C-theorems and entanglement entropy

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Sketch of the talk

- Renormalization Group.
- What is a c-theorem. What do we need?
- Zamolodchikov theorem in (1+1) dim.
- Entanglement entropy in QFT: properties
- Entropic c-theorem in (1+1) dim.
- Comparison between c-functions.
- Entropic F-theorem – (2+1) dim.
  Subtleties: Mutual Information.
- Dimensionally continued c-theorem: some numbers.
- Proposals for generalization to higher dimensions.
- Conclusions.

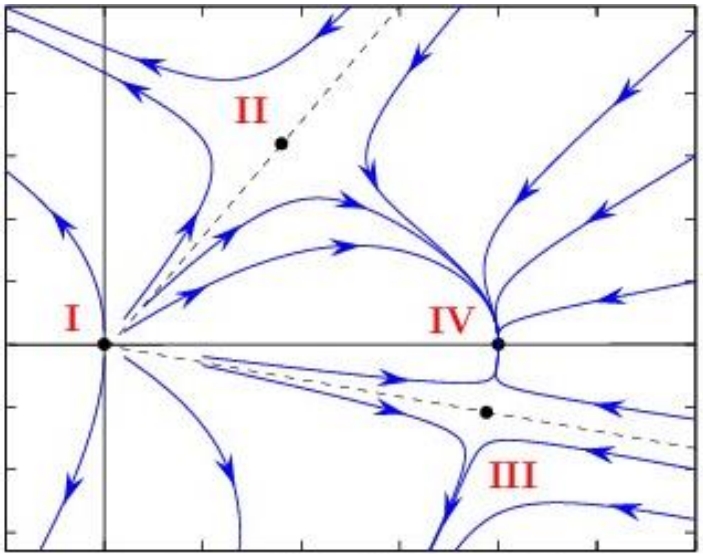
Renormalization group flow in the space of QFT

→ Changes in the physics with scale through the change of coupling constants {g_i} with the RG flow.

$$\tau \frac{dg_i}{d\tau} = \beta_i(\{g(\tau)\})$$

At fixed points there is scale invariance: the theory "looks the same" at all scales.

The RG flow Interpolates between fixed points.



In this picture, the theories with mass scales, for example, are obtained by perturbing critical points and following the renormalization group (RG) trajectories. However, interestingly, in the flow not all critical points can be joined with each other: there are constraints provided by c-theorems.

C – Theorem : General constraint for the RG flow. Ordering of the fixed points.

In the flow, not all critical points can be joined with each other: there are constraints provided by c-theorems. These state that certain c-charge decreases from the UV to the IR fixed points showing the irreversible character of the RG flow.

C-Theorem: What is needed?

1) A regularization independent quantity C, well defined in the space of theories.
2) C dimensionless and finite at the fixed points.
3) C decreases along the renormalization group trajectories. In particular

$$C_{\rm UV} \geq C_{\rm IR}$$

small size $C(r) \rightarrow C_{UV}$

large size $C(r) \rightarrow C_{IR}$

- A universal dimensionless decreasing function C(r , g , τ) will do the job

From (1) $$\tau \frac{\partial}{\partial \tau} C = -\sum_i \beta_i(g) \frac{\partial}{\partial g_i} C$$

From (2) $$\left(r \frac{\partial}{\partial r} - \tau \frac{\partial}{\partial \tau} \right) C = 0$$

$$r \frac{dC(r)}{dr} = -\sum \beta_i(g) \frac{\partial}{\partial g_i} C$$

Zamolodchikov's C-theorem in 1+1 dimensions (1986)

⟶ At the fixed point : C is the Virasoro central charge of the conformal field theory. It can be extracted from the two point function of the stress tensor at the fixed point.

$$\langle T_{\mu\nu}(0) T_{\alpha\beta}(x)\rangle = \frac{C}{|x|^4} I_{\mu\nu,\alpha\beta}(\vec{x})$$

Using conservation of the stress tensor and Lorentz symmetry an interpolating function can be constructed

$$\Theta(x) = T^{\mu}_{\mu}(x)$$

$$C(0) = C_{UV}$$

$$C(r) = \frac{3}{4\pi}\int_r^{\infty} d^2x \; x^2 \,\langle \Theta(0)\Theta(x)\rangle + C_{IR}$$

$$C'(r) = -\frac{3}{2} r^3 \,\langle \Theta(0)\Theta(x)\rangle \leq 0 \longrightarrow$$

Reflection positivity: unitarity in the Euclidean correlation functions

$$\langle 0| \int dx\, \alpha^*(x)\Theta(x) | \int dy\, \alpha(y)\Theta(y)|0\rangle \geq 0$$

