Quantization of Super Teichmüller Spaces

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based on joint work with Joerg Teschner and Michal Pawelkiewicz

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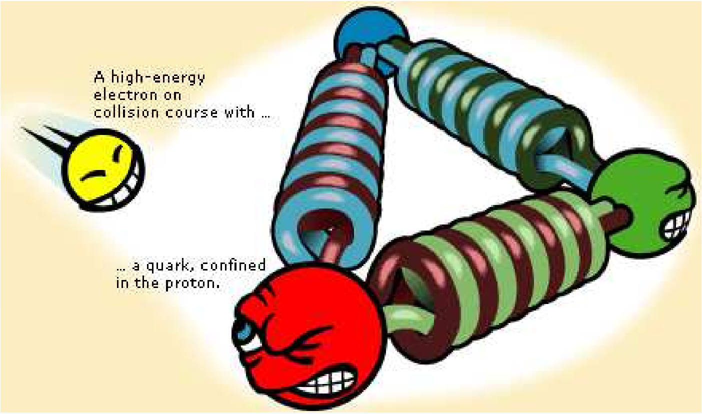
DESY theory workshop

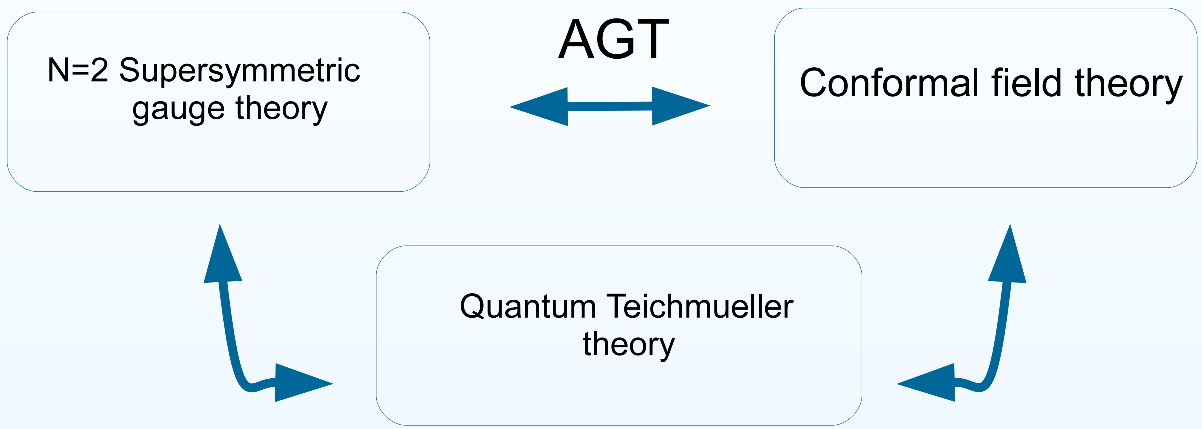
Universität Hamburg

DER FORSCHUNG | DER LEHRE | DER BILDUNG



- Gauge theories describe the interactions that bind the quarks into hadrons.
- Interactions behavior is much less understood at low energies.
- Understanding strong coupling behaviour remains as an important challenge.
- Simplify computational difficulties, add supersymmetry, add more mathematical structure.
- Certain physical quantities can be calculated: expectation values of loop observables (Wilson loop).
- Quantum Teichmüller theory is a mathematical framework for describing these loop observables in certain class of SUSY gauge theory.

AGT correspondence [Alday,Gaiotto,Tachikawa ’10]



◀ There exists a rich class of theories in four dimensions SUSY field theory $\mathcal{N} = 2$ whose properties are encoded in the geometry of certain Riemann surfaces.

◀ We focus on the proper mathematical terminology related to the Riemann surfaces.

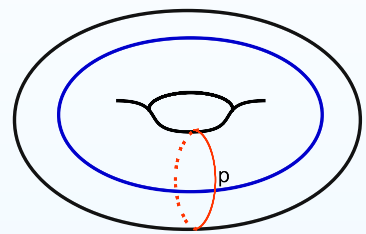
◀ Goal: Generalisation on the Super-Riemann surfaces.

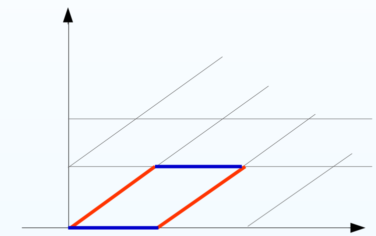
Teichmüller space of Riemann surface $\mathcal{T}_{g,n}$

Two dimensional surface Σ with complex structure is Riemann surface $\mathcal{C}_{g,n}$.

