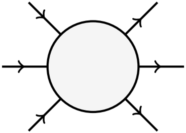
Worldsheet Models in QFT

Yvonne Geyer

September 29th, 2016
Rethinking Quantum Field Theory
DESY Theory Workshop

arXiv:1507.00321, 1511.06315, 1607.08887
YG, L. Mason, R. Monteiro, P. Tourkine

Scattering Amplitudes



$$\sum_{i \in \text{In}} k_i - \sum_{j \in \text{Out}} k_j = 0$$

- tree-level: classical
- loop-level: quantum

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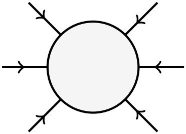

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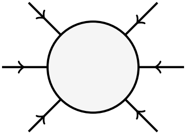

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- good: clear physical picture
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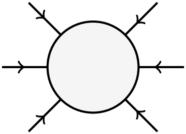

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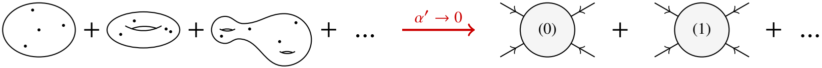
⇒ "Rethinking Quantum Field Theory"

Motivation: worldsheet models

String Theory



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- map: $\Sigma \to M$

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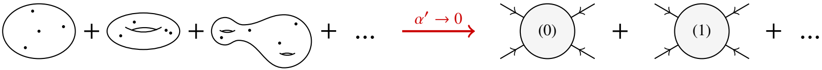

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Worldsheet models of QFT

[Witten, Berkovits, RSV; Hodges, Cachazo-YG, Skinner-Mason]

- map: $\Sigma \to \mathbb{T} \cong \mathbb{CP}^{3|4}$
- $D = 4$, maximal supersymmetry

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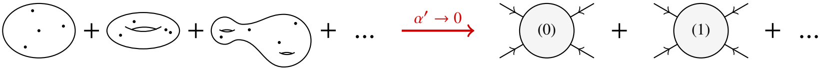

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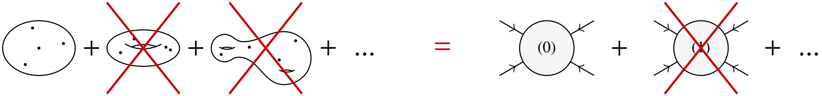

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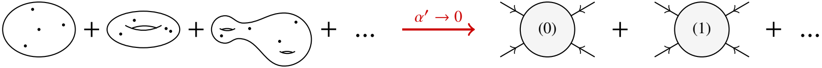

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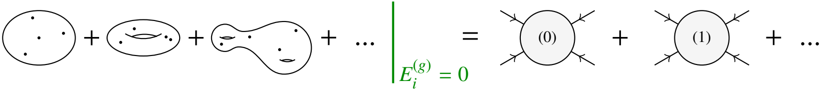

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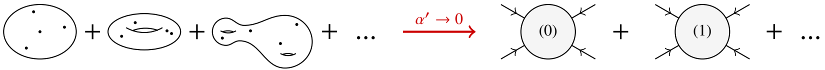

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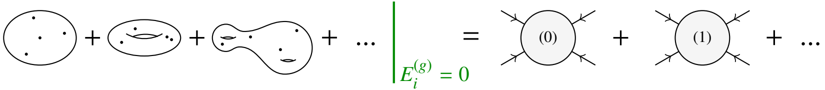


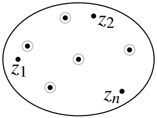
- integration fully localised on the Scattering Equations $\{E_i^{(g)}\}$
- map: $\Sigma \to \mathbb{A}$ = {phase space of complex null rays}
- upshot: generic for massless QFTs

Scattering Equations

[Cachazo-He-Yuan, Mason-Skinner]

Construction: for n null momenta k_i, define
$P_\mu(\sigma, \sigma_i) \in \Omega^0(\Sigma, K_\Sigma)$

$$P_\mu(\sigma) = \sum_{i=1}^{n} \frac{k_{i\mu}}{\sigma - \sigma_i} d\sigma \,.$$



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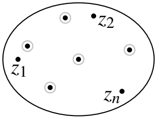

Scattering Equations at tree-level

Enforce $P^2 = 0$ on Σ:

$$E_i = \mathrm{Res}_{\sigma_i} P^2(\sigma) = k_i \cdot P(\sigma_i) = \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0\,.$$

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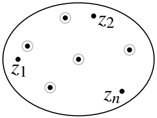