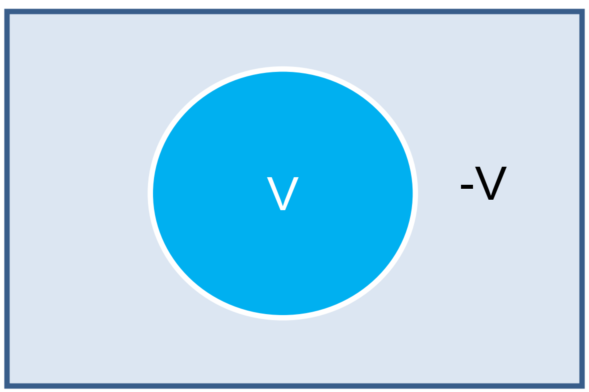
Θ is zero for CFT and it drives the C-function out of the fixed point

Reduced density matrix

Quantum system → Density matrix $\rho = |0\rangle\langle 0|$

Spatial set V → Partition of the total Hilbert space $H_V \otimes H_{-V}$

$$\rho_V = \mathrm{tr}_{(H_{-V})}\rho$$



→ Partial trace over the degrees of freedom localized outside V

→ Observer with access to a subset of the complete set of observables of a quantum system:

Horizon of events

Entanglement Entropy

$$S(V) = -\mathrm{tr}\rho_V \log \rho_V$$

→ Measure of correlations between the degrees of freedom living in and outside V

In QFT : Geometrical structure of divergences : Geometric entropy

ϵ is the UV cutoff

QFT: Continuum limit

$\epsilon \to 0$

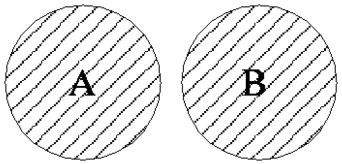
$$S(V) = g_{d-1}[\partial V]\epsilon^{-(d-1)} + \cdots + g_1[\partial V]\epsilon^{-1} + g_0[\partial V]\log\epsilon + S_0(V)$$

"Area law" → Black hole physics

Mutual Information

$$I(A,B) = S(A) + S(B) - S(A\cup B)$$

→ Boundary terms get subtracted

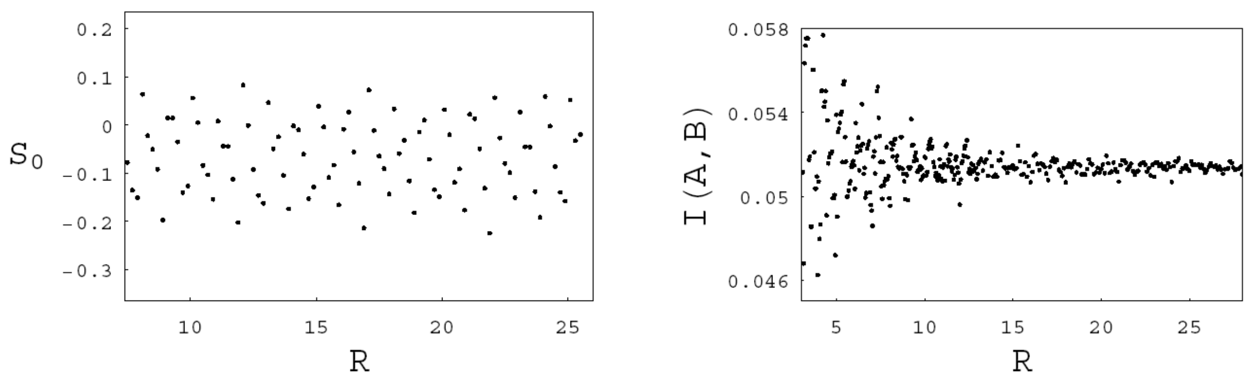


The divergences cancel in I(A,B) which is finite and well defined

Universality

$I(A,B) \neq 0$ → S cannot be finite

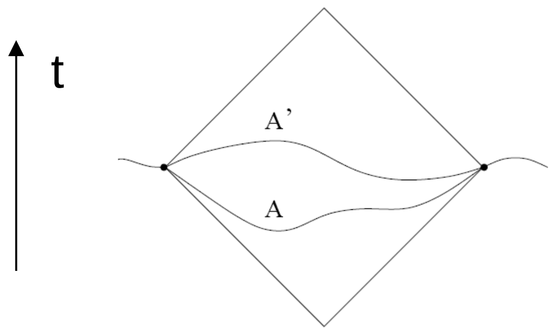
Example



Circles in a square lattice, massless scalar in 2+1.

Independence of the "Cauchy surface"

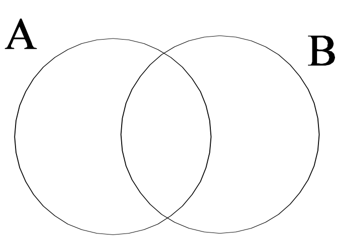
Two sets with the same causal domain of dependence



S is a function of the "diamond shaped region" or equivalently the region boundary .

$$S(A) = S(A') \quad (\rho_A = \rho_{A'})$$

Strong Subadditivity



$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B)$$

(Lieb, Ruskai (1973))

Conditions for use of SSA in spacetime : Cauchy (spatial) surface passing through A and B. Boundaries must be spatial to each other.