Example: Punctured Torus $\mathcal{C}_{1,1}$

Different Λ will give different complex structures.





$$\mathcal{C}_{1,1} \sim \mathbb{C}/\Lambda$$

◀ How many inequivalent complex structures are there?

$\mathcal{T}_{g,n}$: Space of deformations of complex structures.

Uniformisation: On each Riemann surface $\mathcal{C}_{g,n}$ there exists a metric of constant negative curvature. Locally:

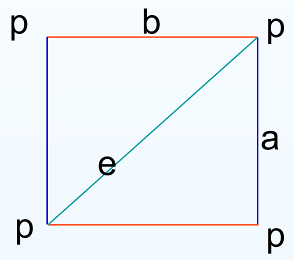
$$ds^2 = e^{2\varphi(z_i)} dz_i d\bar{z}_i, \quad \text{where} \quad \frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_i}\varphi(z_i, \bar{z}_i) = \mu e^{2\varphi(z_i, \bar{z}_i)}$$

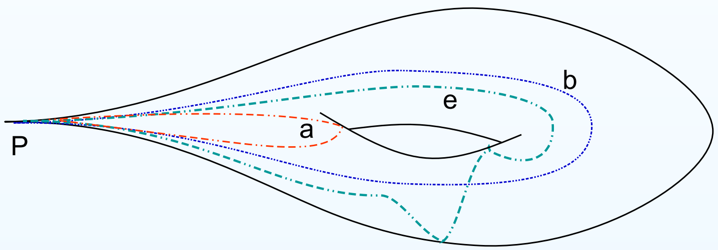
$\mathcal{T}_{g,n}$: Space of all metrics of constant negative curvature modulo diffeomorphisms isotopic to identity.

System of coordinates

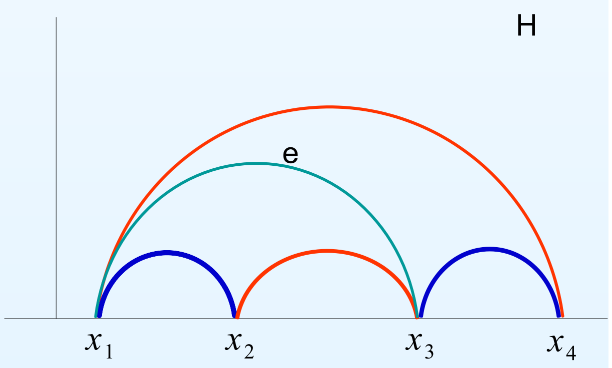
[Penner,Thurston, Fock,...]

An ideal triangulation of $\mathcal{C}_{g,n}$: Geodesics that start and end at the punctures.





- Conformal cross ratio: $e^z = \frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_3-x_2)}$ invariant under the action of $PSL(2,\mathbb{R})$.
- Shear coordinates: Assign e^z to the edge e.
- Coordinates admit Poisson structure.



System of coordinates

[Penner,Thurston, Fock,...]

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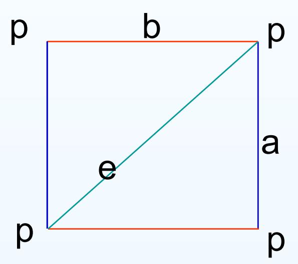
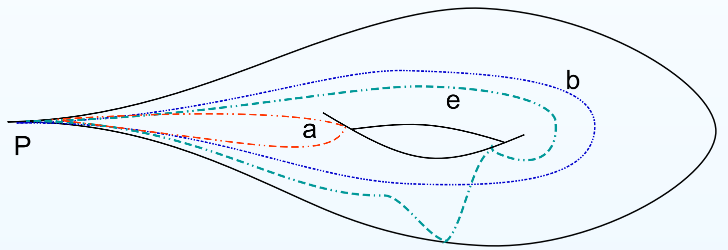




Change of triangulations:

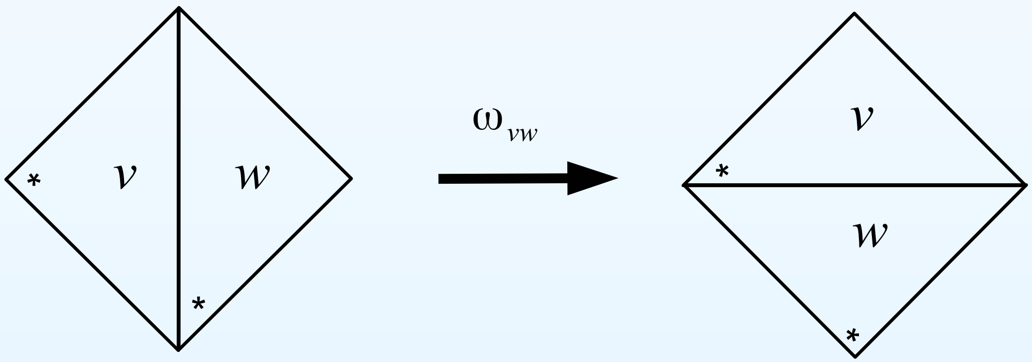
- For a given Riemann surface, any two triangulations can be connected by sequence of elementary transformations.
- Coordinates change under the change of triangulation.

