High-Energy Scattering in $\mathcal{N} = 4$ Super Yang-Mills and Single-Valued Polylogarithms

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based on arXiv:1512.04963, 1606.08411
with J. Broedel and A. Torres Orjuela

ETH zürich

Scattering amplitudes in planar $\mathcal{N} = 4$ SYM

$$A_6(s_{\mathrm{i}\ldots\mathrm{j}}) = A_{6,\mathrm{tree}}\, e^{A_{\mathrm{BDS}} + R_6(u_{\mathrm{i}})}$$

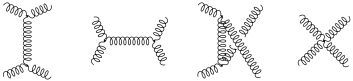
- 4, 5 – gluon amplitudes known to all loop orders [Bern et al., Drummond et al.]
- remainder function R_6 known to 5 loops by amplitude bootstrap [Dixon et al.]
  - ansatz in known space of functions
  - constraints from kinematic limits fully fix amplitude
  - no Feynman diagrams, no loop integrals!
- results hint at simple all-loop description for R_6
- How can we go beyond order-by-order description?

The Regge limit

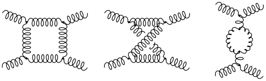
[Balitsky, Fadin, Kuraev, Lipatov]

- consider high-energy limit $s \gg -t$ of four gluon amplitude in YM
- many diagrams are kinematically suppressed

tree level



1 loop



...

n loop ...

- leading contributions $\sim \alpha^n \log^{n-1} s$ (LLA)
- in limit $s \to \infty$: $\alpha \log s \sim \mathcal{O}(1)$
  - need to resum leading contributions from all loop orders
  - emergence of effective particles: Reggeons

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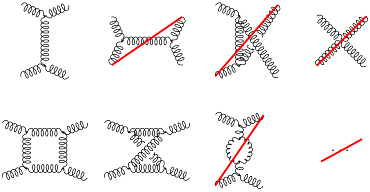

tree level $\sim \alpha \frac{s}{t}$

1 loop $\sim \alpha^2 \log s \, \frac{s}{t}$

n loop $\cdots$ $\sim \alpha^n \log^{n-1} s \, \frac{s}{t}$

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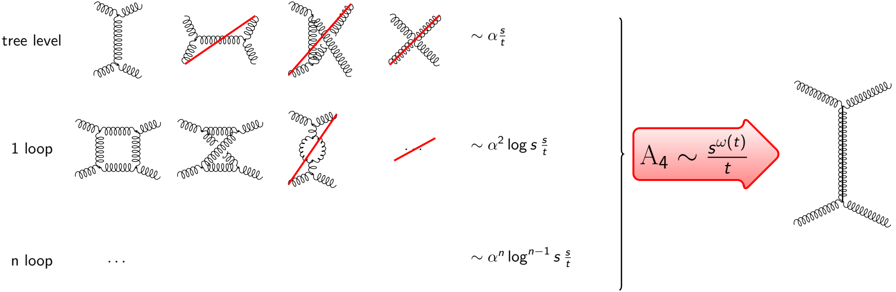

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The remainder of the talk

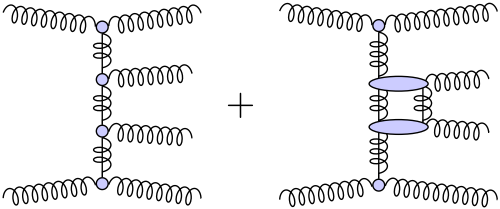
Results

- $R_{6,\mathrm{MRL}}$ is determined by Fourier-Mellin transform to all loop orders
- evaluation of integral tedious
- understanding resulting functions trivializes evaluation of integral
  $\rightarrow$ provide high loop data, proof of open LLA conjecture

An all-loop proposal

[Bartels et al., Lipatov and Prygarin, Lipatov and Fadin, Dixon et al., Caron-Huot, Basso et al.]

