DESY Theory Workshop

Strings & Mathematical Physics Session

Z-theory – non-linear sigma model amplitudes and more from open strings

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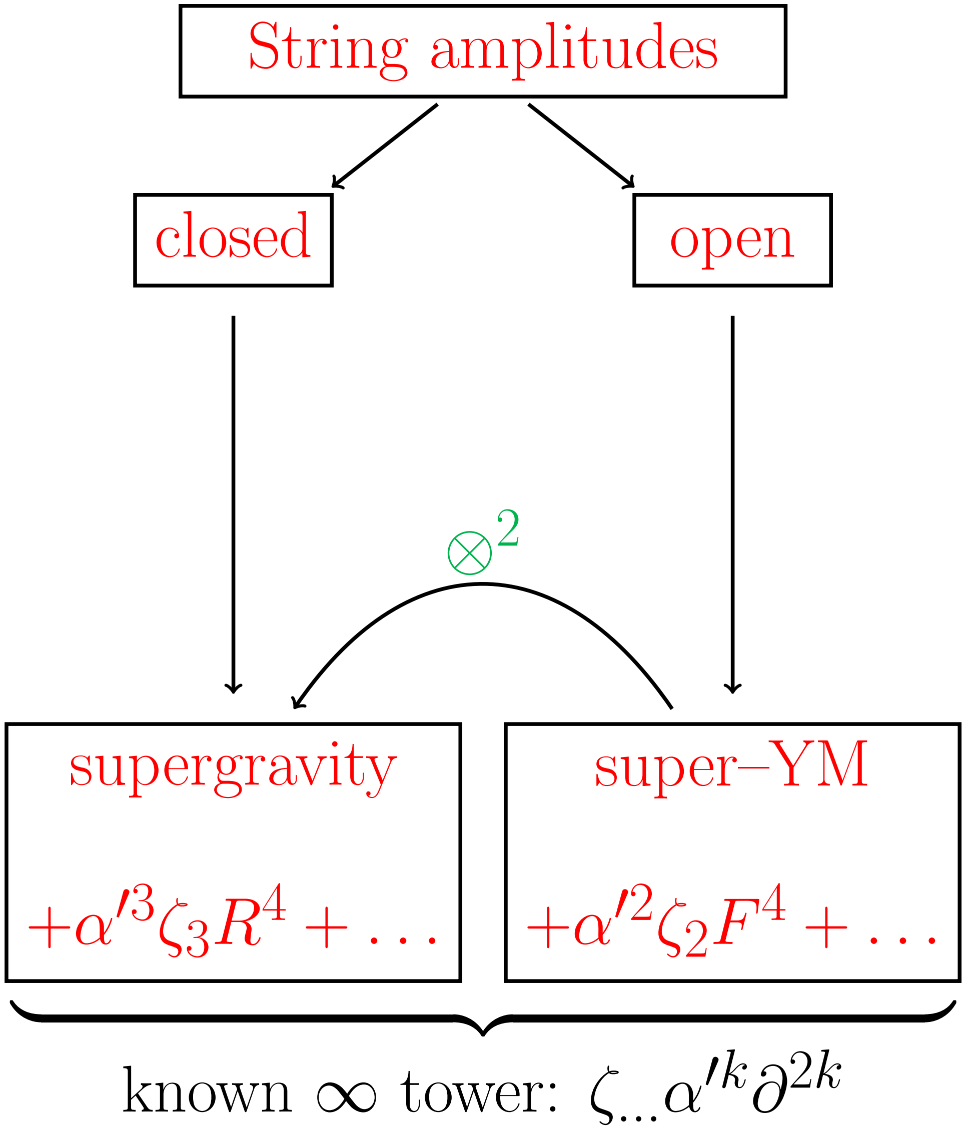
based on arXiv:1608.02569 & 1609.07078 with J.J. Carrasco and C. Mafra

29.09.2016

Roadmap: strings @ low energy (tree-level)