Entropic C-theorem, two dimensional case

c-theorem in (1+1) dimensions

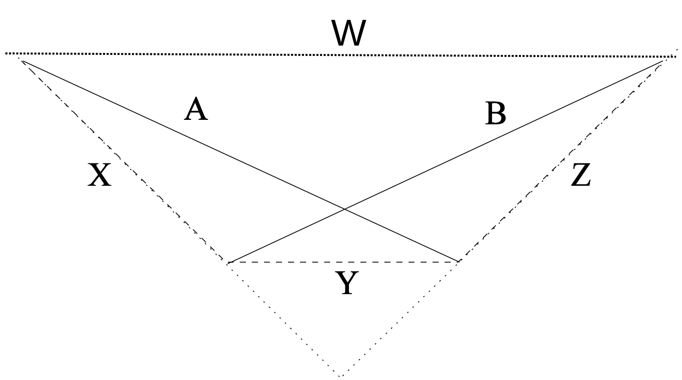
→ There is a universal function c defined for any theory which is dimensionless, decreasing along the renormalization group trajectories and takes finite values at fixed points proportional to the Virasoro central charge.

- A. B. Zamolodchikov, (1986)

→

- «Ordering» in the space of theories
- C – function : measure of degrees of freedom

- Two intervals A and B → $S(A) + S(B) \geq S(A \cap B) + S(A \cup B)$



$$S(XY) + S(YZ) \geq S(Y) + S(XYZ)$$

$$2\,S(\sqrt{rR}) \geq S(R) + S(r)\,.$$

$$rS''(r) + S'(r) \leq 0\,.$$

$$C(r) = rS'(r) \longrightarrow C'(r) \leq 0$$

- H. Casini and M. H., 2004, [hep-th/0405111].

The function C(r) given by

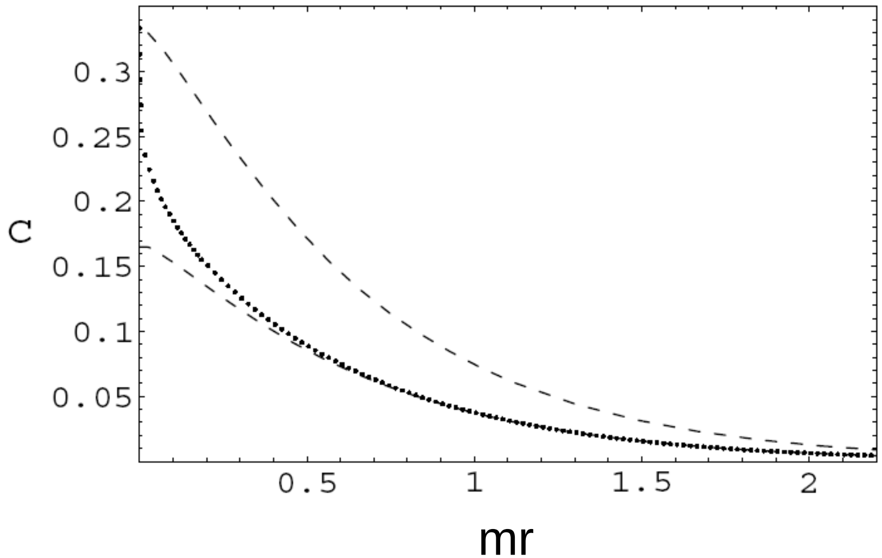
$$C(r) \;=\; rS'(r)$$

satisfies

$$C'(r) \;\leq\; 0$$

→ The central charge of the uv conformal point is larger than the central charge at the ir fixed point: the same result than Zamolodchikov c-theorem

At the conformal point : $S(r) = \frac{c}{3}\log(r/\epsilon) + c_0$ → C (r) = c / 3



From top to bottom: entropic c-functions for a Dirac, real scalar and Majorana fields.

Different c-functions, does it matter?

Is there a best one?
Are they related to each other?
Out of equilibrium interpretation?