Kashaev coordinates

[Kashaev'00]

- Kashaev coordinates: Pair of variables (p_v, q_v) for each triangle.
- Shear coordinates can be expressed in terms of Kashaev coordinates.

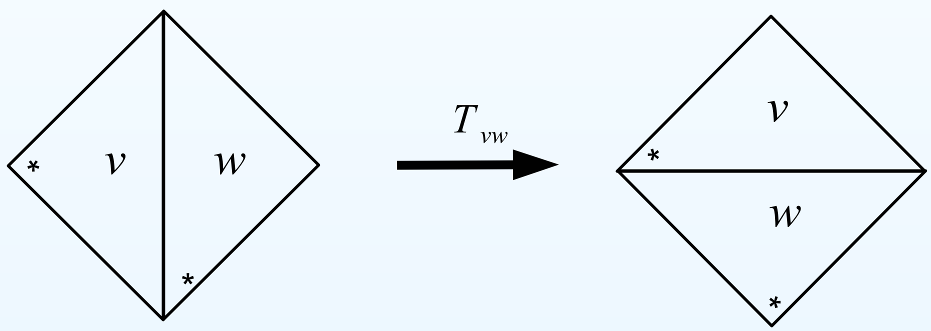
$$\{p_v, p_w\} = 0,$$
$$\{q_v, q_w\} = 0,$$
$$\{p_v, q_w\} = \delta_{v,w},$$



Kashaev coordinates/Quantization of the Teichmüller space

[Kashaev'00]

- Associate Hilbert space $\mathcal{H}_v = L^2(\mathbb{R})$ to each triangle.
- flip map promote to operators T.



$$[p_v, p_w] = 0,$$
$$[q_v, q_w] = 0,$$
$$[p_v, q_w] = \frac{1}{2\pi i}\delta_{vw},$$

$$T_{vw} : \mathcal{H}_v \otimes \mathcal{H}_w \to \mathcal{H}_v \otimes \mathcal{H}_w$$

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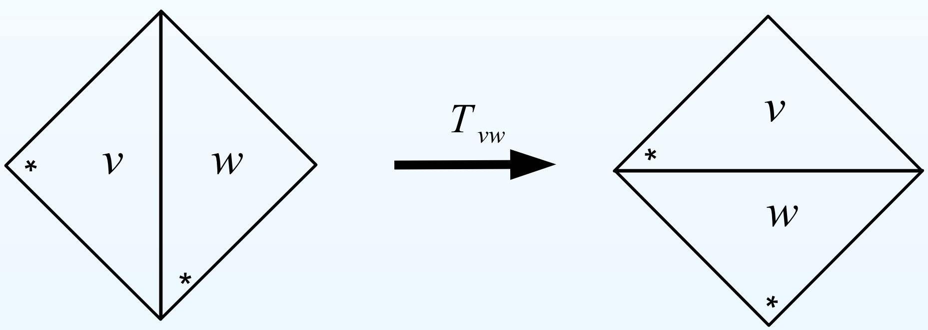