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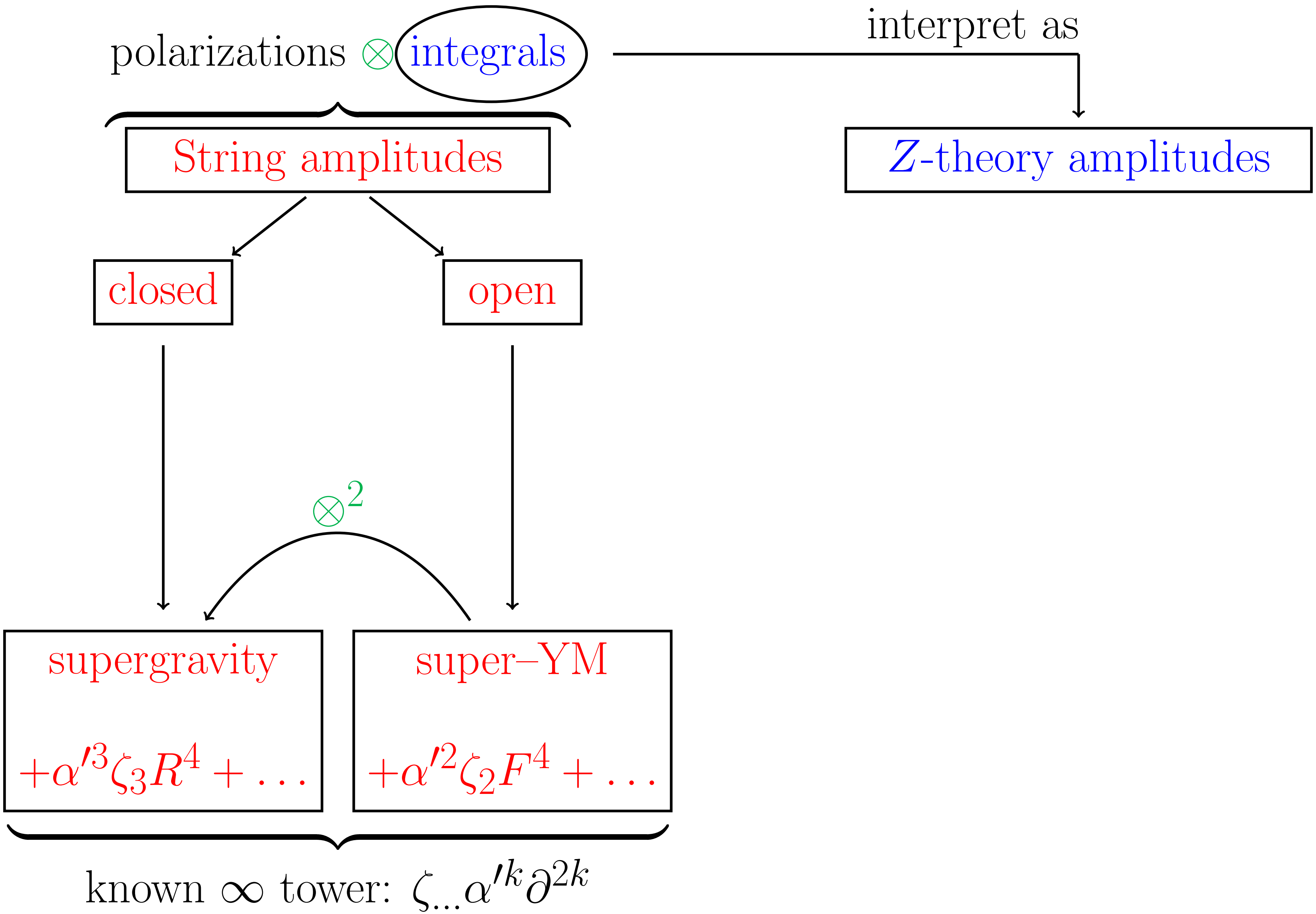

