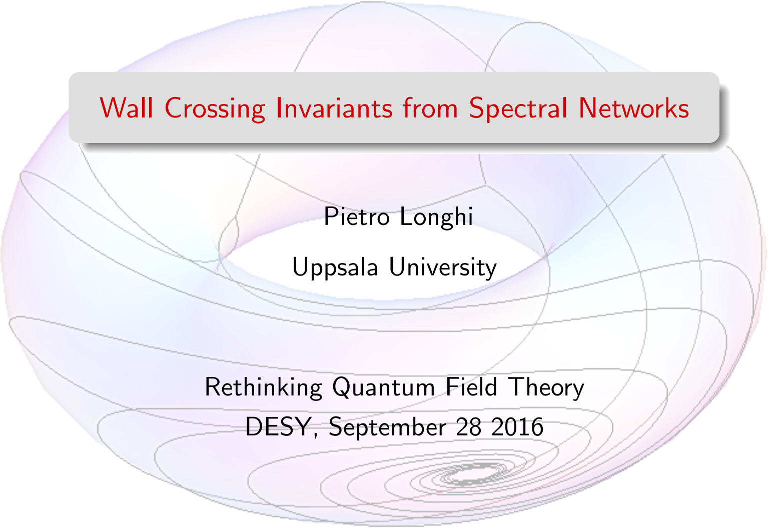
Wall Crossing Invariants from Spectral Networks



Pietro Longhi

Uppsala University

Rethinking Quantum Field Theory

DESY, September 28 2016

Goal of the talk:

A construction of the BPS monodromy for theories of class S, directly from the Coulomb branch geometry.

- Doesn't rely on knowledge of the BPS spectrum
- Manifest wall-crossing invariance

Motivations

- The BPS monodromy $\mathbb{U}$ is of central importance in wall crossing. It is also a spectrum generating function, BPS state counting follows from knowledge of $\mathbb{U}$ **[Kontsevich-Soibelman, Gaiotto-Moore-Neitzke, Dimofte-Gukov]**.
- Relations to various limits of the superconformal index and counts of chiral operators in the SCFT **[Cecotti-Neitzke-Vafa, Iqbal-Vafa, Cordova-Shao, Cecotti-Song-Vafa-Yan]**.
- Graphs encoding $\mathbb{U}$ are an important link in the Network/Quiver correspondence

On Coulomb branches $\mathcal{B}$ of 4d $\mathcal{N}=2$ gauge theories gauge symmetry is spontaneously broken to $U(1)^r$.

At **generic** $u \in \mathcal{B}$ the lightest charged particles are **BPS solitons** $|\psi\rangle = |\gamma, m\rangle \in \mathscr{H}_u^{\mathrm{BPS}}$ characterized by charge $\gamma \in \mathbb{Z}^{2r+f}$ and spin $j_3 = m$

$$M\,|\psi\rangle = |Z_\gamma|\,|\psi\rangle\,, \quad \mathcal{Q}_\vartheta|\psi\rangle = 0 \qquad (\vartheta = \mathrm{Arg} Z_\gamma)\,.$$

$Z_\gamma(u)$ is topological, linear in γ, locally holomorphic in u.
Low energy dynamics on $\mathcal{B}$ is captured by a geometric picture, involving a family of complex curves Σ_u fibered over $\mathcal{B}$ [Seiberg-Witten].

$$\gamma \in H_1(\Sigma_u, \mathbb{Z}) \qquad Z_\gamma = \frac{1}{\pi}\oint_\gamma \lambda$$

On $\mathbb{R}^3 \times S^1_R$ a 3d σ-model into $\mathcal{M} \to \mathcal{B}$, effective action receives quantum corrections $\sim e^{-2\pi R|Z_\gamma|}$ from BPS particles wrapping S^1_R.
The metric on $\mathcal{M}$ therefore encodes the BPS spectrum, which can be extracted with geometric tools like spectral networks [Gaiotto-Moore-Neitzke].

