Complex structures and zero-curvature equations for σ-models

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σ-models

The action of a σ-model describing maps X from a 2D worldsheet $\mathscr{C}$ to a target space $\mathscr{M}$ with metric h is given by

$$\mathcal{S} = \frac{1}{2} \int_{\mathscr{C}} d^2 z \, h_{ij}(X) \, \partial_\mu X^i \, \partial_\mu X^j \tag{1}$$

Its critical points $X(z, \bar{z})$ are called *harmonic maps*.

We will be interested in the case when the target space $\mathscr{M}$ is homogeneous: $\mathscr{M} = G/H$, G compact and semi-simple. We will use the following standard decomposition of the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \tag{2}$$

where $\mathfrak{m} \perp \mathfrak{h}$ with respect to the Killing metric on $\mathfrak{g}$.

Symmetric target spaces

For a reductive homogeneous space one has the following relations:

$$
\begin{aligned}
&[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h} &\Rightarrow\qquad &\mathfrak{h}\text{ is a subalgebra}\\
&[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m} &\Rightarrow\qquad &\mathfrak{m}\text{ is a representation of }\mathfrak{h}
\end{aligned}
$$

A homogeneous space G/H is called *symmetric* if

$$[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{h} \tag{3}$$

Equivalently, there exists a $\mathbb{Z}_2$-grading on $\mathfrak{g}$,
i.e. a Lie algebra homomorphism σ of $\mathfrak{g}$, such that $\sigma(a)=a$ for $a\in\mathfrak{h}$ and $\sigma(b)=-b$ for $b\in\mathfrak{m}$.

Equations of motion. 1

The action of a σ-model with homogeneous target space G/H is globally invariant under the Lie group G. Therefore, there exists a conserved Noether current $K^\mu \in \mathfrak{g}$:

$$\partial_\mu K^\mu = 0 \tag{4}$$

Since the group G acts transitively on its quotient space G/H, the equations of motion are in fact *equivalent* to the conservation of the current.

Equations of motion. 2

It was observed by Pohlmeyer ('76) that in the case when the target space is *symmetric*, the current K is, moreover, flat (with proper normalization):

$$dK - K \wedge K = 0 \tag{5}$$

To get an idea, why this can be the case, recall that the Maurer-Cartan equation has the solution

$$K = -g^{-1}dg, \qquad g \in G \tag{6}$$

What is the relation between g and a point in the configuration space $[\tilde{g}] \in G/H$? The answer is given by Cartan's embedding $G/H \hookrightarrow G$:

$$g = \widehat{\sigma}(\tilde{g})\tilde{g}^{-1} \tag{7}$$

$\widehat{\sigma}$ is a Lie group homomorphism induced by the Lie algebra involution σ.

Equations of motion. 3

Another observation of Pohlmeyer was that the two conditions

$$\begin{aligned} d * K = 0 \qquad & \text{(Conservation)} \\ dK - K \wedge K = 0 \qquad & \text{(Flatness)} \end{aligned} \tag{8}$$

may be rewritten as an equation of flatness of a connection

$$A_u = \frac{1+u}{2}\, K_z dz + \frac{1+u^{-1}}{2}\, K_{\bar{z}} d\bar{z}, \tag{9}$$

where we have decomposed the current $K = K_z dz + K_{\bar{z}} d\bar{z}$. We have

$$dA_u - A_u \wedge A_u = 0 \tag{10}$$

This leads to an associated linear system (Lax pair)

$$(d - A_u)\Psi = 0 \tag{11}$$

Integrability

The existence of a linear system described above is often a sufficient condition for the classical integrability of the model.

The linear system was used by Zakharov & Mikhaylov ('79) to solve the equations of motion for the principal chiral model (target space G), with worldsheet $\mathbb{CP}^1$. A more rigorous approach was developed by Uhlenbeck ('89). Solutions of the e.o.m. for σ-models with symmetric target spaces may be obtained by restricting the solutions of the principal chiral model.

These constructions could not be directly generalized to the case of <u>homogeneous, but not symmetric</u> target spaces (no Cartan involution).