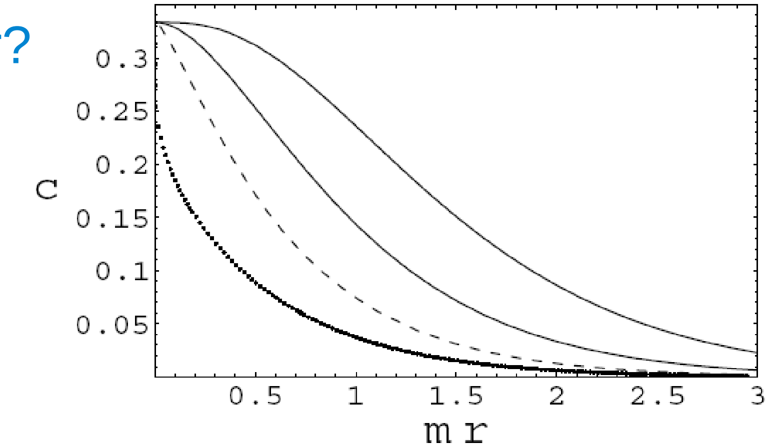


Figure 2. From top to bottom: one third of the Zamolodchikov c-functions for a real scalar and a Dirac field, and entropic c-functions for a Dirac (dashed curve) and a real scalar field (dotted curve).

$$c_D(t) \sim \frac{1}{3} - \frac{1}{3} t^2 \log^2(t) \quad \text{for } t \ll 1$$

$$c_S(t) \sim \frac{1}{3} + \frac{1}{2\log(t)} \quad \text{for } t \ll 1 \, ;$$

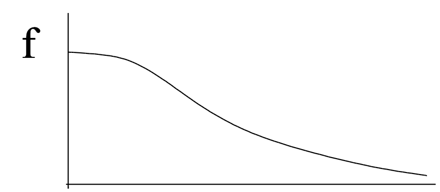
Once we have one c-function we can construct infinitely many other by convoluting with a numerical function. They are highly non-unique.

Cappelli, Friedan, Latorre (1991)
infinitely many c-functions exploiting positivity of spectral density

Spectral representation of the θ correlator

$$\langle \Theta(0)\Theta(x)\rangle = \frac{\pi}{3}\int_0^\infty d\mu\ \rho(\mu)\,\Box^2 G_0(x,\mu)\,, \qquad \rho(\mu)\geq 0$$

$$\tilde{c}(r) = \int d\mu\ \rho(\mu)\, f(\mu r)\,, \qquad f(x)>0,\ f'(x)<0,\ f(0)=1,\ f(\infty)=0$$

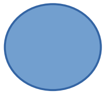


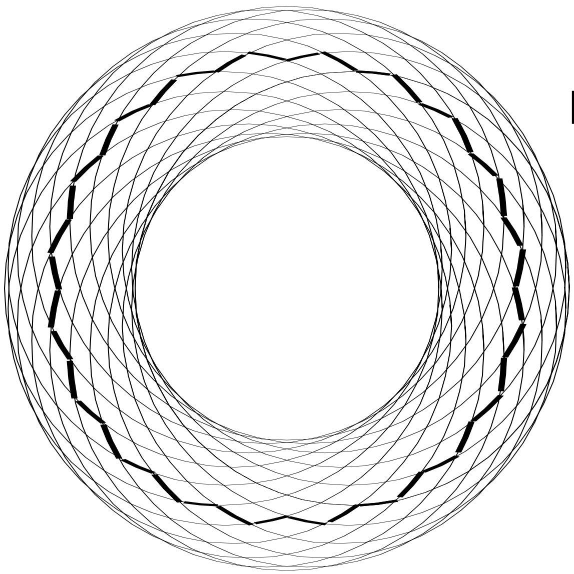
$$\rho_{\text{scalar}} = \rho_{\text{Dirac}} + \frac{1}{2}\partial_\mu(\mu\,\rho_{\text{Dirac}}) \rightarrow \tilde{C}_{\text{scalar}}(r) = \tilde{C}_{\text{Dirac}}(r) - \frac{1}{2}\, r\,\partial_r\, \tilde{C}_{\text{Dirac}}(r)$$

The entropic c-functions do not satisfy this relation!

H. Casini and M.H., 2012

Generalization: Three dimensional case

Starting point → { Sets → 
SSA   Two is not enough! }



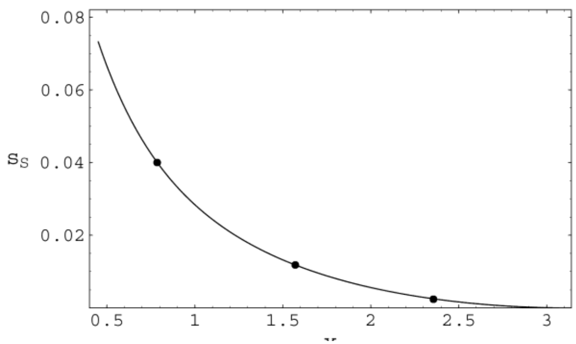
Planar construction { Circles rotated an angle 2πk/N with k = 1, 2, ..N around a point different from the center.
In the infinite N limit, the sets look like circles centered at the same point… }

$$
\begin{aligned}
& S(A)+S(B)+S(C) \ \geq \\
\geq \ & S(A\cap B)+S(A\cup B)+S(C) \\
\geq \ & S(A\cup B\cup C)+S((A\cup B)\cap C)+S(A\cap B) \\
\geq \ & S(A\cup B\cup C)+S(((A\cup B)\cap C)\cup(A\cap B))+S(A\cap B\cap C) \\
= \ & S(A\cup B\cup C)+S((A\cap C)\cup(A\cap B)\cup(B\cap C))+S(A\cap B\cap C)
\end{aligned}
$$

