The Yang-Mills gradient flow — a new tool in non-perturbative QFT

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Introduction

The YM gradient flow offers interesting new ways of studying gauge theories in the framework of the Euclidean functional integral

- ⋆ Conceptually inspiring
- ⋆ Any gauge group and matter content
- ⋆ Many uses, so far mostly in (lattice) QCD

Classical flow equation

Consider the "flow of fields" $B_\mu(t,x)$, $t \geq 0$, defined by

$$B_\mu\big|_{t=0} = A_\mu$$

$$\partial_t B_\mu = D_\nu G_{\nu\mu}, \qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

Atiyah & Bott '82, . . . [Morse theory of field space]

$D_\nu G_{\nu\mu} =$ gradient of the YM action

$\Rightarrow$ B_μ tends to converge to a local minimum of the action

YM gradient flow in QFT

Need a UV regularization (dimensional, lattice)

$\Rightarrow$ B_μ becomes a well-defined function(al) of A_μ at all $t \geq 0$

$\Rightarrow$ May consider correlation functions $\langle \mathcal{O}_1(t_1, x_1) \ldots \mathcal{O}_n(t_n, x_n) \rangle$ of local observables such as

$$G^a_{\mu\nu} G^a_{\mu\nu} \quad \text{or} \quad G^a_{\mu\nu} {}^*G^a_{\mu\nu}$$

and remove the regularization after renormalization

Perturbation theory

At leading order

$$\underbrace{\partial_t B_\mu = \Delta B_\mu}_{\text{heat equation}} - \partial_\mu \partial_\nu B_\nu$$

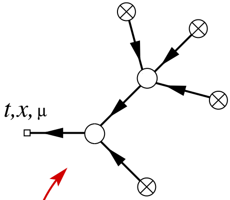
$$\Rightarrow \; B_\mu(t,x) = \int_y K_t(x-y) A_\mu(y) + \text{gauge \& non-linear terms}$$

i.e. $B_\mu = A_\mu$ smoothed in space by a Gaussian with range $\sqrt{8t}$

$$\langle B_{\mu_1}(t_1 x_1) \ldots B_{\mu_n}(t_n, x_n) \rangle =$$

$$g_0^n \int_{y_1,\ldots,y_n} K_{t_1}(x_1 - y_1) \ldots K_{t_n}(x_n - y_n) \underset{\substack{\uparrow \\ \text{gluon } n\text{-point function}}}{G_0(y_1,\ldots,y_n)_{\mu_1 \ldots \mu_n}} + \ldots$$

Perturbation theory



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↑ gluon n-point function

Example

Consider QCD with $N_{\rm f}$ massless quarks

$$\langle G^a_{\mu\nu} G^a_{\mu\nu} \rangle = \frac{3}{\pi t^2} \alpha_s(q) \bigl\{1 + k_1 \alpha_s(q) + k_2 \alpha_s(q)^2 + \ldots\bigr\}, \qquad q = 1/\sqrt{8t}$$

All UV-divergences cancel when the coupling is renormalized!

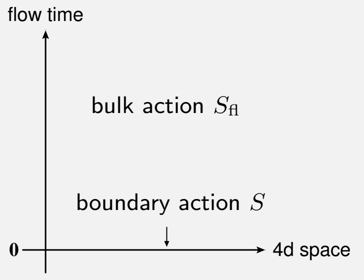
$$k_1 = 1.098 + 0.008\, N_{\rm f}$$ ML '10

$$k_2 = -0.982 - 0.070\, N_{\rm f} + 0.002\, N_{\rm f}^2$$ Harlander & Neumann '16

Equivalent 5d field theory

$$S_{\rm fl} = \int_0^\infty {\rm d}t \int {\rm d}^4x\, L_\mu^a \left\{ \partial_t B_\mu^a - D_\nu G_{\nu\mu}^a \right\}$$

↑ Lagrange multiplier



- Well-behaved local gauge theory
- Can be put on a 5d lattice
- Power-counting renormalizable

Zinn-Justin & Zwanziger '88

Non-renormalization theorem

Local gauge-invariant fields do not require renormalization at $t > 0$
and their correlation functions have no short-distance singularities

- Holds to all orders in perturbation theory
- Ample numerical evidence at the non-perturbative level

ML '10, ML & Weisz '11, Hieda et al. '16, . . .