BPS particles interact, forming boundstates

$$E_{bound} = |Z_{\gamma_1+\gamma_2}| - |Z_{\gamma_1}| - |Z_{\gamma_2}| \leq 0$$

Boundstates form/decay at $codim_{\mathbb{R}}$-1 marginal stability loci

$$MS(\gamma_1, \gamma_2) := \{u \in \mathcal{B} \mid \mathrm{Arg} Z_{\gamma_1}(u) = \mathrm{Arg} Z_{\gamma_2}(u)\}$$

Jumps of the BPS spectrum are controlled by an $\mathrm{Arg}\, Z_\gamma$-ordered product of quantum dilogarithms **[Kontsevich-Soibelman]**

$$\prod_{\gamma,m}^{\mathrm{Arg}Z(u)\nearrow} \Phi((-y)^m Y_\gamma)^{a_m(\gamma,u)} = \prod_{\gamma,m}^{\mathrm{Arg}Z(u')\nearrow} \Phi((-y)^m Y_\gamma)^{a_m(\gamma,u')}$$

- non-commutative: DSZ-twisted product $Y_{\gamma_1} Y_{\gamma_2} = y^{\langle\gamma_1,\gamma_2\rangle} Y_{\gamma_1+\gamma_2}$
- BPS degeneracies $a_m(\gamma, u) = (-1)^m \mathrm{dim}\mathscr{H}^{BPS}_{u,\gamma,m}$ count $|\gamma, m\rangle$

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2d-4d system:

- 2d $\mathcal{N} = (2,2)$ theory on $\mathbb{R}^{1,1} \subset \mathbb{R}^{1,3}$
- chiral matter in a representation of a global symmetry G
- 4d vector multiplets couple to 2d chirals, gauging G

Vevs of 4d VM scalars on $\mathcal{B}$ correspond to twisted masses for 2d chirals. Therefore Coulomb moduli control the 2d effective superpotential $\widetilde{W}(u)$. For u generic, $\widetilde{W}(u)$ has a finite number of massive vacua $\widetilde{W}_i(u)$, $i = 1, \dots, d$.

2d-4d BPS states: BPS field configurations interpolating between vacua (ij) on the defect, carrying both topological (2d) and flavor (4d) charges

$$Z_{ij,\gamma}(u) \sim \widetilde{W}_j(u) - \widetilde{W}_i(u) + Z_\gamma(u)\,, \qquad M_{ij,\gamma} = |Z_{ij,\gamma}|\,.$$

[Hanany-Hori, Dorey, Gaiotto, Gaiotto-Moore-Neitzke, PL, Gaiotto-Gukov-Seiberg]

2d-4d vacua are fibered nontrivially over the space of 4d vacua $\mathcal{B}$.
Both the chiral ring and central charges $Z_{ij,\gamma}$ depend on u, through $\widetilde{W}(u)$.

2d-4d wall-crossing : The 2d-4d BPS spectrum also depends on u, because marginal stability occurs when $Z_{ij,\gamma}(u) \,||\, Z_{jk,\gamma'}(u)$.

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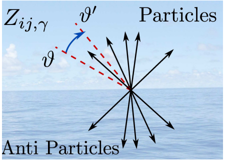
$$F(\vartheta, u) = \sum_{ij,\gamma} \Omega(\vartheta, u, ij, \gamma; y)\, Y_{ij,\gamma}$$

Formal generating series of 2d-4d BPS states preserving $\mathcal{Q}_\vartheta$.
Piecewise-constant in ϑ; jumps across 2d-4d BPS rays, at phases $\mathrm{Arg}\, Z_{ij,\gamma}$



$$F(\vartheta', u) = \left[\prod \Phi((-y)^m Y_{ij,\gamma})^{a_m(ij,\gamma)}\right] F(\vartheta, u) \left[\prod \Phi((-y)^m Y_{ij,\gamma})^{a_m(ij,\gamma)}\right]^{-1}$$

[Gaiotto-Moore-Neitzke]

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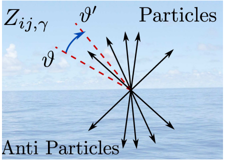

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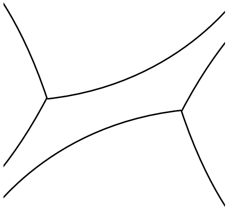
[Gaiotto-Moore-Neitzke]

