N=2 SUSY field theories: A window into nonperturbative QFT?

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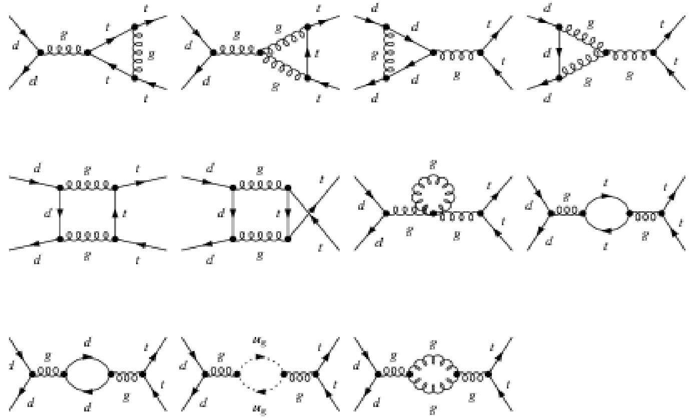
The strong coupling problem



Our goal: Calculate scattering amplitudes/ expectation values/ correlation functions as functions of coupling constants and other parameters (masses etc.).

Perturbation theory yields formal series expansions in coupling constants,

- generically non-convergent (factorial growth),
- asymptotic, but generically not Borel-summable.

Questions:

1) Does a given QFT exist at all?

2) Do we need **extra** non-perturbative info to even define QFT's?

   Is the information provided by a given classical action enough to define a corresponding QFT uniquely?

Examples from low-dimensional QFT (e.g. Liouville theory, $L = \partial_\mu\phi\partial^\mu\phi - \mu e^{2b\phi}$) indicate that the answers can be very delicate (constructive versus bootstrap).

Exact results from localization method

Having unbroken SUSY may open the possibility to compute some important quantities exactly using localization:

Given SUSY-generators Q such that $Q^2 = P$, P: bosonic symmetry, action $S = S[\Phi]$ such that $QS = 0$, fermionic functional $V = V[\Phi]$ such that $PV = 0$. Consider

$$\begin{aligned} \frac{d}{dt}\int [\mathcal{D}\Phi]\; e^{-S-tQV} &= \int [\mathcal{D}\Phi]\; e^{-S-tQV} QV \\ &= \int [\mathcal{D}\Phi]\; Q(e^{-S-tQV} V) \;=\; 0\,, \end{aligned}$$

if path-integral measure is SUSY-invariant, $\int [\mathcal{D}\Phi]\; Q(\ldots) = 0$. This means that

$$\int [\mathcal{D}\Phi]\; e^{-S} = \lim_{t\to\infty} \int [\mathcal{D}\Phi]\; e^{-S-tQV}\,.$$

If V is such that QV has positive semi-definite bosonic part, only non-vanishing contributions are field configurations satisfying $QV = 0$.

Exact results - example: $N = 2^*$-theory.

Fields: $SU(2)$ gauge field A_μ, scalar ϕ, fermionic partners,
$+$ two complex scalars, fermionic partners (mass parameter m).

Coupling constant $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$.

SUSY $\Rightarrow$ Localization $\Rightarrow$ exact results on expectation values `(Pestun;...)`,

e.g. $N = 2^*$ on S^4_ϵ: $x_0^2 + \epsilon_1^2(x_1^2 + x_2^2) + \epsilon_2^2(x_3^2 + x_4^2) = 1$. $\Rightarrow \exists$ unbroken SUSY Q.

Wilson loop $W_\mathcal{C}$ commutes with Q if $\mathcal{C} = \mathcal{C}_i$, $\mathcal{C} : (x_0, x_3, x_4) = 0$, $\mathcal{C}_2 : (x_0, x_1, x_2) = 0$.

Field configurations satisfying $QV = 0$: $\phi = \mathrm{const.} \equiv a$, other fields zero $\Rightarrow \cdots \Rightarrow$

$$\langle W_{\mathcal{C}_i} \rangle_{S^4} = \int_\mathbb{R} da\, a^2 \,|\, \mathcal{Z}^{\mathrm{inst}}(a, m; \tau; \epsilon) \,|^2\, 2\cosh(2\pi a/\epsilon_i)\,,$$

$$\mathcal{Z}^{\mathrm{inst}}(a, m; \tau; \epsilon) = \sum_{k=0}^{\infty} q^k\, \mathcal{Y}_k^{\mathrm{inst}}\big(a; m; \epsilon\big)\,, \qquad q = e^{2\pi i \tau}\,, \quad \epsilon = (\epsilon_1, \epsilon_2)\,.$$

We know **explicit** formulae for $\mathcal{Y}_k^{\mathrm{inst}}(a, m; \epsilon)$ (Nekrasov...).