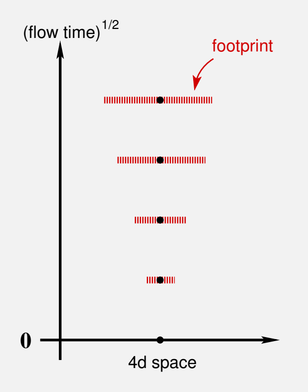
Small t expansion

Let $\mathcal{O}(t,x)$ be a local gauge-invariant field

$$\Rightarrow \quad \mathcal{O}(t,x) \underset{t\to 0}{\sim} \sum_k c_k(t)\phi_k(x)$$

$\phi_k(x)$: renormalized local fields at $t=0$

$c_k(t) \propto t^{\frac{1}{2}(d_k - d_{\mathcal{O}})} \times$ logarithms



May be used to easily construct renormalized fields at $t=0$

Example

Energy-momentum tensor in the pure YM theory

$$T_{\mu\nu} = \frac{1}{4\pi\alpha_s}\{1 + \mathrm{O}(\alpha_s)\}\{G^a_{\mu\rho}G^a_{\nu\rho} - \tfrac{1}{4}\delta_{\mu\nu}G^a_{\rho\sigma}G^a_{\rho\sigma}\}$$

$$+ \frac{1}{8}\{b_0 + \mathrm{O}(\alpha_s)\}\delta_{\mu\nu}\{G^a_{\rho\sigma}G^a_{\rho\sigma} - \langle G^a_{\rho\sigma}G^a_{\rho\sigma}\rangle\} + \mathrm{O}(t)$$

$b_0 = 1^{\mathrm{st}}$ coefficient of the β-function, α_s at scale $q = 1/\sqrt{8t}$

Suzuki '13, Makino & Suzuki '14

- ⋆ Do not need to deal with UV-divergent quantities
- ⋆ Statistical variances tend to be small *since they are UV-finite too!*

Application: Compute energy, pressure and entropy

$$e = -\langle T_{00} \rangle, \qquad p = \langle T_{11} \rangle, \qquad s = (e+p)/T$$

in lattice QCD as a function of temperature T

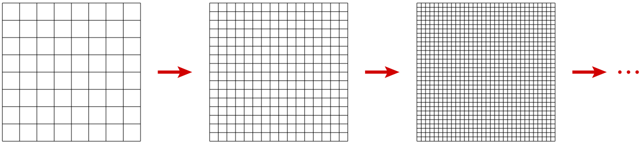
Asakawa et al. '14, Taniguchi et al. '16

Other applications of the YM gradient flow

- ⋆ Non-perturbative renormalization
  - ◇ Scale setting
  - ◇ Discrete RG ("step scaling")
- ⋆ Understanding topology in QCD
- ⋆ Chiral gauge theories beyond perturbation theory

Scale setting in lattice QCD

Extrapolations to the continuum limit . . .



require knowing the lattice spacings in units of some reference length l_{ref}

l_{ref} should

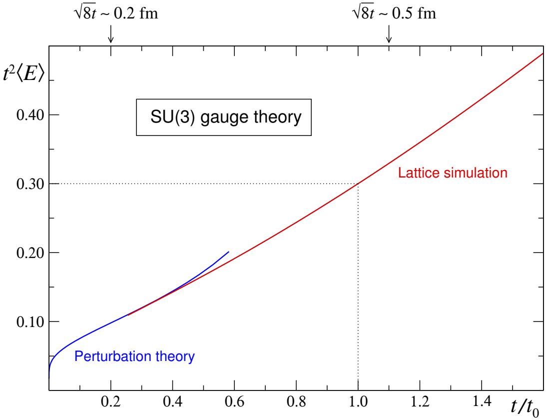
- ⋆ *be characteristic of the low-energy physics*
- ⋆ *have small statistical variance*
- ⋆ *not be affected by systematic errors*