The 2d-4d degeneracies $a_m(ij,\gamma)$ control jumps in ϑ (at fixed u). Conversely, comparing $F(\vartheta, u)$ to $F(\vartheta+\pi, u)$ gives the **whole 2d-4d spectrum at u**:

$$F(\vartheta+\pi, u) = \mathbb{U}_{2d-4d} F(\vartheta, u) \mathbb{U}_{2d-4d}^{-1}$$

1. For **canonical defects** of Class S theories, the generating function $F(\vartheta, u)$ is computed by the combinatorics of networks on the Gaiotto (class S) curve

- The shape of a network is controlled by the **geometry** of Σ_u, and by an angle ϑ
- Edges carry **soliton data** counting 2d-4d BPS states. $a_m(ij, \gamma)$ determined by global topology
- **Finite edges** appear at $\vartheta = \mathrm{Arg} Z_\gamma$, corresponding to 4d BPS states



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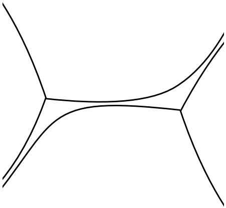
(varying ϑ, u fixed)



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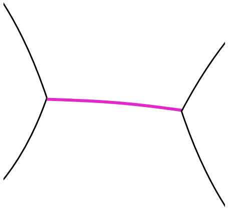

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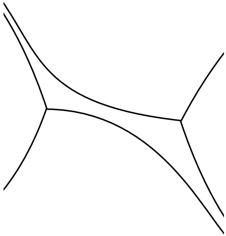

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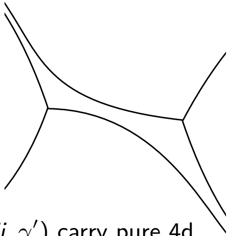

2. Through **2d-4d mixing** (boundstates of (ij, γ) and (ji, γ') carry pure 4d charge), the 2d-4d spectrum **encodes the 4d spectrum**: "$\mathbb{U}_{2d-4d} \supseteq \mathbb{U}$".

[Gaiotto-Moore-Neitzke]

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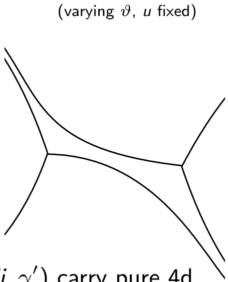

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[Gaiotto-Moore-Neitzke]

Then use spectral networks to compute $F(\vartheta, u)$, $F(\vartheta + \pi, u)$ and obtain $\mathbb{U}$.

- still choosing a chamber of $\mathcal{B}$, with some 4d BPS spectrum
- still difficult, due to complexity of 2d-4d wall crossing

Marginal Stability

Let $\mathcal{B}_c \subset \mathcal{B}$ be a locus where central charges of **all 4d BPS particles** have **the same phase**

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- $F(\vartheta, u_c)$ now has a single jump, occurring at $\vartheta_c(u_c)$
- This jump captures the full BPS monodromy $\mathbb{U}$
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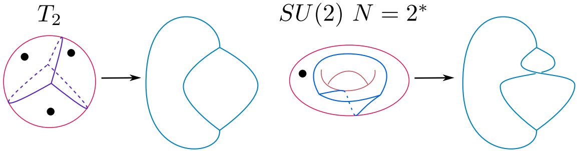
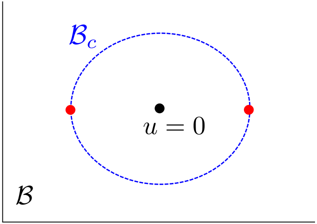
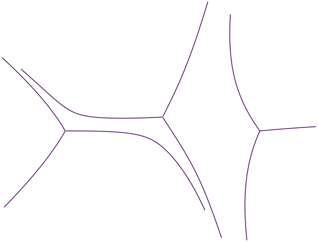
The **graph** topology, together with a notion of framing, **determines** $\mathbb{U}$.