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- $\sum \sigma_i^a E_i = 0$ for $a = 0, 1, 2$
  - $\rightarrow$ SL$(2, \mathbb{C})$ invariant
  - $\rightarrow$ $(n-3)$ independent equations = dim$(\mathfrak{M}_{0,n})$
  - $\rightarrow$ $(n-3)!$ solutions
- factorisation properties

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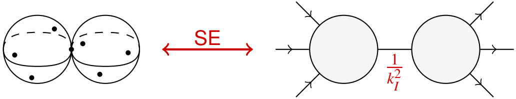
- Factorisation:



boundary of $\mathfrak{M}_{0,n}$

factorisation channel (unitarity, locality)

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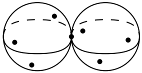

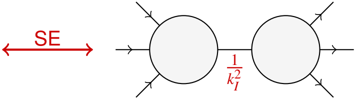

boundary of $\mathfrak{M}_{0,n}$ — factorisation channel (unitarity, locality)

- With $\sigma_i = \sigma_I + \varepsilon x_i$ for $i \in I$, the pole is given by

$$\sum_{i \in I} x_i E_i^{(I)} = \sum_{i,j \in I} x_i \frac{k_i \cdot k_j}{x_i - x_j} = \frac{1}{2} \sum_{i,j \in I} k_i \cdot k_j = k_I^2 \,.$$

Scattering Equations

Universality for massless QFTs

Scattering Equations at tree-level:

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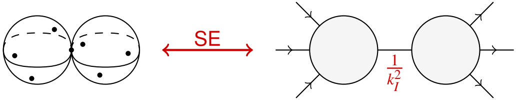

boundary of $\mathfrak{M}_{0,n}$

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- Upshot: amplitudes take the form

$$\mathcal{M} = \sum_{\sigma_j | E_i(\sigma_j)=0} \frac{\mathcal{I}(\sigma_i, k_i, \epsilon_i)}{J(\sigma_i, k_i)}\,.$$

Ambitwistor space $\mathbb{A}$ = space of (complex) null rays in $M_{\mathbb{C}}$

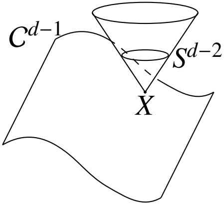
- symplectic quotient of cotangent bundle of spacetime

$$\mathbb{A} := \left\{(X^\mu, P_\mu, \Psi_r^\mu) \in T^*M \,\middle|\, P^2 = \Psi_r \cdot P = 0\right\} \Big/ \{\mathcal{D}_0, \mathcal{D}_r\}$$

with Hamiltonian vector fields $\mathcal{D}_0 = P \cdot \nabla$, $\mathcal{D}_r = \Psi_r \cdot \nabla + P \cdot \partial_{\Psi_r}$

- $\mathbb{A}$ is a symplectic holomorphic manifold, with symplectic potential

$$\Theta = P \cdot \mathrm{d}X + \frac{1}{2} \sum_r \Psi_r \cdot \mathrm{d}\Psi_r$$



The Ambitwistor String

- Construct a theory describing maps $\Sigma \to \mathbb{A}$:

$$S = \frac{1}{2\pi} \int P \cdot \bar{\partial} X + \frac{1}{2} \sum_r \Psi_r \cdot \bar{\partial} \Psi_r - \frac{e}{2} P^2 - \chi_r P \cdot \Psi_r \,,$$

where $P_\mu \in \Omega^0(\Sigma, K_\Sigma)$, $\Psi_i^\mu \in \Pi\Omega^0(\Sigma, K_\Sigma^{1/2})$.

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- Geometrically:
  - action from symplectic potential Θ
  - gauge fields e and χ_r impose the constraints reducing to $\mathbb{A}$
  - gauge freedom: $\delta X^\mu = \alpha P^\mu$, $\delta P_\mu = 0$, $\delta e = \bar{\partial}\alpha$.