Target spaces

We will consider a different class of models, with target spaces $\mathcal{M}$ of the following type:

- $\mathcal{M} = G/H$ is a homogeneous space; for simplicity we take G compact and semi-simple

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$$

- $\mathcal{M}$ has an integrable G-invariant complex structure $\mathscr{I}$

$$\mathfrak{m} = \mathfrak{m}_+ + \mathfrak{m}_-, \qquad [\mathfrak{h}, \mathfrak{m}_\pm] \subset \mathfrak{m}_\pm, \qquad [\mathfrak{m}_\pm, \mathfrak{m}_\pm] \subset \mathfrak{m}_\pm$$

- The Killing metric h is Hermitian (i.e. of type $(1,1)$) w.r.t. $\mathscr{I}$

$$h(\mathfrak{m}_\pm, \mathfrak{m}_\pm) = 0$$

Target spaces. 2

Complex homogeneous spaces were classified by Wang ('54) a long time ago. They are toric bundles over flag manifolds.

Consider for simplicity the case of $G = SU(N)$. Then the relevant manifolds are of the form

$$\mathcal{M} = \frac{SU(N)}{S(U(n_1) \times \ldots \times U(n_m))}, \qquad \sum_{i=1}^{m} n_i \leq N\,,$$

If $\sum\limits_{i=1}^{m} n_i = N$, this is the manifold of partial flags in $\mathbb{C}^N$. Otherwise it is a $U(1)^{2s}$-bundle over a flag manifold, where $2s = N - \sum\limits_{i=1}^{m} n_i$.

The action

Given a homogeneous space of the type just described, one can introduce the action of the model:

$$\mathcal{S} = \int\limits_{\mathscr{C}} d^2 z \, \|\partial X\|^2 + \int\limits_{\mathscr{C}} X^* \omega =$$

$$= \int\limits_{\mathscr{C}} d^2 z \, \left(h_{ij} \partial_\mu X^i \partial_\mu X^j + \epsilon_{\mu\nu} \omega_{ij} \partial_\mu X^i \partial_\nu X^j \right),$$

where $\omega = h \circ \mathscr{I}$ is the Kähler form. Note, however, that, in general, the metric h is <u>not Kähler</u>, hence the form ω is not closed: $d\omega \neq 0$. Therefore the second term in the action contributes to the e.o.m.!

The action

Let K be the Noether current constructed using the above action. As we already discussed, the e.o.m. are equivalent to its conservation:

$$d * K = 0$$

The key observation is that, for the models considered, it is also flat:

$$dK - K \wedge K = 0$$

These two equations mean, in essence, that the described models are sub-models of the principal chiral model (PCM). In particular, the solutions of these models are a subset of solutions of the PCM. The Lax pair representation can be constructed in parallel with the Pohlmeyer procedure.

Relation to the case of symmetric spaces

Complex symmetric spaces fall in our category, with characteristic property $[\mathfrak{m}_+, \mathfrak{m}_+] = 0$. In fact, this implies $[\mathfrak{m}_+, \mathfrak{m}_-] \subset \mathfrak{h}$. Symmetric spaces of the group $SU(N)$ are the Grassmannians

$$\mathbb{G}_{n|N} := \frac{SU(N)}{S(U(n) \times U(N-n))}$$

In this case the canonical one-parametric family of flat connections is

$$\widetilde{A}_\lambda = \frac{1-\lambda}{2}\,\widetilde{K}_z dz + \frac{1-\lambda^{-1}}{2}\,\widetilde{K}_{\bar{z}} d\bar{z},$$

where $\widetilde{K}$ is the canonical Noether current, i.e. the one constructed using the standard action

$$\mathcal{S} = \frac{1}{2} \int\limits_{\mathscr{C}} d^2 z\, h_{ij}(X)\, \partial_\mu X^i\, \partial_\mu X^j \qquad (12)$$

Relation to the case of symmetric spaces. 2

The models, which we described above, feature an additional term in their action: $\int_{\mathscr{C}} X^*\omega$, the integral of the Kähler form. Therefore the Noether current K defined using this action will be different from $\widetilde{K}$, the difference being a 'topological' current:

$$K = \widetilde{K} + *dM$$

Nevertheless both K and $\widetilde{K}$ are flat. The one-parametric family of connections that we constructed earlier has the form

$$A_u = \frac{1+u}{2}\, K_z dz + \frac{1+u^{-1}}{2}\, K_{\bar{z}} d\bar{z},$$

A natural question arises: How are $\widetilde{A}_\lambda$ and A_u related?

