The semi-classical energy of rotating Nambu-Goto strings

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Rethinking Quantum Field Theory
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A motivating example: The hydrogen atom

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- Write Hamiltonian in cylindrical coordinates in terms of $L_3 = \hbar\ell_3 > 0$:

$$H = \frac{m}{2}\left(\dot{\rho}^2 + \dot{z}^2\right) + \frac{\hbar^2\ell_3^2}{2m\rho^2} - \frac{e^2}{\varepsilon_0 r}.$$

- For perturbations $\delta\rho$, δz of the circular orbit with ang. mom. L_3, obtain

$$H = -\frac{1}{2ma_0^2}\frac{\hbar^2}{\ell_3^2} + \frac{m}{2}\left(\delta\dot{\rho}^2 + \delta\dot{z}^2\right) + \frac{1}{2ma_0^4}\frac{\hbar^2}{\ell_3^6}\left(\delta\rho^2 + \delta z^2\right) + \mathcal{O}(\delta^3).$$

- Given L_3, the kth excited energy level is $k+1$ times degenerate, given by

$$E_{k,L_3} = -\frac{\hbar^2}{2ma_0^2}\left(\frac{1}{\ell_3^2} - \frac{2(k+1)}{\ell_3^3}\right).$$

- The actual kth excited state with $L_3 = \hbar\ell_3 > 0$ is $k+1$ times degenerate:

$$E_{k,\ell_3} = -\frac{\hbar^2}{2ma_0^2}\frac{1}{(\ell_3 + k + 1)^2} \simeq -\frac{\hbar^2}{2ma_0^2}\left(\frac{1}{\ell_3^2} - \frac{2(k+1)}{\ell_3^3}\right) + \mathcal{O}(\ell_3^{-4}).$$

- Found the correct degeneracy and energy levels at sub-leading order.

The Nambu-Goto string

There are a few similarities between the hydrogen atom and the Nambu-Goto string, defined by the action

$$\mathcal{S}_{NG} = \gamma \int_\Sigma \sqrt{|g|}\mathrm{d}^2 x.$$

- The Lagrangian is not smooth.
- The classical ground state is at a singularity.

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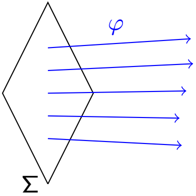
- In the covariant quantization scheme, the ground state energies for a given angular momentum $\ell_{1,2} > 0$ lie on the Regge trajectory

$$E^2_{\ell_{1,2}} = 2\pi\gamma(\ell_{1,2} - a).$$

- The Regge intercept a is free parameter, constrained by $a \leqslant 1$ for $D \leqslant 25$ and $a = 1$ for $D = 26$.
- a is a quantum correction of sub-leading order in $\ell_{1,2}$.
- Computable in a semi-classical approximation!?

The perturbative Nambu-Goto string



- Let $X : \Sigma \to M$ be a classical solution.
- With $\varphi \in \Gamma^\infty(\Sigma, TM)$ describe dynamical configuration as

$$\tilde{X}(x) = \exp_{X(x)}(\gamma^{-\frac{1}{2}}\varphi(x)) = X(x) + \gamma^{-\frac{1}{2}}\varphi(x).$$

- φ is field on Σ, eom depending only on induced metric $g_{\mu\nu} = \partial_\mu X^a \partial_\nu X_a$ and 2nd fundamental form $\Pi^a_{\mu\nu} = \nabla_\mu \partial_\nu X^a$ of the embedding X.
- Use QFT on curved space-times, in particular locally covariant renormalization techniques [Hollands, Wald 01].
- The re-parametrization invariance can be gauge fixed without anomalies, for any D. Can also ensure Poincaré covariance [Bahns, Rejzner, Z 14].
- Consistent as effective theory, analogous to perturbative quantum gravity.
- At $\mathcal{O}(\varphi^2)$, the longitudinal modes $\varphi_{\text{long}} \in \Gamma^\infty(\Sigma, T\Sigma)$ drop out of $\mathcal{S}_{NG}$.
- For the determination of the energy at $\mathcal{O}(\varphi^2)$, gauge fixing not necessary.
- May work with the transversal modes $\varphi \in \Gamma^\infty(\Sigma, N\Sigma)$ only.

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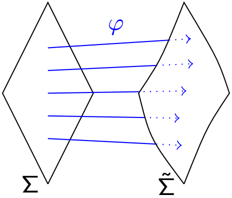

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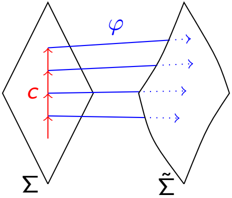

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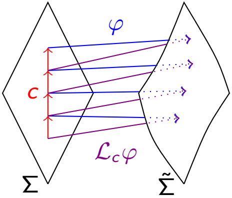

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The rotating open Nambu-Goto string

- The classical solution

$$X(\tau,\sigma) = R(\tau, \cos\tau\cos\sigma, \sin\tau\cos\sigma, 0)$$

with $\sigma \in [0,\pi]$ has the induced metric and scalar curvature

$$g_{\mu\nu} = R^2 \sin^2\sigma\ \eta_{\mu\nu}, \qquad \mathcal{R} = -\frac{2}{R^2\sin^4\sigma}.$$

