Progress in constructing epsilon form of differential equations for master integrals with Fuchsia

Oleksandr Gituliar
oleksandr.gituliar@ifj.edu.pl

Institute of Nuclear Physics
Polish Academy of Sciences
Cracow, Poland

DESY Theory Workshop
Rethinking Quantum Field Theory
Hamburg (Germany)
28 September 2016

Fuchsia project

Gituliar, Magerya [arXiv:1607.00759]

Fuchsia is a program for reducing differential equations for master integrals to the **epsilon form** Henn '13:

- based on the **Lee algorithm** Lee '14
- open-source and free (no proprietary software dependencies)
- implemented in Python, SageMath, Maxima

The idea is to find a **rational transformation** in three reduction steps:

1. **Fuchsification** decrease Poincaré rank to 0 at all singular points (i.e. get rid of irregular singularities)
2. **Normalization**: balance eigenvalues to $n\epsilon$ form
3. **Factorization**: reduce to the epsilon form
4. Optimization for **block-triangular systems** (NEW!)

For more details see:

- Guide & tutorial http://gituliar.net/fuchsia/fuchsia.pdf
- Lee algorithm R.Lee, JHEP 1504 (2015) 108 [arXiv:1411.0911]
- Announcement at Loops&Legs '16 Gituliar, Magerya [arXiv:1607.00759]

Problem

Calculate Feynman integrals of the form

$$f_i(x,\epsilon) = \int \underbrace{\mathrm{d}^d l_1 \ldots \mathrm{d}^d l_n}_{\text{loops}} \; \underbrace{\mathrm{d}^d p_1 \delta(p_1^2) \ldots \mathrm{d}^d p_m \delta(p_m^2)}_{\text{legs}} \; \frac{1}{D_1^{n_1} \ldots D_k^{n_k}}$$

Numerical methods:

- sector decomposition (loops & legs)
- various subtraction schemes (legs)

Analytical methods:

- Integration-By-Parts (IBP) reduction to master integrals
  - Laporta algorithm: `AIR`, `FIRE`, `Reduze`
  - Symbolic reduction: `LiteRed`
- Feynman/Schwinger/Mellin-Barnes parametrization
- Ossola-Papadopoulos-Pittau (OPP) reduction for one-loop integrals
- Differential Equations for master integrals in the epsilon form
  - Lee algorithm: `Fuchsia`

Problem

Calculate Feynman integrals of the form

$$f_i(x,\epsilon) = \int \underbrace{\mathrm{d}^d l_1 \ldots \mathrm{d}^d l_n}_{\text{loops}} \; \underbrace{\mathrm{d}^d p_1 \delta(p_1^2) \ldots \mathrm{d}^d p_m \delta(p_m^2)}_{\text{legs}} \; \frac{1}{D_1^{n_1} \ldots D_k^{n_k}}$$

Let us consider a system of ODEs

$$\frac{\mathrm{d}\bar{f}}{\mathrm{d}x} = \mathbb{A}(x,\epsilon)\,\bar{f}(x,\epsilon),$$

- $\bar{f}(x,\epsilon)$ is a vector of unknown master integrals
- $\mathbb{A}(x,\epsilon)$ is a singular matrix of rational functions

We are looking for the rational transformation $\mathbb{T}(x,\epsilon)$ such that a new basis $\bar{g}(x,\epsilon)$ is given by

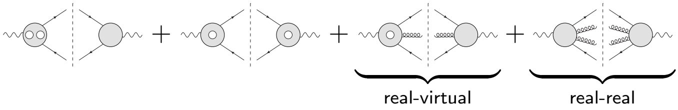
$$\bar{f}(x,\epsilon) = \mathbb{T}(x,\epsilon)\bar{g}(x,\epsilon)$$

with a new system in the **epsilon form** (easy to solve)

$$\frac{\mathrm{d}\bar{g}}{\mathrm{d}x} = \epsilon\,\mathbb{A}(x)\,\bar{g}(x,\epsilon)$$

Example I: Splitting Functions at NLO in QCD

Splitting Functions for DGLAP evolution equations can be extracted from the $e^+e^- \to \gamma^* \to$ partons annihilation process:

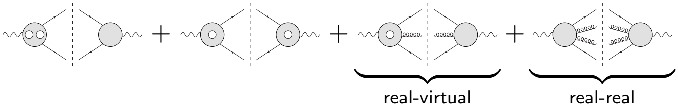


Real-real contribution: initial form

$$
\begin{pmatrix}
\frac{(2\epsilon-1)(2x-1)}{x(1-x)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3\epsilon-2}{x(1-x)} & \frac{1-3\epsilon}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{(2\epsilon-1)}{\epsilon^2 x(1-x)(x+1)} & 0 & \frac{1-6\epsilon}{x+1} & \frac{2}{x+1} & 0 & 0 & 0 & 0 \\
\frac{(2\epsilon-1)}{\epsilon x^2(x+1)} & -\frac{(2\epsilon-1)(3\epsilon-1)}{x^2} & \frac{2(6\epsilon-1)}{x(x+1)} & \frac{2\epsilon(x^2+3x-2)}{(1-x)x(x+1)} & 0 & 0 & 0 & 0 \\
\frac{2(x^2+4x+1)}{\epsilon^2(1-x)x^3(x+1)^3} & \frac{2(2\epsilon-1)(x-1)}{\epsilon x^3(x+1)^2} & \frac{2(6\epsilon-1)(x-1)}{x^2(x+1)^3} & \frac{4(x^2+1)}{x^2(x+1)^3} & -\frac{(2\epsilon+1)(2x+1)}{x(x+1)} & 0 & 0 & 0 \\
0 & -\frac{(2\epsilon-1)}{\epsilon(1-x)x} & 0 & 0 & 0 & \frac{2\epsilon}{1-x} & 0 & 0 \\
-\frac{4}{\epsilon^2(1-x)^3x^3(x+1)} & -\frac{2(2\epsilon-1)(x-2)}{\epsilon(1-x)^2x^3} & -\frac{2(6\epsilon-1)}{x^2(1-x)(x+1)} & \frac{4(x^2+1)}{(1-x)^2x^2(x+1)} & 0 & -\frac{4\epsilon}{(1-x)^2x} & \frac{(2\epsilon+1)(2x-1)}{(1-x)x} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(2\epsilon+1)(2x-1)}{(1-x)x}
\end{pmatrix}
$$

Example I: Splitting Functions at NLO in QCD

Splitting Functions for DGLAP evolution equations can be extracted from the $e^+e^- \to \gamma^* \to$ partons annihilation process:



Real-real contribution: epsilon form by Fuchsia

$$\boxed{\epsilon}\begin{pmatrix}
\frac{2}{1-x}-\frac{2}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{19x} & -\frac{3}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{19(1-x)} & \frac{2}{1-x} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{1-x}-\frac{2}{x} & 0 & 0 & 0 & 0 \\
\frac{3}{x} & -\frac{19}{x} & 0 & 0 & \frac{2}{1-x}+\frac{4}{x} & -\frac{1}{x} & 0 & 0 \\
\frac{6}{1+x}+\frac{6}{1-x}+\frac{16}{x} & \frac{76}{1+x}-\frac{152}{x} & 0 & 0 & \frac{12}{1+x}+\frac{12}{1-x}+\frac{32}{x} & -\frac{2}{1+x}-\frac{8}{x} & 0 & 0 \\
\frac{2}{19(1-x)}-\frac{4}{19x} & \frac{4}{(1-x)}+\frac{8}{x} & \frac{76}{x}-\frac{76}{1-x} & \frac{4}{19(1-x)}-\frac{16}{19x} & 0 & \frac{3}{19x} & \frac{2}{1-x}-\frac{2}{x} & 0 \\
\frac{8}{19x}-\frac{6}{19(1+x)} & \frac{4}{1+x}-\frac{4}{x} & 0 & 0 & \frac{4}{19(1-x)}+\frac{16}{19x}-\frac{8}{19(1+x)} & \frac{2}{19(1+x)}-\frac{3}{19x} & 0 & -\frac{2}{1+x}-\frac{2}{x}
\end{pmatrix}$$

Now the system can be easily solved, hence giving initial master integrals O.Gituliar [JHEP 1602 (2016) 017].

Example II: Two-loop planar pentabox

Papadopoulos, Tommasini, Wever [JHEP 1604 (2016) 078]

The latest optimization for block-triangular matrices allows Fuchsia to handle even more complicated problems:

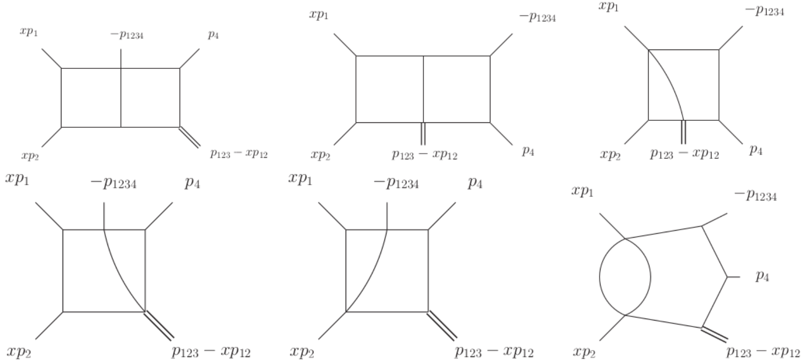


Figure 3. The five-point Feynman diagrams, besides the pentabox itself in Figure 1, that are contained in the family P_1. All external momenta are incoming.