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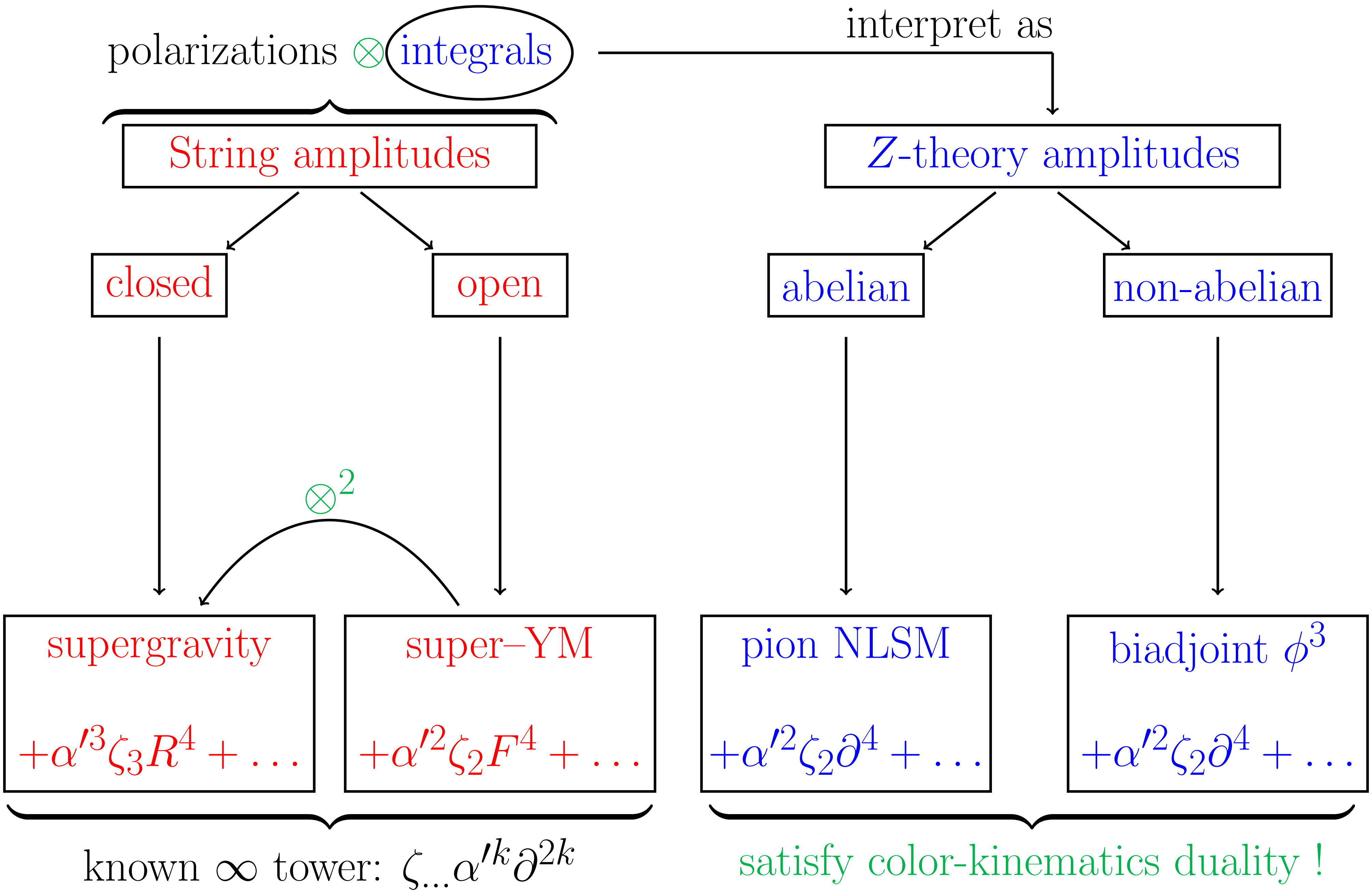