Relation to the case of symmetric spaces. 3

The answer is: $\widetilde{A}_\lambda$ and A_u are related by a gauge transformation Ω:

$$\widetilde{A}_\lambda = \Omega A_u \Omega^{-1} - \Omega d\Omega^{-1}$$

Ω can be written out explicitly ($\tilde{g}$ is the 'dynamical' group element):

$$\Omega = \tilde{g}\Lambda\tilde{g}^{-1}, \quad \text{where} \quad \Lambda = \mathrm{diag}(\underbrace{\lambda^{-1/2}, \ldots, \lambda^{-1/2}}_{n}, \underbrace{\lambda^{1/2}, \ldots, \lambda^{1/2}}_{N-n})$$

Rather important is the nontrivial relation between the spectral parameters:

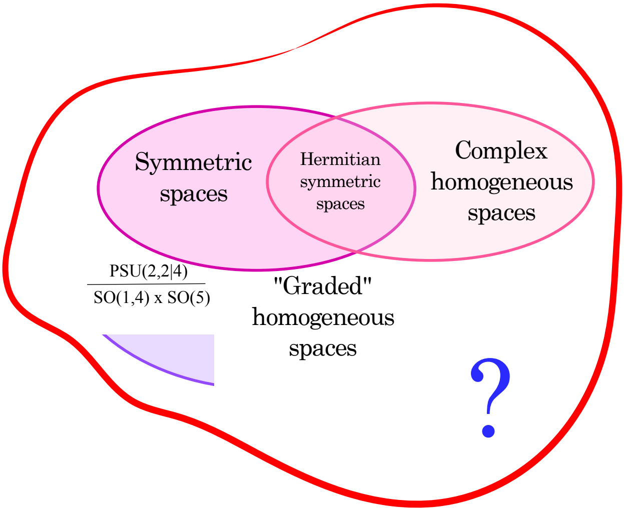
$$\lambda = u^{1/2}$$

This relation may be confirmed by analyzing the limiting behavior of the holonomies of the connection as $u \to 0$ (such analysis can be borrowed from Hitchin ('90)).

Outlook

- Zero-curvature representations were known for σ-models with symmetric target spaces
- We have considered modified σ-models with complex homogeneous target spaces, for which there exist Lax pairs
- A concrete example of such model has been put forward, when the target space is the flag manifold $\frac{U(3)}{U(1)^3}$. When the worldsheet is a sphere $\mathbb{CP}^1$, all solutions of the e.o.m. have been constructed
- Crucial test of integrability: construct solutions, when the worldsheet is a torus $S^1 \times S^1$ (as in Hitchin ('90) for $\mathcal{M} = SU(2)$)
- What is the true role of Lie algebra gradings, such as $\mathbb{Z}_2$, $\mathbb{Z}_4$?

What is the space of integrable σ-models?



An example: the flag manifold

We will consider the simplest homogeneous, but non-symmetric target space – the flag manifold

$$\mathcal{F}_3 = \frac{U(3)}{U(1)^3} \tag{13}$$

It is the space of ordered triples of lines through the origin in $\mathbb{C}^3$, and can be parametrized by three orthonormal vectors

$u_i, \quad i = 1, 2, 3$
$\bar{u}_i \circ u_j = \delta_{ij}$, modulo phase rotations: $u_k \sim e^{i\alpha_k} u_k$.

Complex structures on the flag manifold

To formulate the model, we need to pick a particular complex structure on $\mathcal{F}_3$. The (co)tangent space to $\mathcal{F}_3$ is spanned at each point by the one-forms

$$J_{ij} := u_i \circ d\bar{u}_j, \quad i \neq j \tag{14}$$

One can pick any three non-mutually conjugate one-forms and *define* the action of the complex structure operator I on them:

$$I \circ J_{12} = \pm i J_{12}, \quad I \circ J_{23} = \pm i J_{23}, \quad I \circ J_{31} = \pm i J_{31} \tag{15}$$

Altogether there are $2^3 = 8$ possible choices, so that there are 8 invariant almost complex structures. However, only 6 of them are *integrable*.