Example II: Two-loop planar pentabox

Papadopoulos, Tommasini, Wever [JHEP 1604 (2016) 078]

The resulting matrix for 74 master integrals

$$M_{IJ} = N_{IJ}(\varepsilon)\left(\sum_{i=1}^{20}\sum_{j=1}^{2}\sum_{k=0}^{1}\frac{C_{IJ;ijk}\varepsilon^k}{(x-l_i)^j} + \sum_{j=0}^{1}\sum_{k=0}^{1}\tilde{C}_{IJ;jk}\varepsilon^k x^j\right)$$

contains twenty letters l_i given by

$$0,\ \ 1,\ \ \frac{s_{45}}{s_{45}-s_{23}},\ \ \frac{s_{45}}{s_{12}},\ \ 1-\frac{s_{34}}{s_{12}},\ \ 1+\frac{s_{23}}{s_{12}},$$
$$1-\frac{s_{34}-s_{51}}{s_{12}},\ \ \frac{s_{45}-s_{23}}{s_{12}},\ \ -\frac{s_{51}}{s_{12}},\ \ \frac{s_{45}}{-s_{23}+s_{45}+s_{51}},\ \ \frac{s_{45}}{s_{34}+s_{45}},$$
$$\frac{s_{12}s_{23}-2s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}-s_{45}-s_{51})},\ \ \frac{s_{12}s_{23}-s_{12}s_{45}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_2}}{2s_{12}(s_{23}-s_{45}-s_{51})},$$
$$\frac{s_{12}s_{23}-s_{12}s_{51}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}\pm\sqrt{\Delta_1}}{2s_{12}(s_{23}+s_{34}-s_{51})},\ \ \frac{s_{12}s_{45}\pm\sqrt{\Delta_3}}{s_{12}s_{34}+s_{12}s_{45}},\ \ \frac{s_{45}}{s_{12}+s_{23}},$$

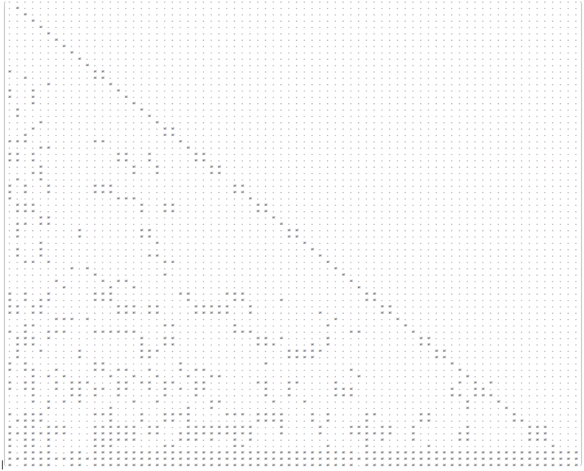
where

$$\Delta_1 = (s_{12}(s_{51}-s_{23})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2 + 4s_{12}s_{45}s_{51}(s_{23}+s_{34}-s_{51})$$
$$\Delta_2 = (s_{12}(-s_{23}+s_{45}+s_{51})+s_{23}s_{34}+s_{45}(s_{51}-s_{34}))^2 - 4s_{12}s_{45}s_{51}(-s_{23}+s_{45}+s_{51})$$
$$\Delta_3 = -(s_{12}s_{34}s_{45}(s_{12}-s_{34}-s_{45}))$$

Example II: Two-loop planar pentabox

Papadopoulos, Tommasini, Wever [JHEP 1604 (2016) 078]

Initial matrix:



- optimizations for block-triangular shape
- 1.5 hours at i5 CPU
- using highly-optimized `Maple` routines

Summary

Fuchsia — a tool for reducing differential equations for master integrals to the epsilon form Henn '13:

- complete implementation of Lee '14 algorithm
- open-source and free (Python, SageMath, Maxima)
- powerful, e.g., two-loop planar pentabox
- http://github.com/gituliar/fuchsia
- http://gituliar.net/fuchsia/fuchsia.pdf

Prospects

- completely eliminate Maple dependencies
- more parameters (not only x, ϵ)

Questions?