$$e^{R_6 + i\pi\delta}\Big|_{\mathrm{MRL}} = \cos\omega_{ab} + i\frac{\lambda}{2}\sum_{n\in\mathbb{Z}}(-1)^n \left(\frac{w}{w^*}\right)^{\frac{n}{2}} \int_{\mathbb{R}} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} |w|^{2i\nu} \; \Phi(\nu, n) \left(-(1-u_1)\frac{|w|}{|1+w|^2}\right)^{-\omega(\nu, n)}$$

$$e^{R_6 + i\pi\delta} =$$ 

- impact factor $\Phi(\nu, n) = 1 + \lambda\,\Phi_{\mathrm{NLLA}}(\nu, n) + \ldots$
- BFKL eigenvalue $\omega(\nu, n) = -\lambda\,\omega_{\mathrm{LLA}}(\nu, n) - \lambda^2\,\omega_{\mathrm{NLLA}}(\nu, n) + \ldots$
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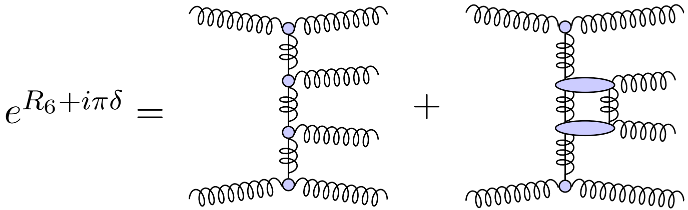

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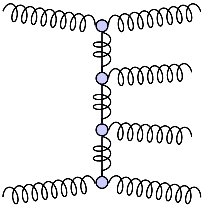
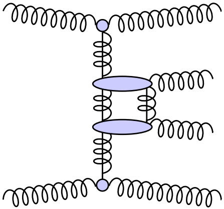
$$e^{R_6+i\pi\delta} =$$  $$+$$ 

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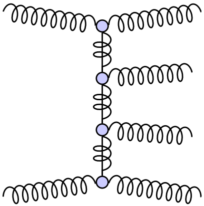
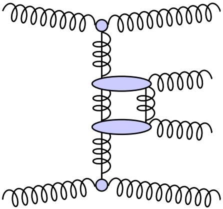
$$e^{R_6 + i\pi\delta} =$$  $$+$$ 

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Harmonic polylogarithms

[Remiddi and Vermaseren]

- generalization of classical polylogarithms $\mathrm{Li}_n(z)$
- appear frequently in scattering amplitudes ($\mathcal{N} = 4$ SYM, QCD,. . .)

$$\mathrm{H}_{x_1,x_2,\ldots,x_n}(z) := \int_0^z \frac{dt}{t - x_1} \mathrm{H}_{x_2,\ldots,x_n}(t), \quad x_i \in \{0, 1\}$$

$$\mathrm{H}_{\underbrace{0,\ldots,0}_{n}}(z) := \frac{1}{n!} \log^n(z)$$

- examples: $\mathrm{H}_{\underbrace{1,\ldots,1}_{n}}(z) = \frac{(-1)^n}{n!} \log^n(1 - z), \quad \mathrm{H}_{\underbrace{0,\ldots,0,1}_{n}}(z) \sim \mathrm{Li}_n(z)$
- series expansion: e.g. $\mathrm{H}_{1,1}(z) = \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell} \mathrm{Z}_1(\ell - 1)$, where $\mathrm{Z}_1(n) = \sum_{k=1}^{n} \frac{1}{k}$

A two-loop example

- simplest example: two-loop LLA $\sim \lambda^2 \log(1 - u_1)$
- sum over residues at $i\nu = \frac{n}{2} + m, \; m \in \mathbb{N}_0$

$$\boxed{R^{(2)}_{6,\mathrm{LLA}}(w, w^*)} \sim \frac{1}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{w}{w^*}\right)^{\frac{n}{2}} \int_{-\infty}^{+\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} |w|^{2i\nu} \underbrace{\left(-\frac{1}{2} \frac{|n|}{(\nu^2 + \frac{n^2}{4})} + \psi\left(1 + i\nu + \frac{|n|}{2}\right) + \psi\left(1 - i\nu + \frac{|n|}{2}\right) - 2\psi(1) \right)}_{\omega_{\mathrm{LLA}}(\nu,n)}$$