The action simplified

Pick the integrable complex structure $\mathscr{I}$, in which J_{12}, J_{13}, J_{23} are holomorphic one-forms. Then the action can be written as (DB '14)

$$\mathcal{S} = \int d^2 z \, \left(|(J_{12})_{\bar{z}}|^2 + |(J_{13})_{\bar{z}}|^2 + |(J_{23})_{\bar{z}}|^2 \right) \tag{16}$$

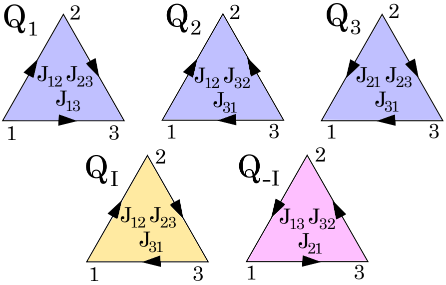
The e.o.m. are:

$$\mathscr{D}_z (J_{12})_{\bar{z}} = 0, \qquad \mathscr{D}_z (J_{31})_{\bar{z}} = 0, \qquad \mathscr{D}_z (J_{23})_{\bar{z}} = 0 \tag{17}$$

From the action (16) it is clear that the holomorphic curves defined by $(J_{12})_{\bar{z}} = (J_{13})_{\bar{z}} = (J_{23})_{\bar{z}} = 0$ minimize the action, hence are solutions of the e.o.m. From (17) it follows that $(J_{12})_{\bar{z}} = (J_{31})_{\bar{z}} = (J_{23})_{\bar{z}} = 0$ is a solution as well. This defines a curve, holomorphic in a different, non-integrable almost complex structure I.

We have seen that the curves, holomorphic in at least two different almost complex structures, satisfy the e.o.m. As we discussed, there are 8 almost complex structures on the flag manifold. Are there any other holomorphic curves that still solve the e.o.m.?

The answer is YES. The relevant complex structures are:



Holomorphic curves. 2

We have already discussed why the $\mathcal{Q}_I$-holomorphic curves and $\mathcal{Q}_1$-holomorphic curves satisfy the e.o.m.

To see why the $\mathcal{Q}_2$- and $\mathcal{Q}_3$-holomorphic curves satisfy the e.o.m., one should note that the differences between the respective Kähler forms are closed forms, i.e. for example $\omega_1 - \omega_2 = \Omega_{top}$ with $d\Omega_{top} = 0$. Therefore the two actions $\mathcal{S}_1$ and $\mathcal{S}_2$ differ by a topological term:

$$\mathcal{S}_1 - \mathcal{S}_2 = \int_{\mathscr{C}} \Omega_{top} \tag{18}$$

Holomorphic curves. 3

This leads to an interesting bound on the instanton numbers of the holomorphic curves. To see this, note that the flag manifold may be embedded as

$$i : \mathcal{F}_3 \hookrightarrow \mathbb{CP}^2 \times \mathbb{CP}^2 \times \mathbb{CP}^2 \tag{19}$$

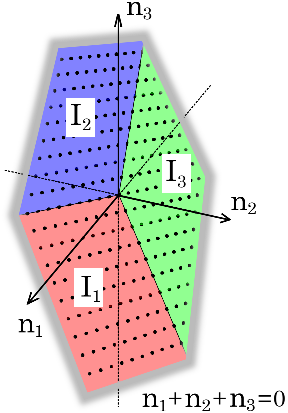
The second cohomology $H^2(\mathcal{F}_3, \mathbb{R}) = \mathbb{R}^2$ can be described via the pull-backs of the Fubini-Study forms of the $\mathbb{CP}^2$'s, and the corresponding instanton numbers are $n_i = \int\limits_{\mathscr{C}} i^*(\Omega_{FS}^{(i)}),\ i = 1, 2, 3.$

These are subject to the condition

$$n_1 + n_2 + n_3 = 0. \tag{20}$$

Holomorphic curves. 4

The bounds on the topological numbers n_i for the holomorphic curves, which follow from the non-negativity of the actions $\mathcal{S}_i$, are:



Solutions for $\mathscr{C} = \mathbb{CP}^1$

The main point of introducing the action (16) is that, as it turns out, the corresponding Noether current is flat, in full analogy with what happens for σ-models with *symmetric* target-spaces.