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- BRST quantisation: $Q = \oint cT + \tilde{c} P^2 + \gamma_r P \cdot \Psi_r$

  Vertex operators: $V = c\tilde{c}\delta^2(\gamma)\, \epsilon_\mu \epsilon_\nu \Psi_1^\mu \Psi_2^\nu e^{ik\cdot X}$

$$[Q, V] = 0 \quad \Rightarrow \quad k^2 = \epsilon \cdot k = 0$$

Geometry of the Scattering Equations

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  so $P_\mu(\sigma) = \sum_{i=1}^n \frac{k_{i\mu}}{\sigma-\sigma_i}\,d\sigma\,.$
- Moduli of gauge field e forces $P^2 = 0$;

$$\text{Res}_{\sigma_i}P^2(\sigma) = k_i\cdot P(\sigma_i) = 0\,.$$

scattering equations $\Leftrightarrow$ map to $\mathbb{A}$

CHY formulae

The correlation functions yield the CHY formulae:

Tree-level S-matrix of massless theories:

$$\mathscr{M}_{n,0} = \int_{\mathfrak{M}_{0,n}} \frac{\mathrm{d}\sigma^n}{\mathrm{vol}\, G} \prod_i \bar{\delta}\left(\sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} \right) \mathcal{I}_n$$

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- Integration over moduli space $\mathfrak{M}_{0,n}$ fixed completely by imposing the $(n-3)$ scattering equations E_i
- $\mathcal{I}_n$ from fermion correlators:

$$\mathcal{I}_n = \text{Pf}'(M)\,\text{Pf}'(\tilde{M}) \text{ for } M = M(k_i, \epsilon_i, \sigma_i)$$

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- progress on evaluation without solving the SE
  [Cachazo-Gomez, Baadsgaard et al, Sogard-Zhang, ...]
- family of massless models

Comment: Colour-Kinematic Duality

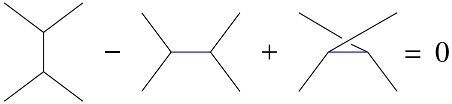
Gravity$\sim$ YM^2

[Bern-Carrasco-Johansson]

- Biadjoint scalar: colour $c_i \otimes$ colour c_j
- Gauge theory: colour $c_i \otimes$ kinematics n_i
- Gravity: kinematics $n_i \otimes$ kinematics n_i

Gauge theory amplitude: $$\mathcal{A} = \sum_{\Gamma_i} \frac{n_i c_i}{d_i}$$

With c_i satisfying the Jacobi identity



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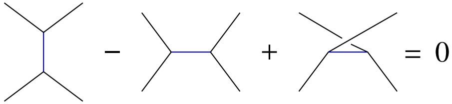

$$f^{dae}f^{ebc} - f^{abe}f^{ecd} + f^{ace}f^{edb} = 0$$

Find n_i satisfying Jacobi, then

$$\mathcal{M} = \sum_{\Gamma_i} \frac{n_i n_i}{d_i}$$

CHY formulae and the Double Copy

[Cachazo-He-Yuan]

Tree-level S-matrix of massless theories:

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- Gravity: $\mathcal{I}_n = \mathrm{Pf}'(M)\,\mathrm{Pf}'(\tilde{M})$
- Yang-Mills theory: $\mathcal{I}_n = C_n\,\mathrm{Pf}'(M)$
- Bi-adjoint scalar: $\mathcal{I}_n = C_n\,\tilde{C}_n$

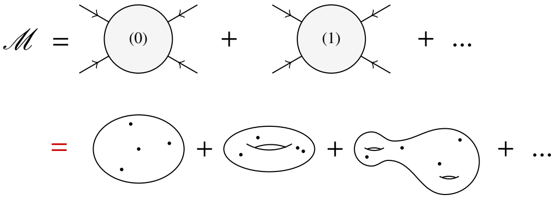
with building blocks

- Parke-Taylor factor: $C_n(1,\ldots,n) = \frac{\mathrm{tr}(T^{a_1}T^{a_2}\ldots T^{a_n})}{\sigma_{1\,2}\ldots\sigma_{n-1\,n}\sigma_{n\,1}} +$ non-cyclic
- Reduced Pfaffian: $\mathrm{Pf}'(M) = \frac{(-1)^{i+j}}{\sigma_{ij}}\mathrm{Pf}(M^{ij}_{ij}), \qquad M = M(k_i, \epsilon_i, \sigma_i)$

Loop Integrands from the Riemann Sphere

Upshot – Loops

Worldsheet models of QFT



$$E_i^{(g)} = 0$$

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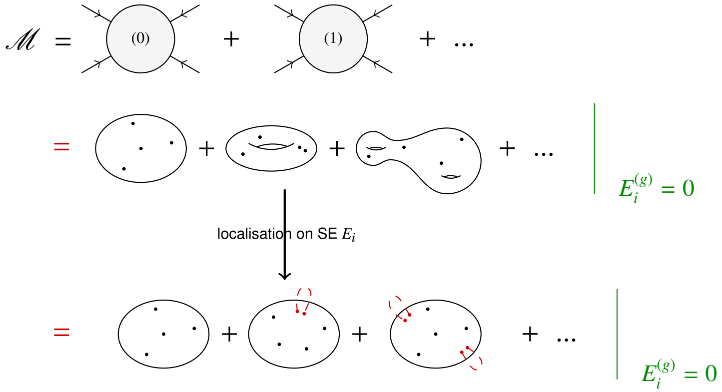
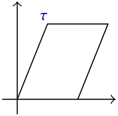