Application I: Extremal correlators

`Baggio, Niarchos, Papadodimas ([BNP]); Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski, Pufu ([GGIKKP])`

Correlators of **local** fields like $\mathcal{O}_1 = -4\pi i \,\mathrm{Tr}\,\varphi^2$ (chiral ring generator)

$$G_2(\tau,\bar\tau) = |x|^{2\Delta}\langle \mathcal{O}(x)\overline{\mathcal{O}}(0)\rangle_{\mathbb{R}^4}$$

can be computed via

$$G_2 = 16\,\partial_\tau\partial_{\bar\tau}\ln Z[S^4]\,, \quad Z[S^4] = \int_{\mathbb{R}} da\, a^2\, |\, \mathcal{Z}^{\text{inst}}(a,m;\tau;\epsilon)\,|^2$$

where $\epsilon_1 = \epsilon_2 = 1$, giving results like

$$\begin{aligned} G_2(\tau,\bar\tau)_{\text{pert}} &= \frac{6}{(\text{Im}\tau)^2}\sum_{n=0}^{\infty}\frac{a_n}{(\text{Im}\tau)^n} \\ &= \frac{6}{(\text{Im}\tau)^2} - \frac{135\zeta(3)}{2\pi^2}\frac{1}{(\text{Im}\tau)^4} + \frac{1575\zeta(5)}{4\pi^3}\frac{1}{(\text{Im}\tau)^5} + \mathcal{O}\left(\frac{1}{(\text{Im}\tau)^6}\right) \end{aligned}$$

... checked to two loop order `[BNP]`, or

$$G_2(\tau,\bar\tau)_{\text{1-inst}} = \cos\theta\, e^{-\frac{8\pi^2}{g^2}}\left(\frac{6}{(\text{Im}\tau)^2} + \frac{3}{\pi}\frac{1}{(\text{Im}\tau)^3} - \frac{135\zeta(3)}{2\pi^2}\frac{1}{(\text{Im}\tau)^4} + \mathcal{O}\left(\frac{1}{(\text{Im}\tau)^5}\right)\right)\,.$$

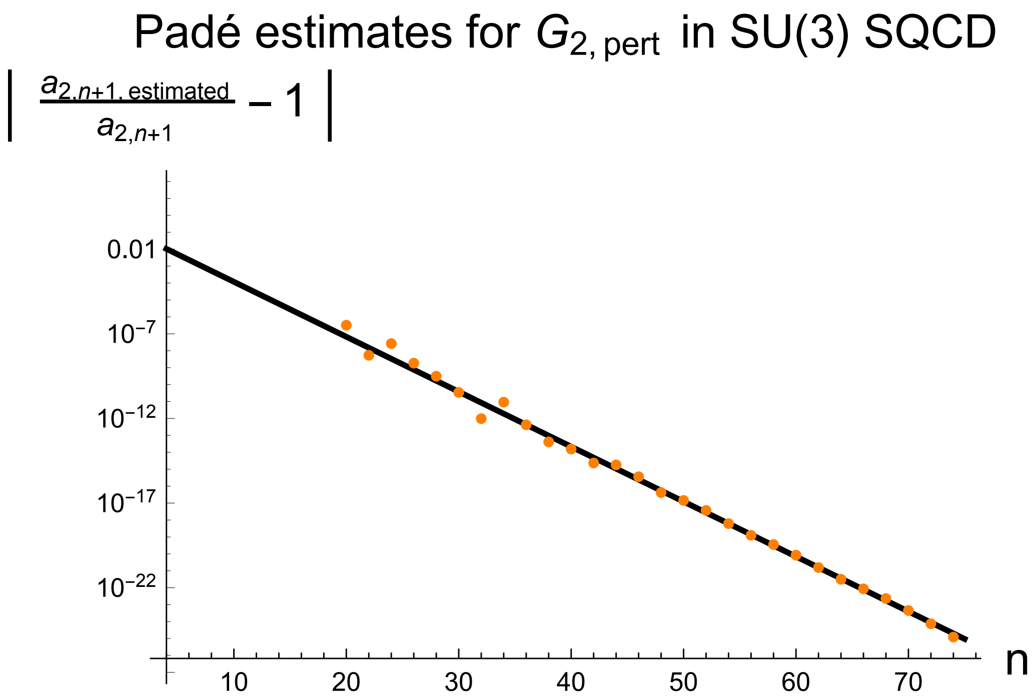
Extremal correlators II

Testing ground for ideas and conjectures on large order behaviour:

Conjecture (for QCD: `Karliner`)

$$\left| \frac{a_{n+1,\text{estimated}}}{a_{n+1}} - 1 \right| < C e^{-\sigma n} \, ,$$

where $a_{n+1,\text{estimated}}$: Padé approximation of order $(n/2, n/2)$. Check (`[GGIKKP]`):



Application II: Checking dualities

There is evidence that QFT at strong coupling can sometimes be described using dual descriptions: Let $\mathcal{F}_z = (\mathcal{A}_z, \langle\, .\, \rangle_z)$, $z \in \mathcal{M}$, be a family of QFT's with

- $\mathcal{M}$: space of QFT parameters z (couplings, masses, ...)
- $\mathcal{A}_z$: algebra of observables, $\langle\, \mathcal{O}\, \rangle_z$: vacuum expectation value of $\mathcal{O} \in \mathcal{A}_z$.