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i\cap X_j)) + S(\cup_{\{ijk\}}(X_i\cap X_j\cap X_k)) + \ldots + S(\cap_i X_i)$$

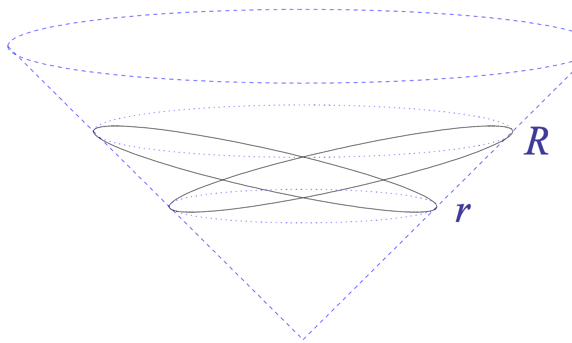
Equal number of regions on both sides of the inequality
Regions on right hand side ordered by inclusion
Totally symmetrical with respect to permutation of the regions

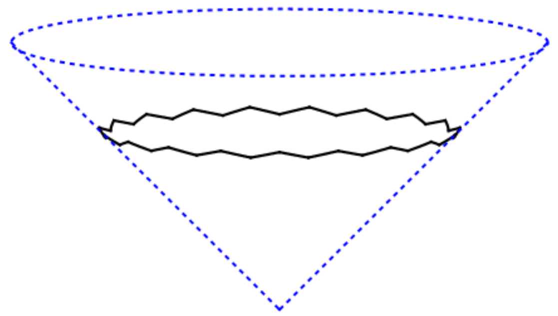
Coefficient of the logarithmically divergent term for a free scalar field.



Motivations: Holographic c-theorems (Myers and Sinha, 2010);
F-Theorem (Jafferis, Klevanov, Pufu and Safdi, 2011);
Renormalized EE (Liu and Mezei , 2012)…

N rotated circles on the light cone





As we approach the light cone the angles go to π
and the perimeters of the wiggled regions approach
the ones of circles of the same radius

Log divergent terms cannot appear for «angles» on a null plane since the feature does not have any local geometric measure

$$N\, S(\sqrt{Rr}) \geq \sum_{i=1}^{N} \tilde{S}\left(\frac{2rR}{R+r-(R-r)\cos(\frac{\pi i}{N})}\right)$$

Infinite N limit

$$S(\sqrt{Rr}) \geq \frac{1}{\pi}\int_0^{\pi} dz\, S\left(\frac{2rR}{R+r-(R-r)\cos(z)}\right)$$

$$S'' \leq 0$$

Infinitesimal inequality $S'' \leq 0$

Running of the constant term

Interpolating function $c_0 = r\, S'(r) - S(r)$

$$c_0' \leq 0$$

Dimensionless and decreasing
C-function proposed by H.Liu and M. Mezei (2012)
Based on holografic and QFT analysis

At fixed points $S(r) = c_1 r - c_0$ and $c_0(r) = c_0$ (coincides with c_0 at fixed point)

$$\Delta c_0 = c_0^{uv} - c_0^{ir} = -\int_0^\infty dr\; r\; S'' \geq 0$$

c_0 is dimensionless decreasing from UV to IR ⟶ "ordering of theories"

Do we have already a c-theorem?

- A dimensionless quantity independent of regularization.
- At the infrared fixed point must not depend on the UV.

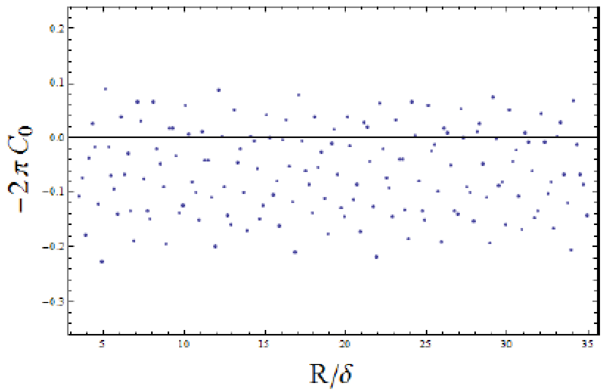
$S(R) = c_1 R - c_0$ ⟶ R has an ambiguity in its definition (cutoff dependent)