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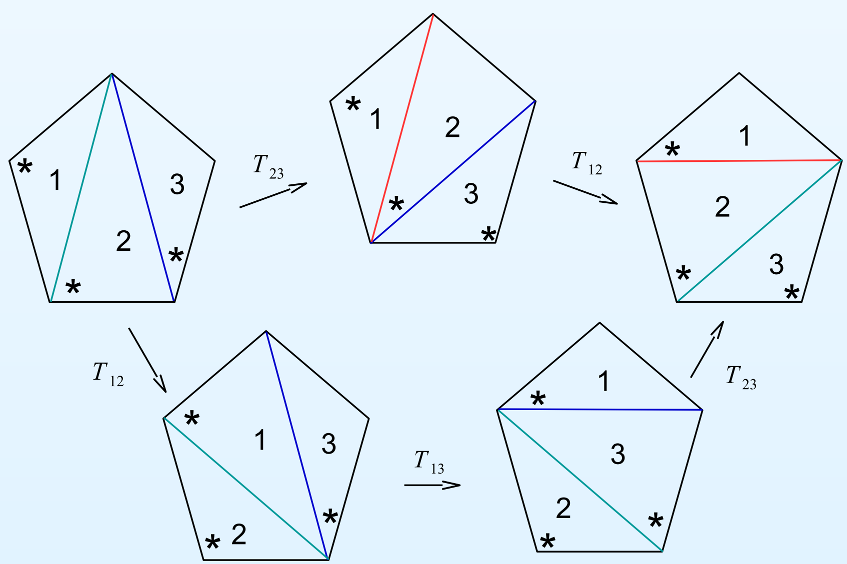
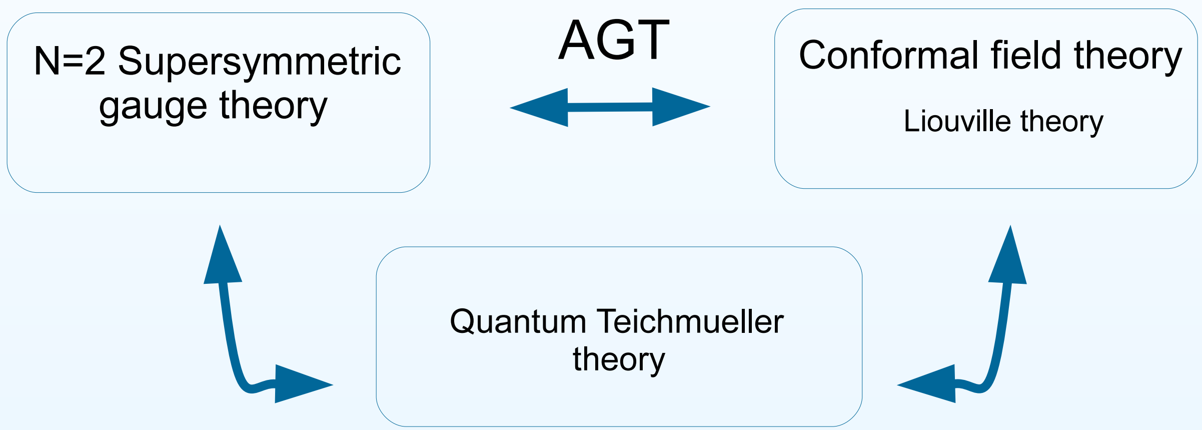


$T_{23}T_{13}T_{12} = T_{12}T_{23}$ Pentagon

Motivation

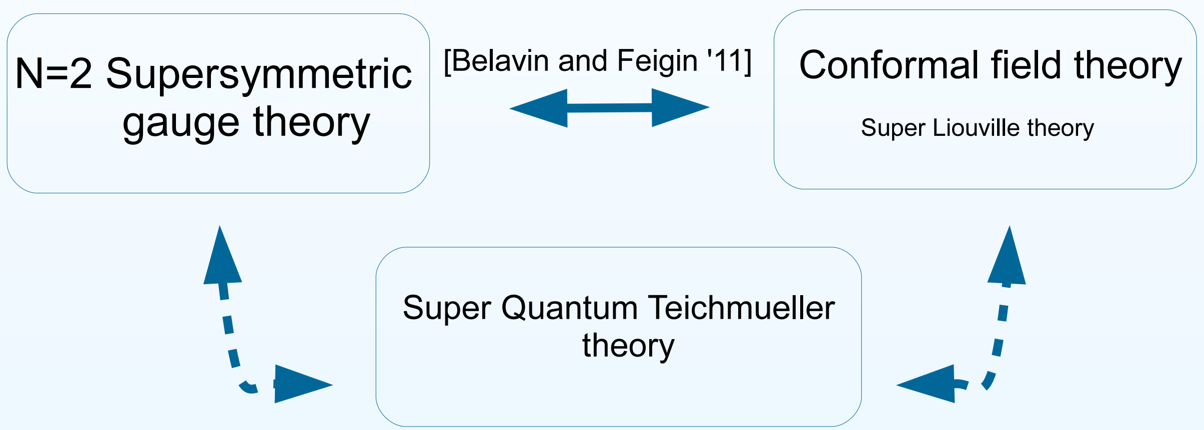
AGT correspondence [Alday, Gaiotto, Tachikawa '10]



- There exists relation between Teichmüller space and a connected component of the moduli space of flat $PSL(2,\mathbb{R})$-connections.
- Space of conformal blocks can be identified by the Hilbert space obtained by the quantization of Teichmüller spaces

Motivation

Teichmüller theory can be generalised to the case of super Riemann surfaces



Outline

- Generalise the coordinates.
- Find super flip using quantum group.
- Find super pentagon relation.