First Example: Argyres-Douglas





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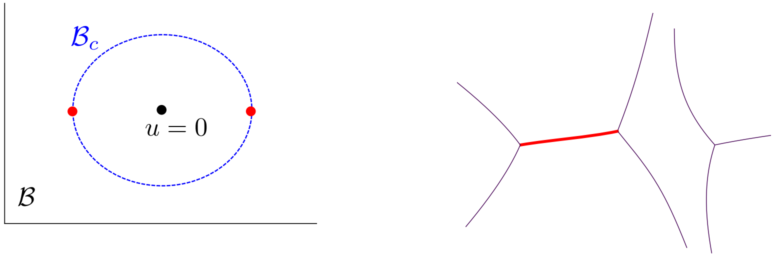

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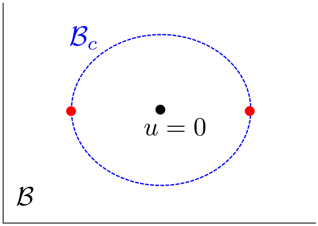

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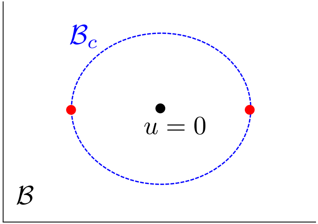

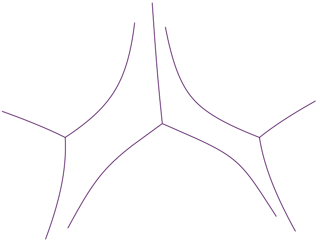

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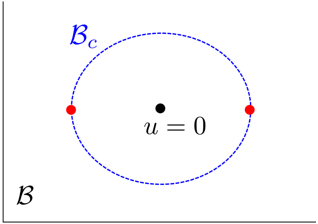

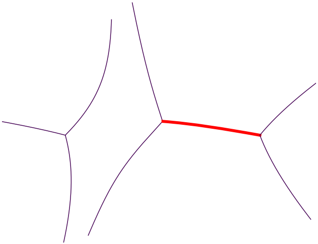

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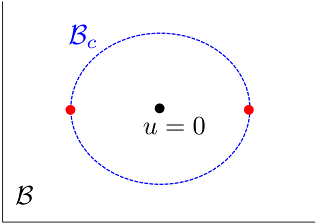

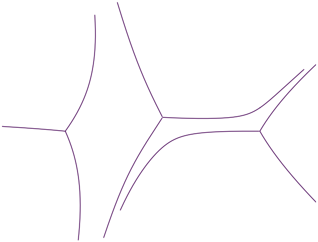

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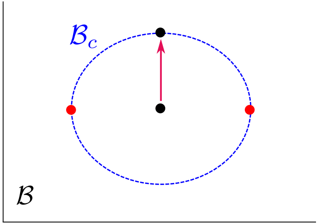

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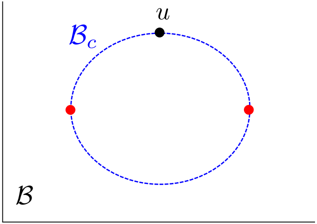

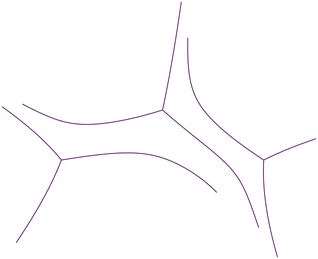

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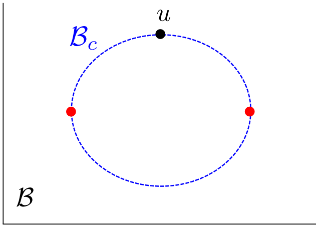

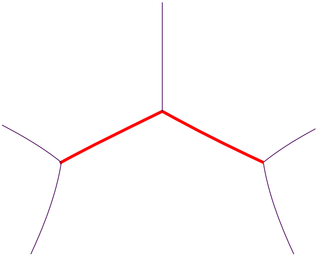

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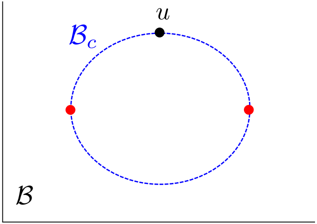
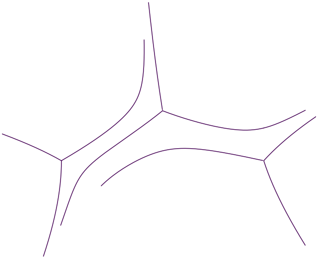