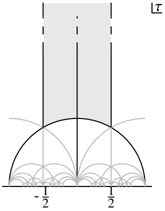
Scattering Equations on the Torus

Genus 1

- moduli space of torus Σ_τ:

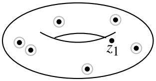


$$z \sim z+1 \sim z+\tau$$
$$\tau \sim \tau+1 \sim -1/\tau$$



- Solve $\bar\partial P = 2\pi i \sum_i k_i \bar\delta(z-z_i)dz$:

$$P_\mu = \left(2\pi i\, \ell_\mu + \sum_i k_{i\mu} \frac{\theta_1'(z-z_i)}{\theta_1(z-z_i)}\right) dz\,,$$



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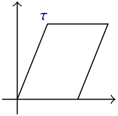

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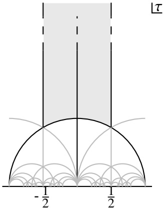
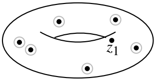


- Solve $\bar\partial P = 2\pi i \sum_i k_i \bar\delta(z - z_i) dz$:

$$P_\mu = \left(2\pi i \, \ell_\mu + \sum_i k_{i\mu} \frac{\theta_1'(z - z_i)}{\theta_1(z - z_i)} \right) dz \,,$$



Scattering Equations on the torus: $P^2(z|\tau) = 0$

$$\mathrm{Res}_{z_i} P^2(z) := 2k_i \cdot P(z_i) = 0 \,,$$
$$P^2(z_0) = 0 \,.$$

The 1-loop Integrand

[Adamo-Casali-Skinner, Casali-Tourkine]

One-loop integrand of type II supergravity

$$\mathcal{M}^{(1)}_{\text{SG}} = \int d^{10}\ell \, d\tau \, \underbrace{\bar{\delta}(P^2(z_0)) \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i))}_{\text{Scattering Equations}} \underbrace{\left(\sum_{\text{spin struct.}} Z^{(1)}(z_i) Z^{(2)}(z_i) \right)}_{\equiv \mathcal{I}_q, \text{ fermion correlator}}$$

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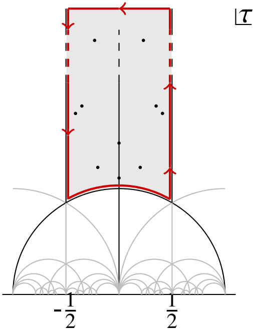
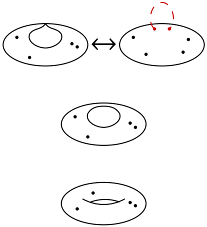
- modular invariance: $\tau \sim \tau + 1 \sim -1/\tau$
- Scattering Equations: $P^2(z|\tau) = 0$
- checks:
  - factorisation
  - correct tensor structure at $n = 4$: $t_8 t_8 R^4$

Why rational function?

- localisation on scattering equations
  $\Rightarrow$ contour integral argument in fundamental domain
- modular invariance: sides and unit circle cancel

$\Rightarrow$ localisation on $q \equiv e^{2i\pi\tau} = 0 \Leftrightarrow \tau = i\infty$

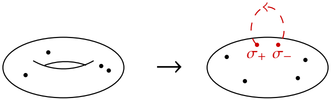




Contour argument in the fundamental domain

One-loop integrand of type II supergravity

$$\mathscr{M}^{(1)}_{\mathrm{SG}} = \int d^d\ell \, d\tau \, \bar{\delta}(P^2(z_0)) \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i)) \, \mathcal{I}_q$$



Residue theorem: elliptic curve $\longrightarrow$ nodal Riemann spere at $q = 0$.