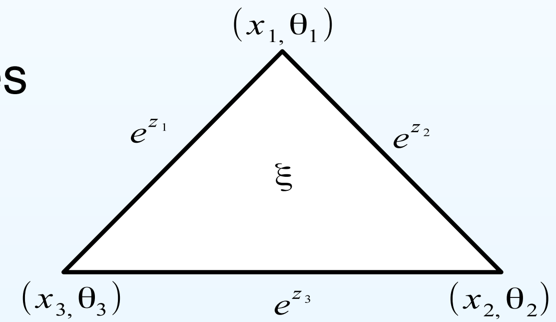
Generalisation

- Uniformisation theory for super Riemann surfaces, as quotient of super upper half plane by discrete subgroup $OSp(1|2)$.
- Points on upper half plane have coordinates (x, θ)

- Superconformal invariants x even , θ odd variables

  even: $e^{z_e} = \frac{X_{12}X_{34}}{X_{14}X_{23}}, \quad X_{ij} = x_i - x_j - \theta_i\theta_j$,

  odd: $\xi = \pm i \frac{x_{23}\theta_1 + x_{31}\theta_2 + x_{12}\theta_3 - \frac{1}{2}\theta_1\theta_2\theta_3}{(X_{12}X_{23}X_{31})^{\frac{1}{2}}}$,

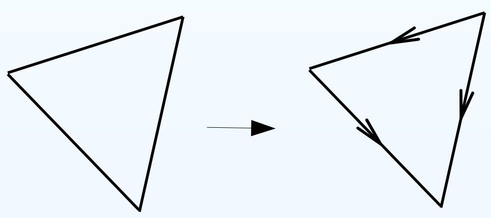


- Extra fermionic variables ξ attached to the face of triangle
- Generalisation of shear coordinates

Generalisation

- Odd variable defined up to a sign, to fix it $\rightarrow$ define the spin structure.
- Spin structure can be encoded by Kasteleyn orientation.

[Cimasoni, Reshetikhin'07]



- Hilbert space associated to the face of the triangle.
- Identify (p, q, ξ) with super Kashaev variables.
- Construct superflip and find super Ptolemy groupoid.

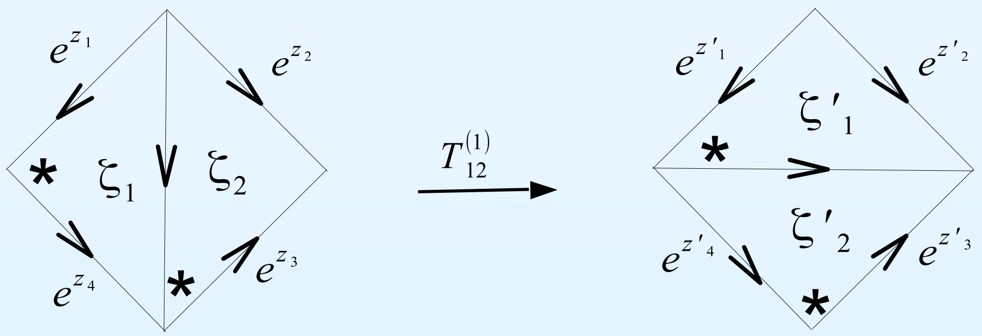
Super flip

From certain infinite dimensional representations of the quantum group on $L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$, one can find a canonical element that gives superflip.

[Aghaei, Pawelkiewicz, Teschner'15]