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$$= \sum_{m=1}^{\infty}\left\{2\frac{|w|^{2m}}{m^2} - 2\frac{(-w)^m+(-w^*)^m}{m^2} + [\log|w|^2 + 2Z_1(m)]\frac{(-w)^m+(-w^*)^m}{m}\right\}$$

$$+ 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^n}{m(m+n)}\left\{w^{m+n}w^{*m} + w^m w^{*m+n}\right\}$$

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&= \log|w|^2\,[\mathrm{H}_1(-w) + \mathrm{H}_1(-w^*)] + 2\mathrm{H}_{0,1}(|w|^2) + 2\mathrm{H}_{1,1}(-w) + 2\mathrm{H}_{1,1}(-w^*) \\
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$\rightarrow$ observation: result is single-valued!

Single-valued harmonic polylogarithms

[F. Brown, in MRL context : Dixon et al.]

- HPLs $\mathrm{H}_a(z)$ can be lifted to single-valued functions $\mathcal{L}_a(z, z^*)$
- explicit construction using Drinfeld associator

$$
\begin{aligned}
\mathcal{L}_0(z) &= \mathrm{H}_0(z) + \mathrm{H}_0(z^*) = \log z + \log z^* \\
\mathcal{L}_1(z) &= \mathrm{H}_1(z) + \mathrm{H}_1(z^*) \\
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&\vdots \\
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&\quad + \mathrm{H}_{1,0}(z)\mathrm{H}_{0,1}(z^*) - 4\zeta_3\mathrm{H}_1(z^*)
\end{aligned}
$$

Using the single-valuedness

- How is single-valuedness encoded in the integral?
- key observation: $\mathcal{L}_a(z, z^*) = \mathrm{H}_a(z) + \ldots$

  [Drummond et al., Drummond and Papathanasiou]
- calculate residue $i\nu = \frac{n}{2}$, restore full result by $\mathrm{H}_a(-w) \to \mathcal{L}_a(-w, -w^*)$

$$R^{(2)}_{6,\mathrm{LLA}}(w, w^*) \sim \frac{1}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{w}{w^*}\right)^{\frac{n}{2}} \int_{-\infty}^{+\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} |w|^{2i\nu} \left(-\frac{1}{2} \frac{|n|}{(\nu^2 + \frac{n^2}{4})} + \psi\left(1 + i\nu + \frac{|n|}{2}\right) + \psi\left(1 - i\nu + \frac{|n|}{2}\right) - 2\psi(1)\right)$$

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$$= \mathrm{H}_0(-w)\,\mathrm{H}_1(-w) - 2\mathrm{H}_{0,1}(-w) + 2\mathrm{H}_{1,1}(-w) + 2\mathrm{H}_{0,1}(-w)$$

Using the single-valuedness

- How is single-valuedness encoded in the integral?
- key observation: $\mathcal{L}_a(z, z^*) = \mathrm{H}_a(z) + \ldots$

  [Drummond et al., Drummond and Papathanasiou]
- calculate residue $i\nu = \frac{n}{2}$, restore full result by $\mathrm{H}_a(-w) \to \mathcal{L}_a(-w, -w^*)$

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- we identified relations among different integrals which hold at $i\nu = \frac{n}{2}$
- all integrals reducible to master integrals with single pole $\to$ trivial

Conclusions and outlook

- studied six-gluon amplitude in $\mathcal{N} = 4$ SYM in multi-Regge limit
- found efficient algorithm for evaluation of R_6 using single-valuedness
- proved all-loop LLA conjecture of [Pennington]
- first steps towards seven-gluon case, full solution for n-gluon LLA in [Del Duca et al.]

next steps:

- simple structure in momentum space beyond LLA?
- n-point amplitudes beyond LLA?
- beyond MRL?