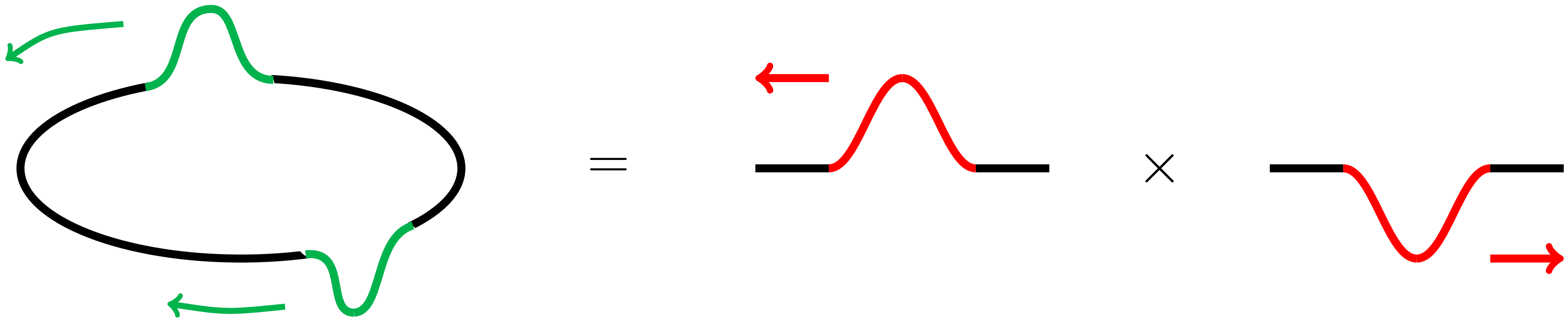
Motivation: double copy QFTs

gravity = (gauge theory)^2 manifested by KLT formula for tree amplitudes

$$M_4^{\rm grav} = A^{\rm YM}(1,2,3,4)\, s_{12}\, \tilde{A}^{\rm YM}(1,2,4,3)$$

partial amplitude $\leftrightarrow$ $\text{Tr}(t^1 t^2 t^3 t^4)$ Mandelstam invariant $s_{12\ldots p} \equiv \sum_{i<j}^{p} k_i \cdot k_j$

- descends from string theory as $\alpha' \to 0$ [Kawai, Lewellen, Tye 1986]



- extension to loop amplitudes: squaring at the level of cubic diagrams

[Bern, Carrasco, Johansson 2008 / 2010; talk of Henrik Johansson]

Motivation: double copy QFTs

n-point KLT formula with $(n-3)!^{\otimes 2}$ matrix $S[\cdot|\cdot]$ with entries $\sim s_{ij}^{n-3}$

$$M_n^{\text{grav}} = \sum_{\rho,\sigma\in S_{n-3}} A^{\text{YM}}(1,\rho,n{-}1,n)\, S[\rho|\sigma]\, \tilde{A}^{\text{YM}}(1,\sigma,n,n{-}1)$$

[Bern et al. 1998; Bjerrum-Bohr et al. 2010]

Consistency requires BCJ relations among $A^{\text{YM}}(\ldots)$ & $\tilde{A}^{\text{YM}}(\ldots)$

$$\sum_{j=2}^{n-1} k_1\cdot(k_2{+}k_3{+}\ldots{+}k_j)\, A^{\text{YM}}(2,3,\ldots,j,1,j{+}1,\ldots,n) = 0$$

[Bern, Carrasco, Johansson 2008; talk of Henrik Johansson]

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Will suppress $S[\cdot|\cdot]$ & permutations ρ,σ of $2,3,\dots,n{-}2$ and write

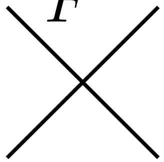
$$M_n^{\text{grav}} = A^{\text{YM}}(\dots) \otimes_{\text{KLT}} \tilde{A}^{\text{YM}}(\dots)$$

$\longrightarrow$ fingerprints of double copy in the tree-level S-matrix !

Born–Infeld = gauge theory $\otimes_{\text{KLT}}$ NLSM

Born–Infeld theory (BI) of **abelian** gauge bosons

$$\mathcal{L}_{\text{BI}} = \frac{1}{\alpha'^2}\left\{\sqrt{\det(\eta_{\mu\nu} - \pi\alpha' F_{\mu\nu})} - 1\right\} =$$



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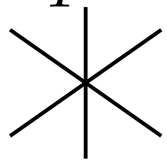


$+ \cdots$

$\alpha' \to 0$ limit of open superstrings **[Metsaev, Rakhmanov, Tseytlin 1987]**

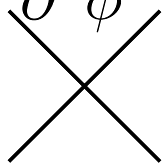
In spite of open-string origin, $\exists$ KLT formula for BI-trees

$$M_n^{\text{BI}} = A^{\text{YM}}(\ldots) \otimes_{\text{KLT}} A^{\text{NLSM}}(\ldots)$$

[Cachazo, He Yuan 2014; talk of Yvonne Geyer]

2nd double-copy component = $SU(N)$ non-linear sigma model (NLSM)

$$\mathcal{L}_{\text{NLSM}} = -\frac{1}{2}\,\text{Tr}\left\{\partial_\mu\phi\,\frac{1}{1-\lambda^2\phi^2}\,\partial^\mu\phi\,\frac{1}{1-\lambda^2\phi^2}\right\} =$$



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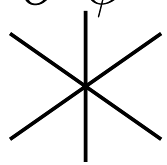