The full consequences of this fact still remain to be investigated, but for the moment we can provide a complete description of the solutions of the e.o.m. for the case when the worldsheet $\mathscr{C} = \mathbb{CP}^1$. To describe these solutions, one should recall that there exist three fibrations

$$\pi_i : \mathcal{F}_3 \to (\mathbb{CP}^2)_i, \quad i = 1, 2, 3, \tag{21}$$

each with fiber $\mathbb{CP}^1$.

Solutions for $\mathscr{C} = \mathbb{CP}^1$. 2

All solutions to the e.o.m. are parametrized by the following data:

- One of the projections $\pi_i : \mathcal{F}_3 \to (\mathbb{CP}^2)_i, \quad i = 1, 2, 3$
- A harmonic map $v_{har} : \mathbb{CP}^1 \to (\mathbb{CP}^2)_i$ to the base of the projection
- A holomorphic map $w_{hol} : \mathbb{CP}^1 \to \mathbb{CP}^1$ to the fiber of the projection,

.

For every triple (i, v_{har}, w_{hol}) there exists a solution of the e.o.m., and all solutions are obtained in this way. (DB '15)

The crucial point is that the harmonic maps to the base manifold $\mathbb{CP}^2$ are known explicitly (Din, Zakrzewski '80) (and the holomorphic maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ are just rational functions).

$\mathbb{Z}_m$-graded spaces. 1.

A $\mathbb{Z}_m$-graded (m-symmetric) space G/H is characterized by the relations

$$\mathfrak{g} = \oplus_{k=0}^{m-1} \mathfrak{g}_k, \qquad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j \bmod m} \tag{22}$$

There exists a Lax representation for $\mathbb{Z}_m$-graded models with the action [Young, 2006]

$$\mathcal{S} = \int\limits_{\mathscr{C}} d^2z \, \|\partial X\|^2 + \int\limits_{\mathscr{C}} X^* \widetilde{\omega}, \tag{23}$$

where the B-field is expressed in terms of the $\mathbb{Z}_m$-graded components $J^{(k)}$ of the current:

$$\widetilde{\omega} = \frac{1}{2} \sum_{k=0}^{m} \frac{(m-k)-k}{m} \operatorname{tr}(J^{(k)} \wedge J^{(m-k)}) \tag{24}$$

$\mathbb{Z}_m$-graded spaces. 2.

But: In general, there are many $\mathbb{Z}_m$-gradings on a given Lie algebra $\mathfrak{g}$.

Example: $su(3)$

$$\mathbb{Z}_2 : \begin{pmatrix} 0 & \mathbf{0} & 1 \\ 0 & 0 & \mathbf{1} \\ \mathbf{1} & 1 & 0 \end{pmatrix}, \quad \mathbb{Z}_3 : \begin{pmatrix} 0 & \mathbf{1} & 2 \\ 2 & 0 & \mathbf{1} \\ \mathbf{1} & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{0} & 1 \\ 0 & 0 & \mathbf{1} \\ \mathbf{2} & 2 & 0 \end{pmatrix},$$

$$\mathbb{Z}_4 : \begin{pmatrix} 0 & \mathbf{1} & 2 \\ 3 & 0 & \mathbf{1} \\ \mathbf{2} & 3 & 0 \end{pmatrix}, \quad \mathbb{Z}_5 : \begin{pmatrix} 0 & \mathbf{1} & 3 \\ 4 & 0 & \mathbf{2} \\ \mathbf{2} & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1} & 2 \\ 4 & 0 & \mathbf{1} \\ \mathbf{3} & 4 & 0 \end{pmatrix},$$

$$\mathbb{Z}_6 : \begin{pmatrix} 0 & \mathbf{1} & 3 \\ 5 & 0 & \mathbf{2} \\ \mathbf{3} & 4 & 0 \end{pmatrix}, \quad \mathbb{Z}_7 : \begin{pmatrix} 0 & \mathbf{1} & 3 \\ 6 & 0 & \mathbf{2} \\ \mathbf{4} & 5 & 0 \end{pmatrix}$$

$\mathbb{Z}_m$-graded spaces. 3.

A question arises: Are the models different for different choices of gradings?

Answer: No, they can all be reduced to our model, with an appropriate choice of complex structure (up to a topological term).

[At least for $G = SU(N)$ and $A_{N-1}^{(1)}$ gradings.]