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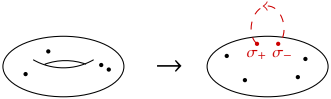


Residue theorem: elliptic curve $\longrightarrow$ nodal Riemann spere at $q = 0$.

$$\mathscr{M}^{(1)}_{SG} = \frac{1}{2\pi i} \int d^{10}\ell \, \frac{dq}{q} \, \bar{\partial}\left(\frac{1}{P^2(z_0)}\right) \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i)) \, \mathcal{I}_q$$

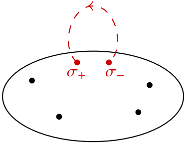
$$= -\int d^{10}\ell \, dq \, \frac{1}{P^2(z_0)} \, \bar{\delta}(q) \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i)) \, \mathcal{I}_0$$

$$= -\int \frac{d^{10}\ell}{\ell^2} \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i)) \, \mathcal{I}_0 \Big|_{q=0} .$$

One-loop off-shell scattering equations

On the nodal Riemann Sphere:

$$P = \left(\frac{\ell}{\sigma - \sigma_{\ell^+}} - \frac{\ell}{\sigma - \sigma_{\ell^-}} + \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i} \right) d\sigma \,.$$



Define $S = P^2 - \left(\frac{\ell}{\sigma - \sigma_{\ell^+}} - \frac{\ell}{\sigma - \sigma_{\ell^-}} \right)^2 d\sigma^2$.

One-loop off-shell scattering equations

$$\mathrm{Res}_{\sigma_i} S = k_i \cdot P(\sigma_i) = \frac{k_i \cdot \ell}{\sigma_i - \sigma_{\ell^+}} - \frac{k_i \cdot \ell}{\sigma_i - \sigma_{\ell^-}} + \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0 \,,$$

$$\mathrm{Res}_{\sigma_{\ell^-}} S = - \sum_j \frac{\ell \cdot k_j}{\sigma_{\ell^-} - \sigma_j} = 0 \,,$$

$$\mathrm{Res}_{\sigma_{\ell^+}} S = \sum_j \frac{\ell \cdot k_j}{\sigma_{\ell^+} - \sigma_j} = 0 \,.$$

The integrand

One-loop integrand on the nodal Riemann sphere

$$\mathscr{M}^{(1)} = -\int \frac{d^d\ell}{\ell^2} \frac{d^{n+2}\sigma}{\text{vol}\,(G)} \underbrace{\prod_{i,\ell^\pm} \bar{\delta}\left(\text{Res}_{\sigma_i} S\right)}_{\text{off-shell scattering equations}} \mathcal{I}$$

- type II sugra: $\mathcal{I} = \mathcal{I}|_{q\to 0}$, limit of $g = 1$ correlator
- manifestly rational
- upshot: widely applicable for supersymmetric and non-supersymmetric theories

The integrand

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Puzzle: Only depends on $1/\ell^2$, remainder $\ell \cdot k_i$, $\ell \cdot \epsilon_i$, ...

Solution: Shifted integrands

- partial fractions: $\frac{1}{\prod_i D_i} = \sum_i \frac{1}{D_i \prod_{j\neq i}(D_j - D_i)}$
- shift: $D_i \to \ell^2$
- formalised: Q-cuts

[Baadsgaard et al]

The integrand

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[Baadsgaard et al]

Example:

$$\frac{1}{\ell^2(\ell+K)^2} = \frac{1}{\ell^2(2\ell\cdot K + K^2)} + \frac{1}{(\ell+K)^2(-2\ell\cdot K - K^2)}$$
$$\to \frac{1}{\ell^2}\left(\frac{1}{2\ell\cdot K + K^2} + \frac{1}{-2\ell\cdot K + K^2}\right)$$

Integrands – Double Copy again

One-loop integrand on the nodal Riemann sphere

$$\mathscr{M}^{(1)} = -\int \frac{d^d\ell}{\ell^2}\frac{d^{n+2}\sigma}{\text{vol}\,(G)} \prod_{i,\ell^\pm} \bar{\delta}(\text{Res}_{\sigma_i} S)\; \mathcal{I}$$

Supersymmetric:

- $\mathcal{I}_{\text{sugra}} = \frac{1}{(\sigma_{\ell^+\ell^-})^4}\, \mathcal{I}_0\, \widetilde{\mathcal{I}}_0$
- $\mathcal{I}_{\text{sYM}} = \frac{1}{(\sigma_{\ell^+\ell^-})^4}\, \mathcal{I}_0\, \mathcal{I}^{PT}$