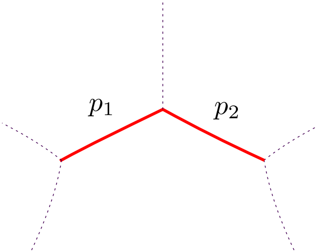


The graph has 2 edges, each contributes an equation

$$Q^{+}(p)\mathbb{U} = \mathbb{U}Q^{-}(p)$$

with

$$Q^{-}(p_1) = 1 + Y_{\gamma_2}$$
$$Q^{-}(p_2) = 1 + Y_{\gamma_1} + Y_{\gamma_1+\gamma_2}$$
$$Q^{+}(p_1) = 1 + Y_{\gamma_2} + Y_{\gamma_1+\gamma_2}$$
$$Q^{+}(p_2) = 1 + Y_{\gamma_1}$$

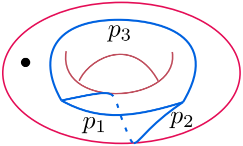


Together, they determine $\mathbb{U} = \Phi(Y_{\gamma_1})\Phi(Y_{\gamma_2}) = \Phi(Y_{\gamma_2})\Phi(Y_{\gamma_1+\gamma_2})\Phi(Y_{\gamma_2})$.

The graph has three edges p_1, p_2, p_3;
each contributes one equation

$$Q^+(p)\mathbb{U} = \mathbb{U}Q^-(p)$$

with



$$Q^-(p_1) = \frac{1+Y_{\gamma_1}+\left(y+y^{-1}\right)Y_{\gamma_1+\gamma_3}+Y_{\gamma_1+2\gamma_3}+\left(y+y^{-1}\right)Y_{\gamma_1+\gamma_2+2\gamma_3}+Y_{\gamma_1+2\gamma_2+2\gamma_3}+Y_{2\gamma_1+2\gamma_2+2\gamma_3}}{\left(1-Y_{2\gamma_1+2\gamma_2+2\gamma_3}\right)^2}$$

$$Q^+(p_1) = \frac{1+Y_{\gamma_1}+\left(y+y^{-1}\right)Y_{\gamma_1+\gamma_2}+Y_{\gamma_1+2\gamma_2}+\left(y+y^{-1}\right)Y_{\gamma_1+2\gamma_2+\gamma_3}+Y_{\gamma_1+2\gamma_2+2\gamma_3}+Y_{2\gamma_1+2\gamma_2+2\gamma_3}}{\left(1-Y_{2\gamma_1+2\gamma_2+2\gamma_3}\right)^2}$$

$Q^\pm(p_2)$ & $Q^\pm(p_3)$ are obtained by cyclic shifts of $\gamma_1, \gamma_2, \gamma_3$.

The solution:

$$\mathbb{U} = \left(\prod^{\nearrow}_{n\geq 0}\Phi\left(Y_{\gamma_1+n(\gamma_1+\gamma_2)}\right)\right)$$

$$\times\Phi\left(Y_{\gamma_3}\right)\Phi\left((-y)Y_{\gamma_1+\gamma_2}\right)^{-1}\Phi\left((-y)^{-1}Y_{\gamma_1+\gamma_2}\right)^{-1}\Phi\left(Y_{2\gamma_1+2\gamma_2+\gamma_3}\right)$$

$$\times\left(\prod^{\searrow}_{n\geq 0}\Phi(Y_{\gamma_2+n(\gamma_1+\gamma_2)})\right)$$

Summary

1. To a class S theory associate a **canonical "critical graph"** on the Gaiotto curve, emerging from a degenerate spectral network at $\mathcal{B}_c$.

2. The **graph's topology + framing encode** equations that characterize the BPS **monodromy** $\mathbb{U}$.

3. Manifestly invariant under wall-crossing: the critical locus $\mathcal{B}_c$ is the **intersection of marginal stability walls**, the BPS spectrum is ill-defined and we never need to compute it.
In fact this is generally **simpler** than building $\mathbb{U}$ by computing BPS spectra.