$+ \cdots$

KLT for open (!) superstrings $\Rightarrow$ Z-theory

n-point superstring tree amplitude of **abelian** gauge bosons

$$M_n^{\text{open}}(\alpha') = A^{\text{YM}}(\ldots) \otimes_{\text{KLT}} Z_\times(\ldots ; \alpha')$$

[Mafra, OS, Stieberger 2011; Brödel, OS, Stieberger 2013]

with disk integrals $Z_\times(\ldots ; \alpha')$

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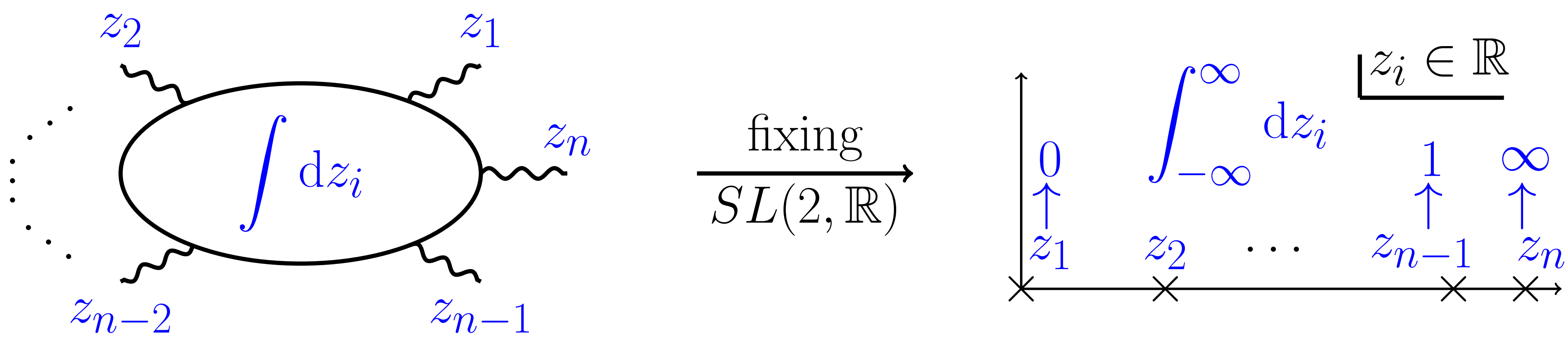
with disk integrals $(z_{ij} \equiv z_i - z_j)$ $\longleftrightarrow$ abelian Z-theory amplitudes

$$Z_\times(b_1,b_2,\ldots,b_n;\alpha') \equiv \alpha'^{n-3} \int_{\mathbb{R}^n} \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\ldots\,\mathrm{d}z_n}{\mathrm{vol}\;SL(2,\mathbb{R})} \frac{\prod_{i<j}^n |z_{ij}|^{\alpha' s_{ij}}}{z_{b_1,b_2}\,z_{b_2,b_3}\,\ldots\,z_{b_n,b_1}}$$



Must recover Born-Infeld KLT in the low-energy limit $\alpha' \to 0$

$$
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\downarrow & & & & \downarrow \\
M_n^{\text{BI}} & = & A^{\text{YM}}(\ldots) & \otimes_{\text{KLT}} & A^{\text{NLSM}}(\ldots)
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Hence, NLSM amplitudes $\overset{!}{=}$ low-energy limit of disk integrals $Z_\times(\ldots)$,

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\boxed{A^{\text{NLSM}}(b_1,b_2,\ldots,b_n) = \lim_{\alpha'\to 0} \frac{1}{\alpha'} \int_{\mathbb{R}^n} \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\ldots\,\mathrm{d}z_n}{\text{vol}\; SL(2,\mathbb{R})} \frac{\prod_{i<j}^n |z_{ij}|^{\alpha' s_{ij}}}{z_{b_1,b_2}\, z_{b_2,b_3}\,\ldots\, z_{b_n,b_1}}}
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[Carrasco, Mafra, OS 1608.02569]

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[Carrasco, Mafra, OS 1608.02569]

BCJ relations among $A^{\text{NLSM}}(\ldots)$ **[Chen Du 1311.1133]** ...