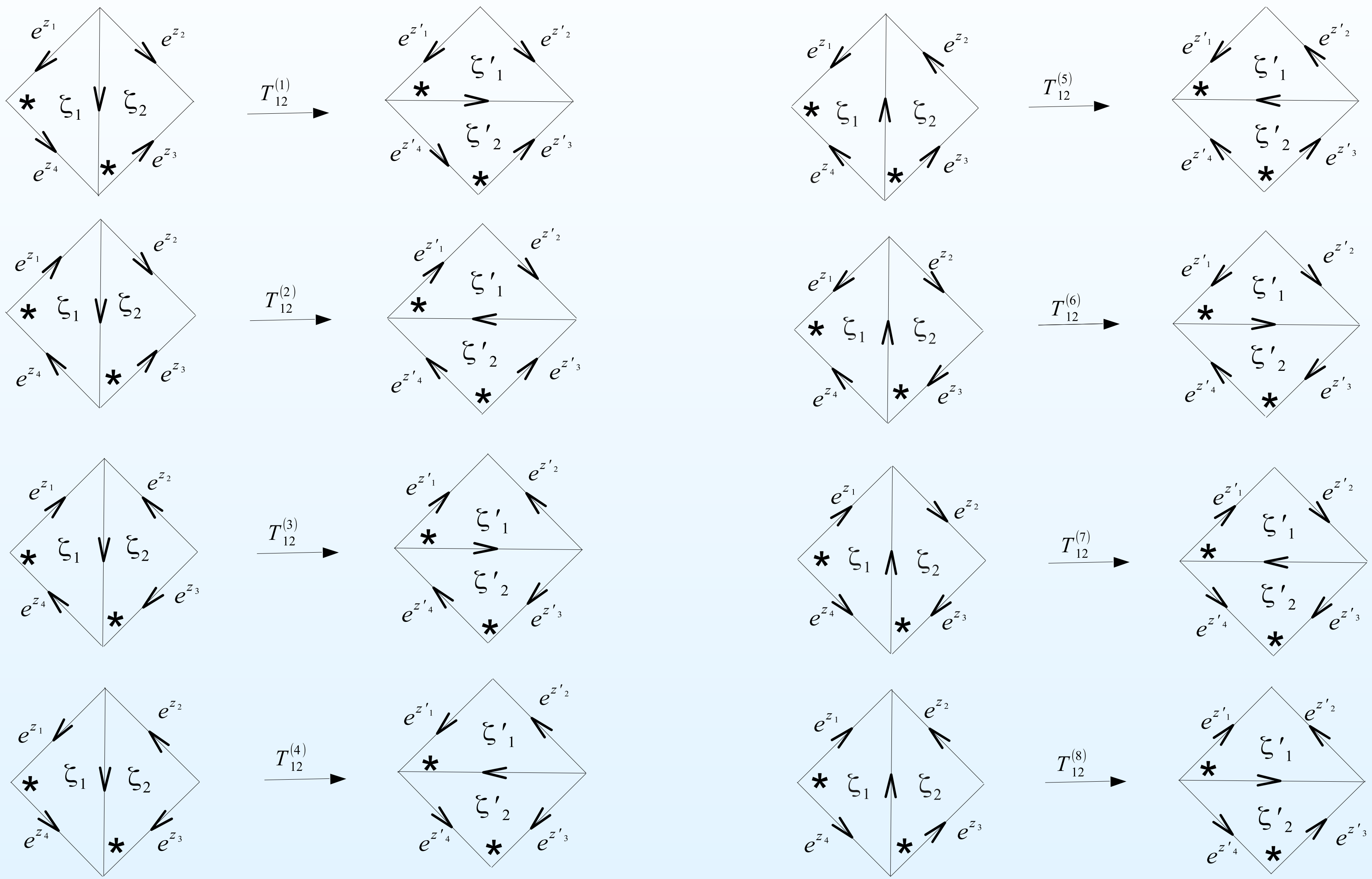
$$T_{12}^{(1)} = [f^+(q_1 + p_2 - q_2)\mathbb{I} \otimes \mathbb{I} - i f^-(q_1 + p_2 - q_2)\kappa \otimes \kappa] e^{-\pi i p_1 q_2}$$

$\kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, f^+, f^- are constructed out of quantum dilogarithm function.
There are different types of superflip depending on the Kasteleyn orientations.



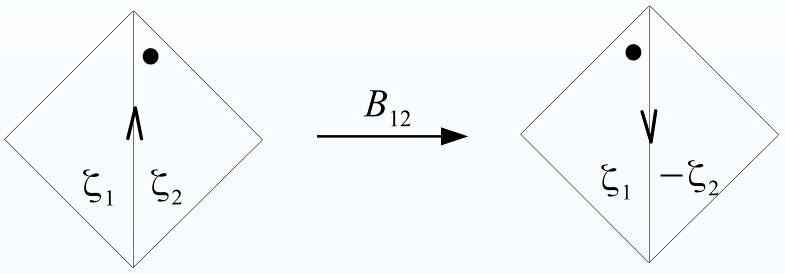
8 Superflips

There are 8 possible superflips:



Pushout operator

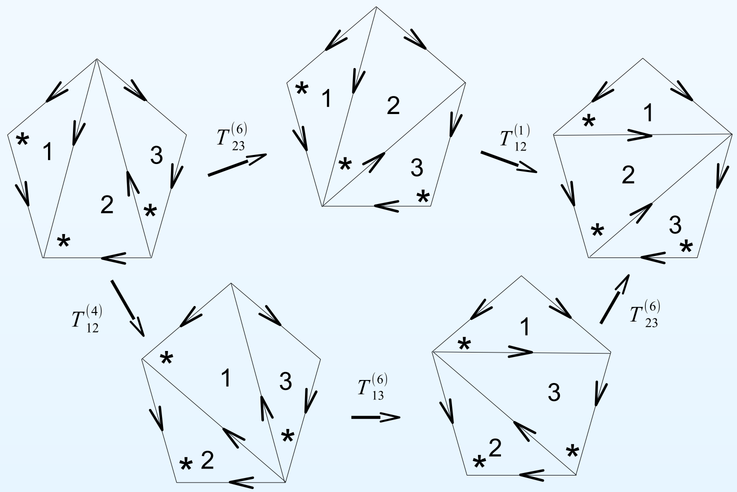
Pushout operator $B_{12} : \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2}$
Relates different Kasteleyn orientations which have the same spin structure.



[Aghaei, Pawelkiewicz, Teschner'15]

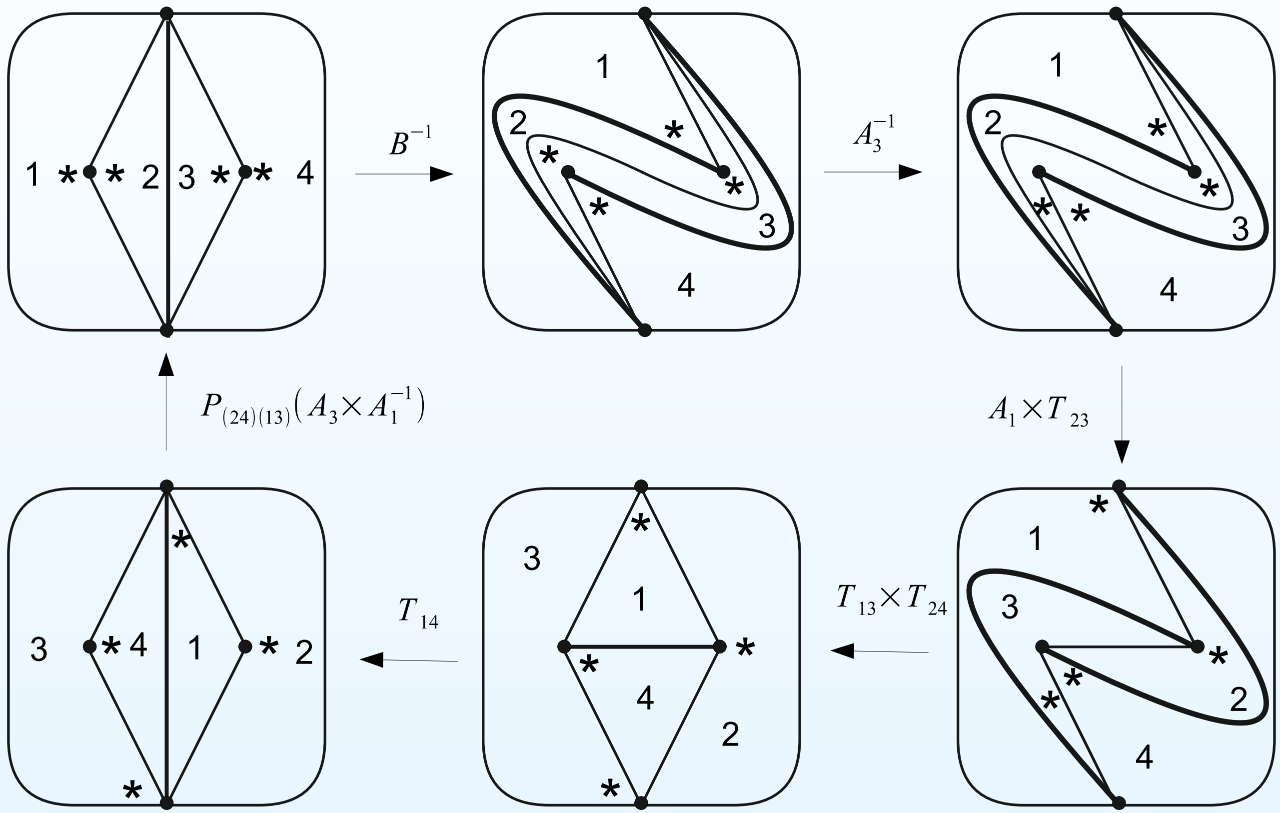
Super Ptolemy groupoid

- Super Pentagon relations
  $T_{23}^{(k)} T_{13}^{(j)} T_{12}^{(i)} = T_{12}^{(m)} T_{23}^{(l)}$.
- Relation between B and $T^{(i)}$'s.