$\mathcal{F}_z$ has action $S_\tau[\Phi]$, Φ: collection of fields, if there exists $z_0 \in \mathcal{M}$, coordinates τ around z_0 in $\mathcal{M}$ such that $\tau(z_0) = 0$, and a functional $\mathcal{O}_\tau[\Phi]$ such that

$$\langle\, \mathcal{O}\, \rangle_z \simeq \int \mathcal{D}[\Phi]\; e^{-S_\tau[\Phi]}\, \mathcal{O}_\tau[\Phi]\,,$$

where $\simeq$ means equality of (trans-)series expansions in τ.

$\mathcal{F}_z$ has a dual actio=n $S'_{\tau'}[\Phi']$, if there exists $z'_0 \in \mathcal{M}$, coordinates τ' around z_0 such that $\tau'(z'_0) = 0$, and a functional $\mathcal{O}'_{\tau'}[\Phi']$ such that

$$\langle\, \mathcal{O}\, \rangle_z \simeq \int \mathcal{D}[\Phi']\; e^{-S'_{\tau'}[\Phi']}\, \mathcal{O}'_{\tau'}[\Phi']\,,$$

Classical example: Sine-Gordon - massive Thirring, $\frac{1}{2\beta^2} - 1 = \frac{g}{\pi}$, $0 < \beta^2 < 1$.

Application II: Checking dualities — in our case:

$$\boxed{\big\langle\, W_{\mathcal{C}_i}\,\big\rangle_{S_\tau} = \big\langle\, T_{\mathcal{C}_i}\,\big\rangle_{S'_{\tau'}}\,, \quad S' = S,\;\; \tau' = -1/\tau\,,}$$

where $\big\langle\, T_{\mathcal{C}_i}\,\big\rangle_{S_\tau}$: expectation value of **'t Hooft loop** $T_{\mathcal{C}}$:

- Expectation values defined by path integral over fields with specific singularity,

$$F \sim \frac{B}{4}\epsilon_{ijk}\frac{x^i}{|\vec{x}|^3}dx^k \wedge dx^j \qquad \text{near } \mathcal{C}\,,$$

where B: diagonal matrix, x^i, $i = 1,2,3$ coordinates for space transverse to $\mathcal{C}$.
Localisation `(Gomis, Okuda, Pestun)` $\Rightarrow$

$$\big\langle\, T_{\mathcal{C}_i}\,\big\rangle_{S_\tau} = \int_{\mathbb{R}} da\, a^2\, \overline{\mathcal{Z}^{\rm inst}(a,m;\tau;\epsilon)}\, \mathcal{D}_{a,i}\mathcal{Z}^{\rm inst}(a,m;\tau;\epsilon)\,,$$

where $\mathcal{D}_{a,i}$: explicitly known difference operator.

To check **duality** we'd have to **resum** all perturbative and non-perturbative corrections!

Application II: Checking dualities – possible using two further ingredients:

1) AGT-correspondence: Observation `(AGT: Alday, Gaiotto, Tachikawa)`:

$$\mathcal{Z}^{\mathrm{inst}}(a,m;\tau;\epsilon) \;=\; \text{Conformal blocks of Virasoro algebra } \mathfrak{Vir}$$

Virasoro algebra: $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$, $c = 1 + 6(\frac{\epsilon_1}{\epsilon_2} + \frac{\epsilon_2}{\epsilon_1})$

(Lie-algebra of conformal symmetry in two dimensions).

Representation theory of $\mathfrak{Vir} \Rightarrow$ Definition of conformal blocks as power series in q.

2) Duality relations (from CFT `(J.T.; Hadasz, Jaskolski, Suchanek)`)

$$\mathcal{Z}^{\mathrm{inst}}(a,m;\tau;\epsilon) \;=\; e^{\frac{\pi i}{12\tau}(\tau^2+1)} \int da' \; S(a,a';m;\epsilon)\, \mathcal{Z}^{\mathrm{inst}}(a,m;-1/\tau;\epsilon)\,,$$

$$\mathcal{D}_{a,i} S(a,a';m;\epsilon) \;=\; 2\cosh(2\pi a'/\epsilon_i) S(a,a';m;\epsilon)\,,$$

allowing us to check $\boxed{\;\langle\, W_i \,\rangle_{S_\tau} = \langle\, T_i \,\rangle_{S_{-1/\tau}}\;}$ **explicitly**!

Upshot:

Localisation $+$ AGT-correspondence $+$ CFT-results

$\Rightarrow$ Quantitative checks of duality conjectures.

Discussion:

The results above consider **partially** SUSY-protected quantities only

but:

- The duality checks are detailed enough to be sensitive to many details of the dual Lagrangians.

  By constraining (determining?) the dual Lagrangians, such results encode info on non-protected quantities

- Exact results on extremal correlation functions allow us to investigate relations between between perturbative approaches, non-perturbative effects and exact results

Within the SUSY QFT we can

- **focus** on particular non-perturbative issues/phenomena present in general QFT,

- and analyse in great detail how some dualities comes about, for example: crucial role of non-perturbative (instanton) effects for emergence of duality phenomena.