Building blocks

- Parke-Taylor: $\mathcal{I}^{PT} = \sum_{i=1}^{n} \frac{\sigma_{\ell^+\ell^-}\, tr(T^{a_1}T^{a_2}\ldots T^{a_n})}{\sigma_{\ell^+ i}\sigma_{i+1\, i}\sigma_{i+2\, i+1}\ldots\sigma_{i+n\, \ell^-}} +$ non-cycl.
- Pfaffian: $\mathcal{I}_0 = \sum_r \text{Pf}\,'(M^r_{\text{NS}}) - \frac{c_d}{\sigma^2_{\ell^+\ell^-}}\text{Pf}\,(M_2)$

Integrands – Double Copy again

One-loop integrand on the nodal Riemann sphere

$$\mathscr{M}^{(1)} = -\int \frac{d^d\ell}{\ell^2}\,\frac{d^{n+2}\sigma}{\text{vol}\,(G)} \prod_{i,\ell^\pm} \bar{\delta}(\text{Res}_{\sigma_i} S)\ \mathcal{I}$$

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Non-supersymmetric

- $\mathcal{I}_{\text{YM}} = (\sum_r \text{Pf}\,'(M^r_{\text{NS}}))\,\mathcal{I}^{PT}$
- $\mathcal{I}_{\text{grav}} =$ $(\sum_r \text{Pf}\,'(M^r_{\text{NS}}))^2 - \alpha\,(\text{Pf}\,(M_3)\big|_{q^0})^2\,.$

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- $\alpha = \frac{1}{2}(d-2)(d-3)+1$: d.o.f. of B-field and dilaton

Integrands – Double Copy again

One-loop integrand on the nodal Riemann sphere

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- $\mathcal{I}_{\text{grav}} =$
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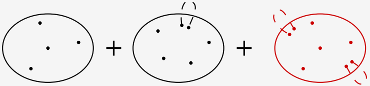
Checks: numerical, factorisation

Question: Beyond one loop?

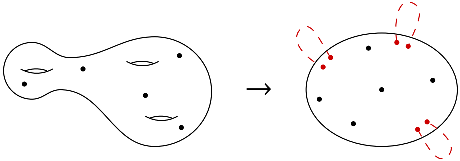
$\mathscr{M} =$



$+ \ \dots \quad \Big| \; E_i(\sigma_j) = 0$

Heuristics: Higher genus

Riemann surface Σ_g $\xrightarrow[\text{contract } g\ a\text{-cycles}]{\text{residue theorems}}$ nodal RS



- fixes g moduli
- remaining $2g - 3 \Leftrightarrow 2g$ new marked points mod $\mathrm{SL}(2,\mathbb{C})$
- 1-form P_μ

$$P = \sum_{r=1}^{g} \ell_r \omega_r^{(g)} + \sum_i k_i \frac{d\sigma}{\sigma - \sigma_i} \,,$$

$\omega_r^{(g)} = \frac{(\sigma_{r^+} - \sigma_{r^-})\, d\sigma}{(\sigma - \sigma_{r^+})(\sigma - \sigma_{r^+})}$: basis of g global holomorphic 1-forms

The Scattering Equations

The multiloop off-shell scattering equations are

$$\mathrm{Res}_{\sigma_A} S = 0\,, \quad A = 1, \ldots, n+2g$$

$$S(\sigma) := P^2 - \sum_{r=1}^{g} \ell_r^2 \omega_r^2 + \sum_{r<s} a_{rs}(\ell_r^2 + \ell_s^2)\omega_r \omega_s\,.$$

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At two-loops:

- Factorisation: $a_{12} = 1$

Two-loop integrand

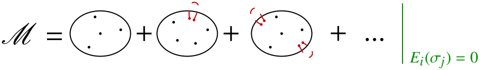
$$\widehat{\mathcal{M}}^{(2)} = \frac{1}{\ell_1^2 \ell_2^2} \int_{\mathfrak{M}_{0,n+4}} \frac{d^{n+4}\sigma}{\mathrm{Vol}\, G} \prod_{A=1}^{n+4} \bar{\delta}\big(\mathrm{Res}_A S(\sigma_A)\big)\, \mathcal{I}\,,$$

- integrands known for $n = 4$ sYM and sugra
- Similar complexity as tree amplitudes with $n + 4$ particles!

Conclusion and Outlook

Summary:

- Worldsheet models describing scattering in massless QFTs
- moduli integrals localised on scattering equations
- framework to derive loop integrands on nodal Riemann spheres



This formalism implies that n-point g-loop scattering amplitudes have the same complexity as $n + 2g$-point tree-level amplitudes!

Outlook:

- sharpen ambitwistor string derivation at higher loop order
- model on nodal Riemann sphere
- new features at higher loops?

Thank you!