The reference flow time t_0

$$E = \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu}$$

$$\left\{t^2 \langle E \rangle\right\}_{t=t_0} = 0.3$$

has all these properties



In (continuum) QCD with physical u, d, s quarks

$$l_{\text{ref}} \equiv \sqrt{8t_0} = 0.414(4)(1)\,\text{fm}$$

Bruno, Korzec & Schaefer ’16

Understanding topology in QCD

The fields in 4d (Euclidean) QFT are typically nowhere continuous!

Colella & Lanford '73

Does the field space in QCD divide into topological (instanton) sectors, or is this merely a feature of the semi-classical approximation?

Related issues

- Divergent UV fluctuations
- Definition of $\langle Q^n \rangle$ is ambiguous
- The space of lattice gauge fields is connected

Topological charge at $t > 0$

The moments

$$\langle Q^n \rangle = \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n \langle q(t,x_1) \ldots q(t,x_n) \rangle, \qquad q = \frac{1}{32\pi^2} G^a_{\mu\nu}{}^*G^a_{\mu\nu}$$

are well defined and do not require renormalization

Moreover, since

$$\partial_t q = \frac{1}{8\pi^2} \partial_\mu \{ \underbrace{\partial_t B^a_\nu {}^*G^a_{\mu\nu}}_{\text{gauge invariant}} \}$$

partial integration shows that $\langle Q^n \rangle$ **is independent of t!**

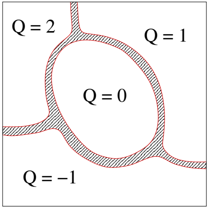
After a lot of work

- ⋆ These are the moments that appear in the chiral WIs
- ⋆ And thus in the Witten–Veneziano formula

$$m_{\eta'}^2 = \frac{2N_\mathrm{f}}{F_\pi^2 V}\,\langle Q^2\rangle\big|_{N_\mathrm{f}=0} + \mathrm{O}(1/N_\mathrm{c})$$

- ⋆ On the lattice, the topological sectors emerge <u>dynamically</u> as $a \to 0$

suppressed like $\sim a^6$ ⟶



field space

ML '81, '04, '10
Ginsparg & Wilson '82
Hasenfratz et al. '98
Giusti et al. '02, '04
Seiler '02
Cè et al. '15
...

Chiral lattice gauge theories?

Consider a theory with a multiplet of left-handed fermions

$$S = S_{\mathrm{G}} + \int_x \overline{\psi} P_+ \not{D} P_- \psi, \qquad P_\pm = \tfrac{1}{2}(1 \pm \gamma_5)$$

$$D_\mu = \partial_\mu + R(A_\mu)$$

Would like a lattice formulation that

- *respects locality and the gauge symmetry*
- *and which does not require tuned counterterms*

New proposal

Add mirror fermions at flow time $t > 0$

$$S_{\mathrm{F}} = \int_x \overline{\psi} \not{D} \psi, \qquad D_\mu = \partial_\mu + P_- R(A_\mu) + P_+ R(B_\mu)$$

and take $t \to \infty$

Grabowska & Kaplan '16

$\not{D}$ is elliptic and gauge covariant

$\Rightarrow$ $\det \not{D}$ is well defined & gauge invariant

$\Rightarrow$ Lattice formulation using domain-wall or Ginsparg–Wilson fermions

Do the mirror fermions decouple as $t \to \infty$?

In (infinite-volume) perturbation theory

- $\ln\det \not{D} = -S_{\rm eff}(A) - S_{\rm eff}(B)^*$ iff R is anomaly-free
- The effects of $S_{\rm eff}(B)$ probably go away like t^{-p}

On the lattice

- Factorization up to $\mathrm{O}(a)$ terms
- B-field becomes weak
- Decoupling still to be shown