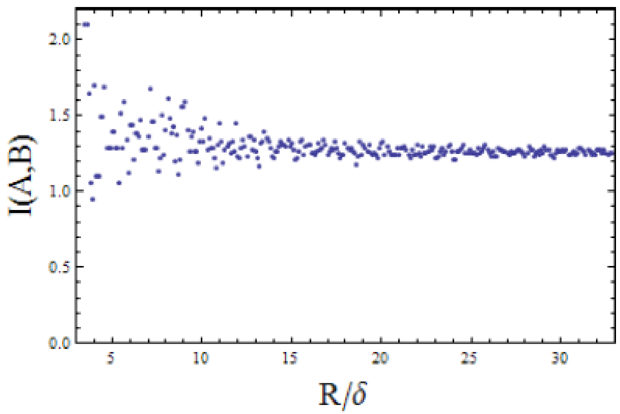
$c_1 \to \frac{a}{\epsilon} + b$ «contamination» from UV to IR

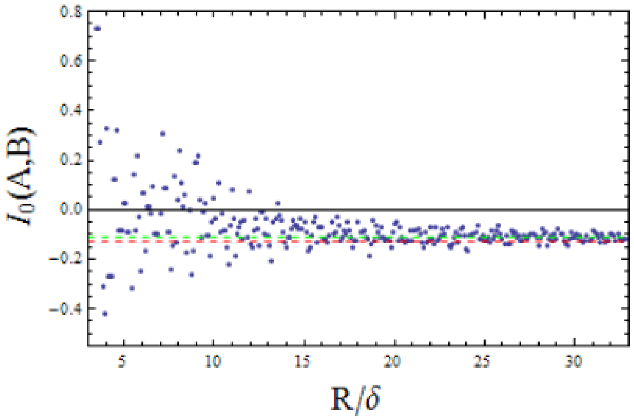
• Example in a lattice:
$R = n\epsilon$ or $R = (n+1)\epsilon$

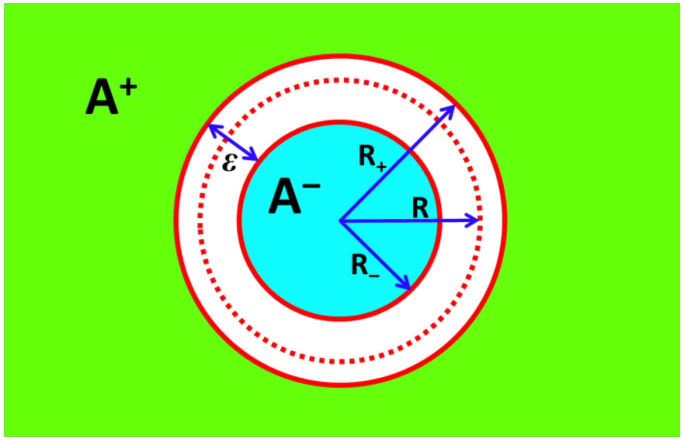
Subtleties in defining at fixed points: Mutual information clarify this issue

H.Casini., M. H., R.C. Myers, A. Yale









$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

The local divergences cancel in I(A,B) which is finite and well defined in QFT

Mutual information as a geometric regulator for EE: all coefficients on the expansion are universal and well defined

$$S(r) = \left(\frac{a}{\delta} + b\right) r + c_0 \longrightarrow I(A^+, A^-) = 2\left(\frac{\tilde{a}}{\epsilon} + \tilde{b}\right) R + 2C_0$$

Locality+symmetry argument

C charge well defined through mutual information
IR , UV values depend only on the CFT
This is a physical quantity calculable with any regularization, including lattice
The constant term coincides with the one in the entropy of a circle for «good enough» regularizations

C-theorem in more dimensions?

Even dimensions

Coefficient of the Euler density term in the trace anomaly at the fixed point, Cardy (1988).

$$\langle \Theta(x) \rangle = \frac{(-1)^{d/2}}{2} a_d \, E(x) + \text{other polynomials of order d/2 in the curvatura tensor}$$

D=4: Proved by Komargodski and Schwimmer (2011) (a-theorem) using the effective action for the dilaton coupled to the theory.