- It has energy and angular momentum

$$E^c = \gamma\pi R, \qquad L^c_{1,2} = \tfrac{1}{2}\gamma\pi R^2.$$

- The world-sheet Hamiltonian is $H = H^0 + \mathcal{O}(R^{-1}\gamma^{-\frac{1}{2}})$, with H^0 of $\mathcal{O}(1)$.
- The quantum correction to the target space energy E fulfills

$$E^q = \tfrac{1}{R}(H + L^q_{1,2}),$$

so that

$$\begin{aligned} E^2 = (E^c + E^q)^2 &= \gamma^2\pi^2 R^2 + 2\gamma\pi(H + L^q_{1,2}) + \mathcal{O}(R^{-2}) \\ &= 2\gamma\pi(\underbrace{L^c_{1,2} + L^q_{1,2}}_{L_{1,2}} + \underbrace{H^0}_{-a}) + \mathcal{O}(L_{1,2}^{-\frac{1}{2}}). \end{aligned}$$

Perturbations of the rotating open Nambu-Goto string

- Decompose fluctuations φ in scalar and planar polarizations:

$$\varphi = \phi_s v_s + \phi_p v_p = \phi_s \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \phi_p \begin{pmatrix} \cot\sigma \\ -\sin\tau/\sin\sigma \\ \cos\tau/\sin\sigma \\ 0 \end{pmatrix}.$$

- For the coefficients, we obtain eom's with boundary conditions

$$-\ddot{\phi}_s = \Delta_s \phi_s := -\phi_s'', \qquad 0 = \phi_s'(0) = \phi_s'(\pi),$$
$$-\ddot{\phi}_p = \Delta_p \phi_p := -\phi_p'' + \tfrac{2}{\sin^2\sigma}\phi_p \quad 0 = \phi_p(0) = \phi_p(\pi) = \phi_p'(0) = \phi_p'(\pi).$$

- Δ_s, Δ_p essentially self-adjoint on domain $C^2([0,\pi])$, so they admit a unique self-adjoint extension. The normalized eigenvectors are ($\omega_n = n$)

$$\phi_{s,n} = \tfrac{\sqrt{2}}{\sqrt{\pi}} \cos n\sigma, \qquad n \geqslant 0$$
$$\phi_{p,n} = \tfrac{\sqrt{2}}{\sqrt{\pi(n^2-1)}} \left(n \cos n\sigma - \cot\sigma \sin n\sigma \right) \qquad n \geqslant 2.$$

- Note absence of planar $n = 1$ mode [Baker, Steinke 01; Zayas, Sonnenschein, Vaman 04].
- Using the normalized modes, one may quantize canonically.
- Scalar $n = 0$ mode is position conjugate to cms momentum. Omit for Regge intercept.

The world-sheet Hamiltonian

- The free Hamiltonian for the perturbations of our classical solution is

$$H^0 = \frac{1}{2}\int_0^\pi \left((D-3)\left[\dot{\phi}_s^2 + {\phi_s'}^2\right] + \dot{\phi}_p^2 + {\phi_p'}^2 + \tfrac{2}{\sin^2\sigma}\phi_p^2 \right) \mathrm{d}\sigma.$$

- Renormalize in a locally covariant way, as in CSQFT [Hollands, Wald 01]

$$\langle \Omega | (\nabla^\alpha \phi \nabla^\beta \phi)(x) | \Omega \rangle = \lim_{x' \to x} \nabla^\alpha \nabla'^\beta \left(w_\Omega(x;x') - h(x;x') \right)$$

  with w_Ω the two-point function of Ω and h the Hadamard parametrix.
- h is determined purely locally. It involves a renormalization scale Λ.
- For the energy density, we obtain

$$\rho(\sigma) := (D-3)\langle H_s^0(\sigma)\rangle + \langle H_p^0(\sigma)\rangle = -\frac{D-2}{24\pi} - \frac{1}{2\pi\sin^2\sigma}\log\frac{\Lambda}{2R\sin^2\sigma}.$$

  Locally finite, but diverges non-integrably at the boundaries.
- Divergence can be treated with geodesic curvature boundary counterterms [Dowker, Kennedy 78; Baker, Steinke 02; Hellerman, Swanson 15]:

$$\langle H^0 \rangle = \lim_{s\to 0}\left[\int_s^{\pi - s} \rho(\sigma)\mathrm{d}\sigma - \frac{1}{\pi}\sqrt{|h^s|}\kappa_s \log\frac{d_s}{\Lambda_{bd}} \right]$$

- The boundary counterterm is of geometric origin, i.e., in the spirit of local covariance. The scale Λ_{bd} is uniquely fixed, thus no ambiguity.

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- The boundary counterterm is of geometric origin, i.e., in the spirit of local covariance. The scale Λ_{bd} is uniquely fixed, thus no ambiguity.

The closed Nambu-Goto string

- The divergence of the curvature and the energy density at the boundaries casts doubt on the validity of the semi-classical approximation.
- There are rotating closed string solutions, for which g is flat:

$$X(\tau, \sigma) = \frac{R}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\tau \\ \sin\tau\cos\sigma \\ -\cos\tau\cos\sigma \\ \cos\tau\sin\sigma \\ \sin\tau\sin\sigma \\ 0 \end{pmatrix}.$$

- We now have three planar modes, coupled by eom.
- For $n = 1$, two planar modes degenerate in the symplectic form. There are supplementary linearly growing modes (also oscillating with $n = 1$). These correspond to positions and momenta in the subspace spanned by $e_1 - e_4$.
- As for scalar $n = 0$ mode, ignored in the computation of the intercept.
- The two missing $n = 1$ modes contribute 1 to the intercept, yielding again

$$a = \tfrac{D-2}{24} + 1.$$

Conclusion

- For the open and the closed string, we find the semi-classical intercept

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  for which the covariantly quantized string is inconsistent for any D.
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THANK YOU FOR YOUR ATTENTION!