... descend from integration by parts relations among disk integrals

$$\sum_{j=2}^{n-1} k_1\cdot(k_2{+}k_3{+}\ldots{+}k_j)\,Z_\times(2,3,\ldots,j,1,j{+}1,\ldots,n) = 0$$

[Brödel, OS, Stieberger 2013]

Examples

α'-expansion of disk integrals $\Rightarrow$ multiple zeta values (MZVs)

$$\zeta_{n_1,n_2,\ldots,n_r} \equiv \sum_{0<k_1<\ldots<k_r} k_1^{-n_1}\, k_2^{-n_2} \ldots k_r^{-n_r} \,, \qquad n_r \leq 2$$

At four points, only Riemann zeta values ζ_n (e.g. $\zeta_2 = \frac{\pi^2}{6}$)

$$Z_\times(1,2,3,4) = 2\, \frac{\big[\sin(\pi\alpha' s_{12}) + \text{cyc}(1,2,3)\big]}{\pi\alpha'\, s_{12}s_{23}} \exp\left(\sum_{k=2}^{\infty} \frac{\zeta_k}{k} (-\alpha')^k \sum_{i<j}^{3} s_{ij}^k \right)$$

$$= \alpha'^2 \pi^2 \underbrace{(s_{12}+s_{23})}_{A^{\rm NLSM}(1,2,3,4)} \left\{ 1 + \frac{\alpha'^2 \zeta_2}{4} \sum_{i<j}^{3} s_{ij}^2 - \alpha'^3 \zeta_3 s_{12}s_{13}s_{23} + \mathcal{O}(\alpha'^4) \right\}$$

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At n-points, get α'-expansion & MZVs from Drinfeld associator, e.g.

[Brödel, OS, Stieberger, Terasoma 2013]

$$Z_\times(1,2,3,4,5,6) = \alpha'^4 \pi^4 \underbrace{\left(s_{12} - \frac{(s_{12}+s_{23})(s_{45}+s_{56})}{2\, s_{123}} + \text{cyc}(1,2,3,4,5,6) \right)}_{A^{\text{NLSM}}(1,2,3,4,5,6)} + \mathcal{O}(\alpha'^6)$$

Z-theory = factory for BCJ-satisfying amplitudes

α'-expansion of abelian Z-theory amplitudes,

$$Z_\times(1,2,\ldots,n) = (\pi\alpha')^{n-2}\Big\{A^{\rm NLSM}(1,2,\ldots,n) + \alpha'^2\zeta_2 A^{\partial^4{\rm NLSM}}(1,2,\ldots,n) + \sum_{w=3}^{\infty}\alpha'^w\zeta_w A^{\partial^{2w}{\rm NLSM}}(1,2,\ldots,n) + \text{MZV-products \& higher depth}\Big\}$$

- BCJ-relations hold separately for any $A^{\partial^{2w}{\rm NLSM}}(\ldots)$
- $A^{\partial^{2w}{\rm NLSM}}(\ldots)$ from eff. Lagrangian $\sim \partial^{2w}{\rm Tr}\big(\partial_\mu\phi\,\frac{1}{1-\lambda^2\phi^2}\,\partial^\mu\phi\,\frac{1}{1-\lambda^2\phi^2}\big)$

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- uniform transcendentality:**[Brödel, Brown, OS, Stieberger, Taylor, Terasoma, ...]**

$$\alpha'^w \quad\longleftrightarrow\quad \zeta_{n_1,n_2,\ldots,n_r} \text{ of weight } n_1+n_2+\ldots+n_r = w$$

- in progress: exact Lagrangian description & efficient expansion methods

Non-abelian Z-theory

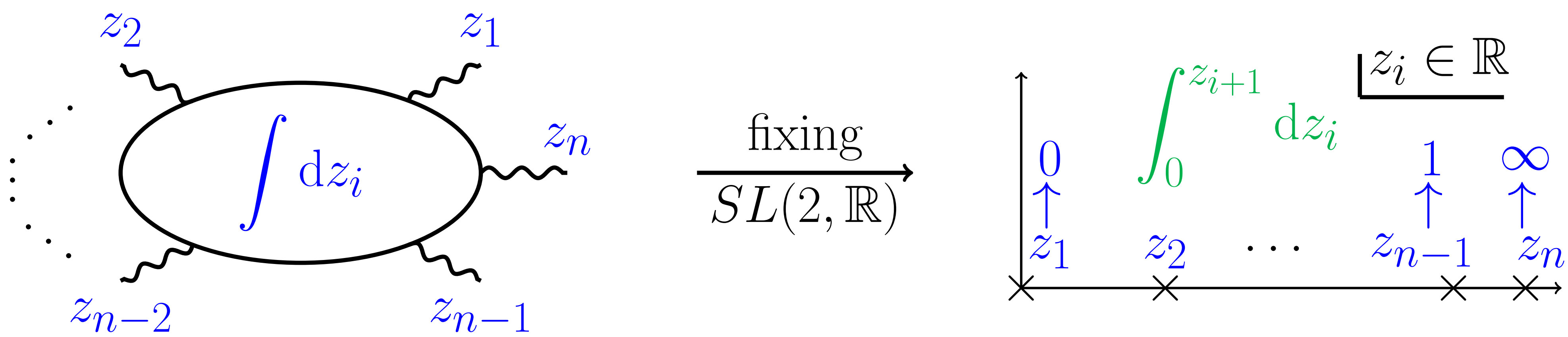
n-point superstring tree amplitude of **non-abelian** gauge bosons