Related to log. div. term in Log Z on a d-sphere

Odd dimensions? No trace anomaly in odd dimensions

Myers-Sinha (2010) Holographic c-theorems

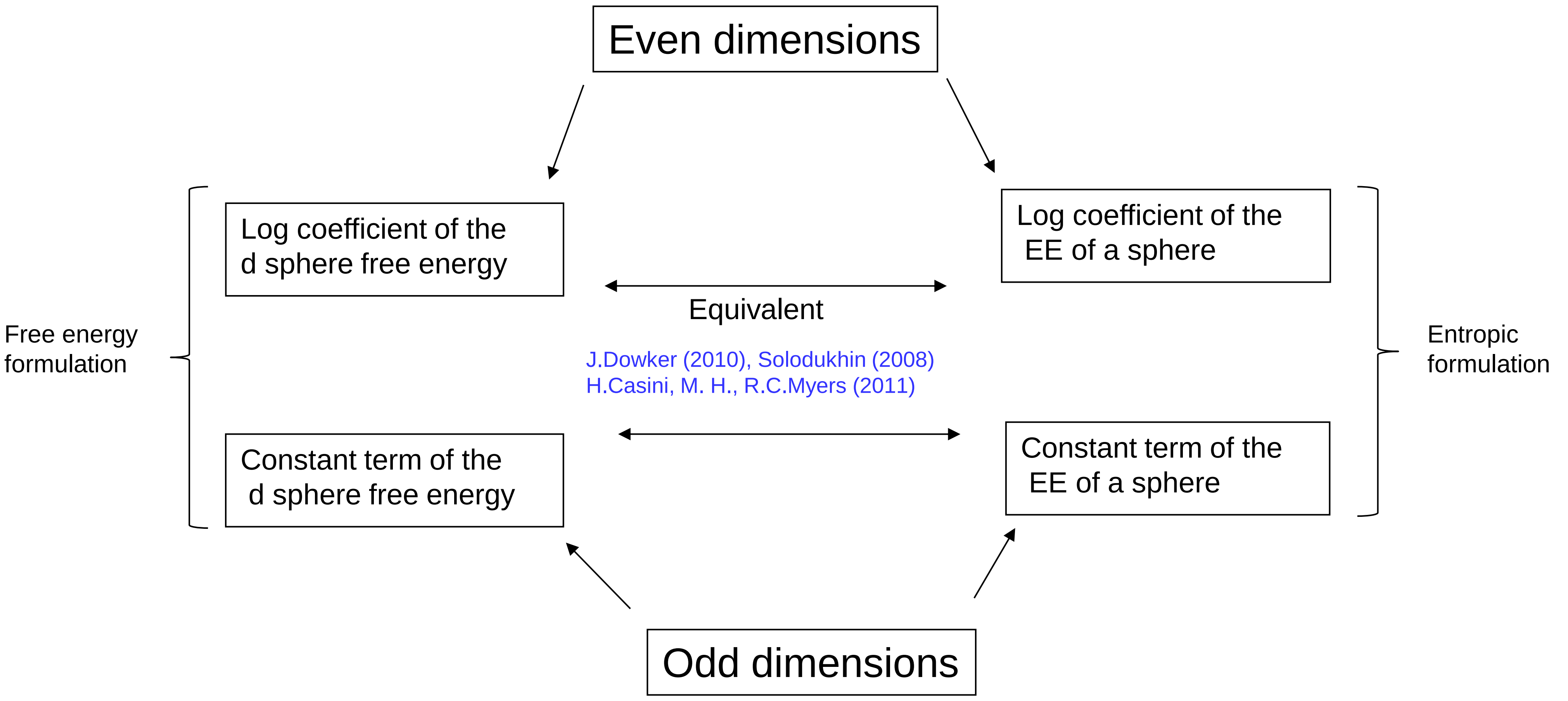
$$a^*_{uv} \geq a^*_{ir}$$

QFT interpretation: For even spacetime dimensions a^* is the coefficient of the Euler term in the trace anomaly (coincides with Cardy proposal for the c-theorem). This is the log coeff. in the EE of the sphere.

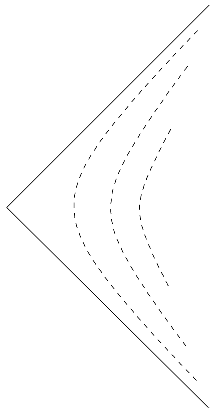
For odd dimensions the constant term of the sphere entanglement entropy is proportional to a^*

F-theorem (Jafferis, Klebanov, Pufu, Safdi (2011)): propose finite term in the free energy F= –log(Z) of a three sphere decreases between fixed points under RG.
Non trivial tests for supersymmetric and non-susy theories .

Relation between EE of spatial (d-2)-sphere / partition function on euclidean d-sphere

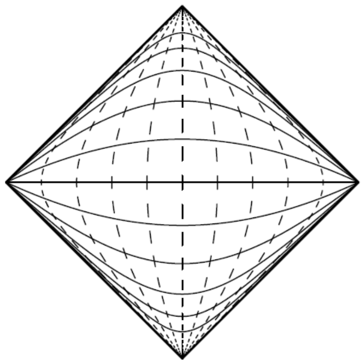


H.Casini, M.H., R.C.Myers (2011)



Conformal mapping

$\longleftrightarrow$



S=entanglement entropy

Causal domain of dependence of a sphere in Minkowski

Rindler Wedge
Unruh temperature
(any QFT)

$$\rho \sim e^{2\pi K}$$

Conformal mapping

$\updownarrow$

Static patch in de Sitter space $ds^2 = -\left(1-\frac{\hat{r}^2}{R^2}\right) d\tau^2 + \frac{d\hat{r}^2}{1-\frac{\hat{r}^2}{R^2}} + \hat{r}^2\, d\Omega_{d-2}^2$ $\longrightarrow$ three sphere