$$T_{23}^{(6)} T_{13}^{(6)} T_{12}^{(4)} = T_{12}^{(1)} T_{23}^{(6)}$$

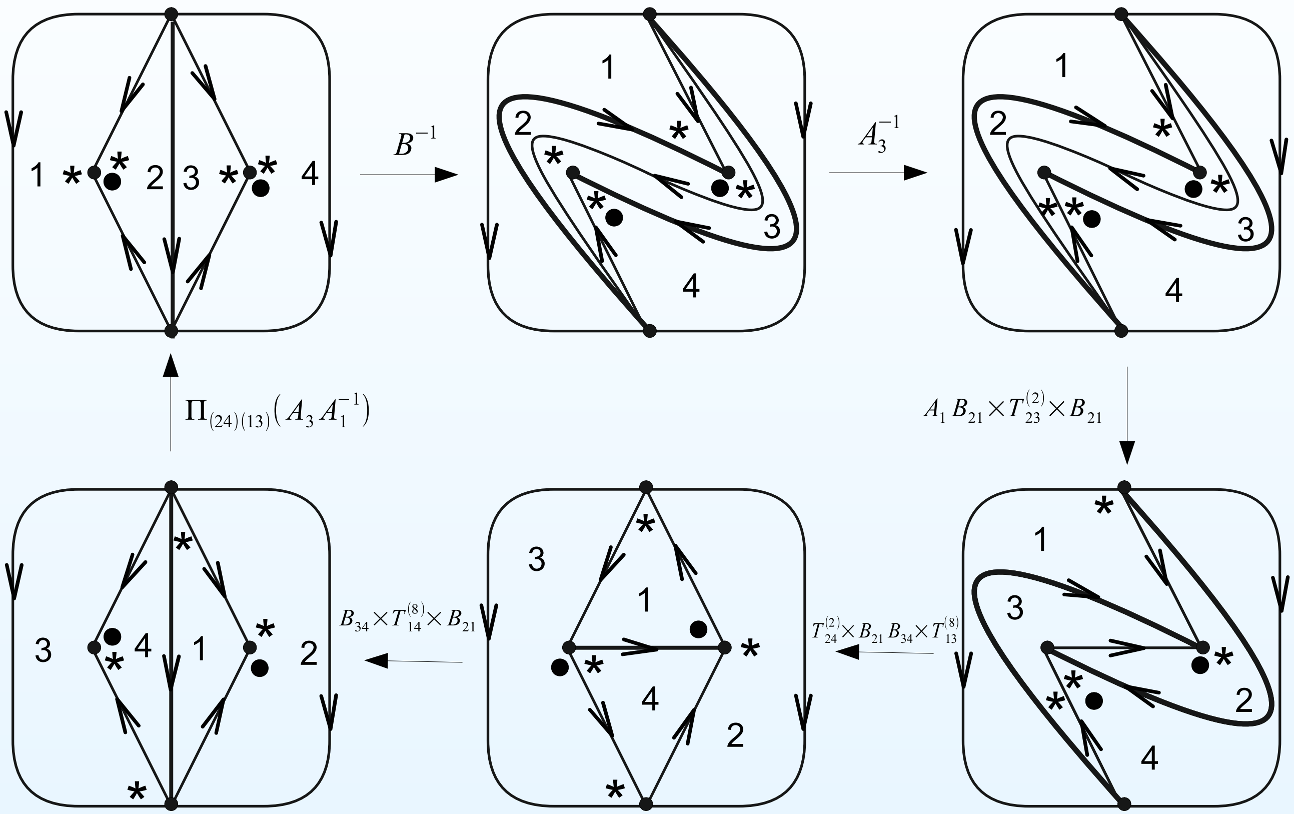
R matrix



$$B \simeq P_{(13)(24)} R_{1234}$$

$$R = R_{1234} \equiv Ad(A_2 T_{12}^{-1} A_4^{-1} T_{43}) T_{24}$$

Current project : R matrix



$$R = R(A_i, T_{ij}^{(m)}, B_{ij})$$

Future outlook

- The MCG [1] representation defined by the representation of the Ptolemy groupoid and its relation to Liouville theory has been studied.$\rightarrow$

  Study the MCG representation defined with the representation of the super Ptolemy groupoid and relate it to $\mathcal{N}=1$ supersymmetric Liouville theory.

- Ordinary Teichmüller theory is closely related to $SL(2,\mathbb{R})$-Chern-Simons theory on $\Sigma\times\mathbb{R}$. $\rightarrow$

  Study the connections between the quantum super Teichmüller theory and the quantum $OSp(1|2)$-Chern-Simons theory.

[1] $MCG(\Sigma)=$ group of isotopy classes of diffeomorphisms of Σ

Thank you for your attention!