$$\underbrace{A^{\rm open}(1,2,\ldots,n)}_{\leftrightarrow \ {\rm Tr}(t^1 t^2 \ldots t^n)} = A^{\rm YM}(\ldots) \otimes_{\rm KLT} Z(1,2,\ldots,n\,|\,\ldots)$$

[Mafra, OS, Stieberger 2011; Brödel, OS, Stieberger 2013]

with **ordered** disk integrals (domain $D(1,2,\ldots,n)$ imposes $z_i < z_{i+1}$)

$$Z(1,2,\ldots,n\,|\,b_1,\ldots,b_n) \equiv \alpha'^{n-3} \int_{D(1,2,\ldots,n)} \frac{{\rm d}z_1\,{\rm d}z_2\,\ldots\,{\rm d}z_n}{{\rm vol}\ SL(2,\mathbb{R})} \frac{\prod_{i<j}^n |z_{ij}|^{\alpha' s_{ij}}}{z_{b_1,b_2}\, z_{b_2,b_3}\,\cdots\, z_{b_n,b_1}}$$



Interpret disk integrals as Z-theory amplitudes

$$Z(1,2,\ldots,n\,|\,b_1,\ldots,b_n) \equiv \alpha'^{n-3} \int_{D(1,2,\ldots,n)} \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\ldots\,\mathrm{d}z_n}{\mathrm{vol}\; SL(2,\mathbb{R})} \frac{\prod_{i<j}^n |z_{ij}|^{\alpha' s_{ij}}}{z_{b_1,b_2}\, z_{b_2,b_3}\,\cdots\, z_{b_n,b_1}}$$

- still: BCJ relations under permutation of integrand $b_1,\ldots,b_n$ (IBP)
- α'-corrected monodromy relations for permutations in domain $D(1,2,\ldots,n)$

[Bjerrum-Bohr, Damgaard, Vanhove 0907.1425 ; Stieberger 0907.2211]

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- In $\alpha' \to 0$ limit: Z-theory $\to$ biadjoint scalars $\Phi = \Phi_{a|b} t^a \otimes \tilde{t}^b$

$$\mathcal{L}_{\phi^3} = \frac{1}{2}\partial_\mu \Phi_{a|b}\,\partial^\mu \Phi_{a|b} + \frac{1}{3} f^{abc}\,\tilde{f}^{dgh}\,\Phi_{a|d}\,\Phi_{b|g}\,\Phi_{c|h}$$

Reproduce doubly-partial amplitude $\leftrightarrow \mathrm{Tr}(\ldots)\,\tilde{\mathrm{Tr}}(\ldots)$

$$\lim_{\alpha'\to 0} Z(a_1,a_2,\ldots,a_n\,|\,b_1,b_2,\ldots,b_n) = M_n^{\phi^3}\Big|_{\mathrm{Tr}(\tilde{t}^{b_1}\tilde{t}^{b_2}\ldots\tilde{t}^{b_n})}^{\mathrm{Tr}(t^{a_1}t^{a_2}\ldots t^{a_n})}$$

[Cachazo, He Yuan 2013; talk of Yvonne Geyer]