T=1/ (2 π R) and $S = \beta E + \log(Z) = \log(Z)$ (no anomaly)

Hence Myers-Sinha and F-theorem (Jafferis,Klebanov,Pufu,Sadfi) proposals are the same. It extends Cardy proposal to odd dimensions

Are odd and even results connected?
Is the choice of the constant term natural in odd dimensions?
Dimensionaly continued c-theorem: some numbers

- Normalize the c-charge to the scalar c-charge in any dimension. For the Dirac field we have for the ratio of c-charge to number of field degrees freedom

d	2	4	6	8	10	12	14
$\frac{c[\text{Dirac}]}{2^{d/2}c[\text{scalar}]}$	$\frac{1}{2}$	$\frac{11}{4}$	$\frac{191}{40}$	$\frac{2497}{368}$	$\frac{73985}{8416}$	$\frac{92427157}{8562368}$	$\frac{257184319}{20097152}$
approx.	0.5	2.75	4.775	6.7853	8.7909	10.7946	12.7971

Fitting as $$\frac{C[\text{Dirac}]}{2^{d/2}C[\text{scalar}]} = (d-2) + k_0 + \frac{k_1}{d} + \frac{k_2}{d^2} + \ldots$$

- Fitting with 100 dimensions gives for d=3 $$\frac{C[\text{Dirac}]}{2^{[d/2]}C[\text{scalar}]} \to 1.7157936606$$

- The correct value $$\frac{C[\text{Dirac}]}{2^{[d/2]}C[\text{scalar}]} = \frac{\frac{\log(2)}{4} - \frac{3\zeta(3)}{8\pi^2}}{\frac{\log(2)}{4} + \frac{3\zeta(3)}{8\pi^2}} = 1.71579366494\ldots$$

Reason? The ratios of free energies on the sphere in zeta regularization

$$F = -\frac{1}{2}\lim_{s\to 0}\left[\mu^{2s}\zeta'(s) + \zeta(s)\log(\mu^2)\right]$$

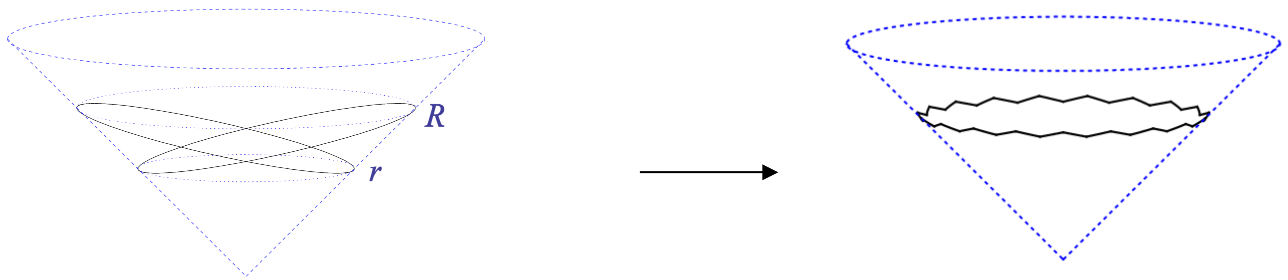
$$\lim_{s\to 0}\frac{\zeta^1(s)}{\zeta^2(s)} = \begin{cases} \dfrac{\zeta^1(0)}{\zeta^2(0)} & \text{even dimensions} \\[2ex] \dfrac{\zeta^{1\,\prime}(0)}{\zeta^{2\,\prime}(0)} & \text{odd dimensions} \end{cases}$$

The same could be expected for the ratios of the entropies of spheres (taking out the most divergent terms)

"Interpolating between a and F"
Giombi,Klebanov,2015

Entropic proof in more dimensions?

Symmetric configuration of boosted spheres in the limit of large number of spheres



Divergent terms do not cancel, trihedral angles, curved dihedral angles. Wiggly spheres not converge to smooth spheres: mismatch between curvatures.

More generally:

a) Strong subadditivity always gives inequalities for second derivatives

b) This inequality should give C'<0. Then C has to be constructed with S and S'

c) C has to be cutoff independent. But at fixed points

$$S(r) = c_2 \frac{r^2}{\epsilon^2} + c_{\log} \log(r/\epsilon) + c_0$$

It is not possible to extract the coeff. of the logarithmic term with S and S'.

New inequalities for the entropy?

Final comments...

Is C a measure of «number of field degrees of freedom»?

In d=3, C is not an anomaly but a small universal term in a divergent entanglement entropy:

"number of field degrees of freedom" vs. "measure of entanglement that is lost under renormalization"

Is there some loss of information interpretation?

Even if the theorem applies to an entropic quantity, there is no known interpretation in terms of some loss of information. Understanding this could tell us whether there is a version of the theorem that extends beyond relativistic theories.

More inequalities seem to be needed for an entropic c-theorem in higher dimensions.