Berends–Giele recursion for biadjoint ϕ^3

Bridge between amplitudes and e.o.m. via Berends–Giele recursion

[YM: Berends, Giele 1987 → ϕ^3: Mafra 2016]

$$\lim_{\alpha'\to 0} Z(a_1,a_2,\ldots,a_n\,|\,b_1,b_2,\ldots,b_n) = \overbrace{s_{a_1a_2\ldots a_{n-1}}}^{\text{amputate propagator}} \underbrace{\phi_{a_1a_2\ldots a_{n-1}|b_1b_2\ldots b_{n-1}}}_{(n-1)\text{-pt BG current}}$$

e.o.m. $\Box\Phi_{a|d} = f^{abc}\tilde{f}^{dgh}\Phi_{b|g}\Phi_{c|h}$ for the generating series "perturbiner"

$$\Phi(x) = \sum_{n=1}^{\infty}\sum_{\substack{a_1,a_2,\ldots,a_n\\ b_1,b_2,\ldots,b_n}} \phi_{a_1a_2\ldots a_n|b_1b_2\ldots b_n} e^{i(k_{b_1}+\ldots+k_{b_n})\cdot x} t^{a_1}t^{a_2}\ldots t^{a_n}\otimes \tilde{t}^{b_1}\tilde{t}^{b_2}\ldots\tilde{t}^{b_n}$$

implies recursion for the currents with initial condition $\phi_{i|j}=\delta_{i,j}e^{ik_j\cdot x}$:

$$\phi_{A|B} = \frac{1}{s_A}\sum_{\substack{PQ=A\\ XY=B}}\left(\phi_{P|X}\phi_{Q|Y} - \phi_{P|Y}\phi_{Q|X}\right)$$

E.g. $\phi_{12|12} = \frac{1}{s_{12}}$ $\Rightarrow$ 3pt amplitude $Z(123|123) = s_{12}\phi_{12|12} = 1$

Z-theory equation of motion

Can extend BG approach to full Z-theory amplitudes including α'

$$\underbrace{Z(a_1, a_2, \ldots, a_n \,|\, b_1, b_2, \ldots, b_n)}_{\text{integral w. known } \alpha'\text{-expansion}} = \overbrace{s_{a_1 a_2 \ldots a_{n-1}}}^{\text{amputate propagator}} \; \underbrace{\phi^{\alpha'}_{a_1 a_2 \ldots a_{n-1} | b_1 b_2 \ldots b_{n-1}}}_{\text{infer } \alpha'\text{-dependent current}}$$

Recursion for currents $\Rightarrow$ e.o.m. for generating series "perturbiner"

$$\Phi(x) = \sum_{n=1}^{\infty} \sum_{A,B} \phi^{\alpha'}_{A|B} e^{i(k_{b_1}+\ldots+k_{b_n})\cdot x} t^{a_1} t^{a_2} \ldots t^{a_n} \otimes \tilde{t}^{b_1} \tilde{t}^{b_2} \ldots \tilde{t}^{b_n}$$

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Schematic results ($t^a,\tilde{t}^b$-dependence & all α'-orders in the reference)

$$\Box\Phi = \Phi^2 \ + \ \alpha'^2\zeta_2\left(\partial^2\Phi^3+\Phi^4\right) \ + \ \alpha'^3\zeta_3\left(\partial^4\Phi^3+\partial^2\Phi^4+\Phi^5\right) \ + \ \mathcal{O}(\alpha'^4)$$

[Mafra, OS 1609.07078]

Extremely efficient expansion methods for $Z(\ldots\,|\,\ldots)$ @ high multiplicity!

Summary

Double copies in field- and string theory include

$$\text{gravity} = \text{YM} \otimes \text{YM}$$

$$\text{Born-Infeld} = \text{YM} \otimes \text{NLSM}$$

$$\text{open string} = \text{YM} \otimes Z\text{-theory}$$

Z-theory = factory for effective scalar field theories with BCJ duality, e.g.

$$\lim_{\alpha' \to 0} (Z\text{-theory}) = \begin{cases} \text{NLSM} & : \text{ abelian (unordered integrals)} \\ \text{biadjoint } \phi^3 & : \text{ non-abelian (ordered integrals)} \end{cases}$$

α'-dependence of Z-theory captured by e.o.m. of schematic form

$$\Box \Phi = \Phi^2 + \alpha'^2 \zeta_2 \left(\partial^2 \Phi^3 + \Phi^4\right) + \alpha'^3 \zeta_3 \left(\partial^4 \Phi^3 + \partial^2 \Phi^4 + \Phi^5\right) + \mathcal{O}(\alpha'^4)$$

[Mafra, OS 1609.07078]

Thank you for your attention !