Rethinking QFT, Desy Hamburg, 28.09.2016

Structure constants in $\mathcal{N} = 4$ SYM from the OPE and from integrability

[BE, F. Paul (hep-th/1608.04222)], [BE, V. Smirnov (hep-th/1607.06427)], [BE, A. Sfondrini (hep-th/1510.01242)]

Burkhard Eden

Institut für Mathematik und Physik, Humboldt-Universität zu Berlin
Zum großen Windkanal 6, 12489 Berlin

Introduction

- **CFT: spectrum** of operators and **structure constants**
- **Three-point** functions of **generic** operators are **hard** to compute at higher loops.
- **Integrand** of **stress tensor four-point function** by **symmetry** and **graph theory** [BE, Heslop, Howe, Korchemsky, Petkou, Sokatchev, Schubert, West]
- Most of the relevant **four-loop integrals** are **unknown**
- **Double OPE** limit: structure constants [BE (2012)], [BE, Paul (2016)] using **expansion by regions**

- **Hexagon: integrability** approach to structure constants [Basso, Komatsu, Vieira (2015)]
- **Match** with OPE [BE, Sfondrini (2015)], [Basso, Gonzalves, Komatsu, Vieira (2015)]
- **Problems** at four loops

- Exact evaluation of **four-loop scalar conformal** integrals by **Henn's method**

The stress-tensor four-point function

Quantum corrections take a **factorised form**: [BE, Petkou, Schubert, Sokatchev (2000)]

$$G_4(1,2,3,4) = G_4^{(0)} + \frac{2\,(N_c^2-1)}{(4\pi^2)^4}\; R(1,2,3,4)\; \Big[aF^{(1)} + a^2F^{(2)} + a^3F^{(3)} + O(a^4)\Big]\,,$$

where **y**'s keep track of indices of the **internal symmetry** $SU(4)$ and $a = \frac{g^2N}{8\pi^2}$.

$$R(1,2,3,4) = \frac{y_{12}^2y_{23}^2y_{34}^2y_{14}^2}{x_{12}^2x_{23}^2x_{34}^2x_{14}^2}(x_{13}^2x_{24}^2 - x_{12}^2x_{34}^2 - x_{14}^2x_{23}^2) + \frac{y_{12}^2y_{13}^2y_{24}^2y_{34}^2}{x_{12}^2x_{13}^2x_{24}^2x_{34}^2}(x_{14}^2x_{23}^2 - x_{12}^2x_{34}^2 - x_{13}^2x_{24}^2)$$
$$+ \frac{y_{13}^2y_{14}^2y_{23}^2y_{24}^2}{x_{13}^2x_{14}^2x_{23}^2x_{24}^2}(x_{12}^2x_{34}^2 - x_{14}^2x_{23}^2 - x_{13}^2x_{24}^2) + \frac{y_{12}^4y_{34}^4}{x_{12}^2x_{34}^2} + \frac{y_{13}^4y_{24}^4}{x_{13}^2x_{24}^2} + \frac{y_{14}^4y_{23}^4}{x_{14}^2x_{23}^2}$$

Integrands at one and two loops:

[BE, Schubert, Sokatchev (2000)], [Bianchi, Kovacs, Rossi, Stanev (2000)]

$$I_4^{(1)} \propto \frac{1}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2}\,, \qquad I_4^{(2)} \propto \frac{x_{12}^2x_{34}^2x_{56}^2 \;+\; (14 \text{ terms})}{(x_{15}^2x_{25}^2x_{35}^2x_{45}^2)(x_{16}^2x_{26}^2x_{36}^2x_{46}^2)x_{56}^2}$$

- Numerator of $I_4^{(l)}$ must have $\mathbf{S_{l+4}}$ **symmetry**. [BE, Heslop, Korchemsky, Sokatchev (2011)]

Three loop ansatz:

$$I_4^{(3)} \propto \frac{P^{(3)}}{(x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2)(x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2)(x_{17}^2 x_{27}^2 x_{37}^2 x_{47}^2) x_{56}^2 x_{57}^2 x_{67}^2}$$

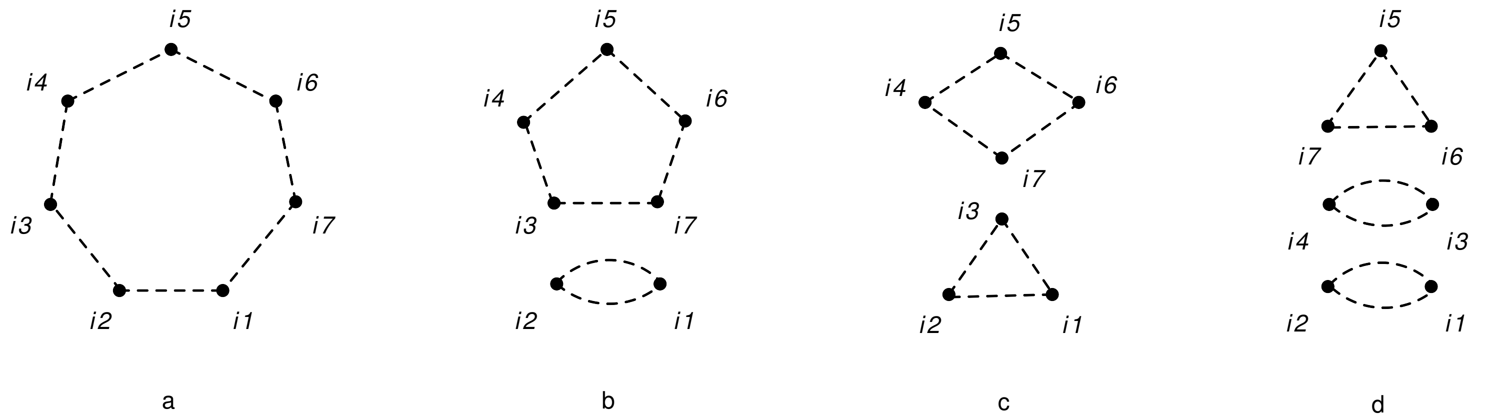
$P^{(3)}(x_{ij}^2)$ should be **S_7 symmetric** and should have conformal **weight -2** at every point. Options:

(a) heptagon: $x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{71}^2 \; + \; S_7$ permutations

(b) 2-gon × pentagon: $(x_{12}^4)(x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{73}^2) \; + \; S_7$ permutations

(c) triangle × square: $(x_{12}^2 x_{23}^2 x_{31}^2)(x_{45}^2 x_{56}^2 x_{67}^2 x_{74}^2) \; + \; S_7$ permutations

(d) 2-gon × 2-gon × triangle: $(x_{12}^4)(x_{34}^4)(x_{56}^2 x_{67}^2 x_{75}^2) \; + \; S_7$ permutations



Coefficients: [Alday, BE, Heslop, Maldacena, Korchemsky, Sokatchev]

- **Planar** integrands at **high loop orders**, also for $\mathcal{O}^L = \mathrm{Tr}(Z^L)$, Z a scalar (stress-tensor $= \mathcal{O}^2$)

Conformal partial wave analysis

- **Double OPE** $x_{12}, x_{34} \to 0$ on G_4
- **Block** for each "exchanged" **primary** with **anomalous dimension** γ and **spin s**:

$$G_4 = \sum_{\gamma,s} \left(N^{\mathcal{O}}_{\mathcal{T}\mathcal{T}}\right)^2 \text{cpwa}(\gamma, s)$$

$$\text{cpwa}(\gamma, s) = N(s) \; u^{\gamma/2} \, Y^s \, {}_2F_1\Big(s+1+\gamma/2, s+1+\gamma/2, 2+2\,s+\gamma; \, Y\Big)$$

- $Y = 1 - v$, and u, v are conformal cross ratios.
- Match expansion in g^2, Y on correlator to **extract** $\gamma(\mathbf{s})$, $N(s)$ as power series in $a = \frac{g^2 N}{8\,\pi^2}$.
  Odd spin absent!

- **Uniform transcendentality**:
  At **l** loops **harmonic sums** and **zeta-values** with total weight $2\,l - 1$ for γ and $2\,l$ for N.

- Apart from a factor related to $(s!)^2/(2s)!$ for example at a^3: [BE (2012)]

$$
\begin{aligned}
N(s) \;\propto\; & \left[-\frac{25}{2}\,S_1\right]\zeta_5 \\
+ \; & \left[-\,3\,S_{-3} - 10\,S_{-2}\,S_1 + \frac{4}{3}\,S_1^3 - 6\,S_1\,S_2 - \frac{4}{3}\,S_3 + 6\,S_{-2,1}\right]\zeta_3\,, \\
+ \; & \Bigg[-\,11\,S_{-6} + \frac{5}{2}\,S_{-3}^2 - 5\,S_{-4}\,S_{-2} - \frac{41}{2}\,S_{-5}\,S_1 - S_{-3}\,S_{-2}\,S_1 - 5\,S_{-4}\,S_1^2 - 2\,S_{-2}^2\,S_1^2 \\
& + \frac{4}{3}\,S_{-3}\,S_1^3 - \frac{13}{2}\,S_{-4}\,S_2 - \frac{3}{2}\,S_{-2}^2\,S_2 - 10\,S_{-3}\,S_1\,S_2 - 2\,S_{-2}\,S_2^2 - S_2^3 - \frac{16}{3}\,S_{-3}\,S_3 \\
& - 8\,S_{-2}\,S_1\,S_3 - 6\,S_1\,S_2\,S_3 - 3\,S_3^2 - 3\,S_{-2}\,S_4 + 9\,S_1^2\,S_4 - 4\,S_2\,S_4 + \frac{15}{2}\,S_1\,S_5 - \frac{13}{2}\,S_6 \\
& + 14\,S_{-5,1} + 11\,S_1\,S_{-4,1} + 9\,S_{-4,2} - 12\,S_1\,S_{-3,-2} + 10\,S_{-2}\,S_{-3,1} - 4\,S_1^2\,S_{-3,1} \\
& + 8\,S_2\,S_{-3,1} + 4\,S_1\,S_{-3,2} + 9\,S_{-3,3} - 10\,S_{-3}\,S_{-2,1} + 14\,S_{-2}\,S_1\,S_{-2,1} - \frac{8}{3}\,S_1^3\,S_{-2,1} \\
& + 4\,S_1\,S_2\,S_{-2,1} + \frac{20}{3}\,S_3\,S_{-2,1} + 10\,S_{-2,1}^2 + 10\,S_{-2}\,S_{-2,2} - 6\,S_1^2\,S_{-2,2} + 6\,S_2\,S_{-2,2} \\
& + 6\,S_1\,S_{-2,3} + 11\,S_{-2,4} - 6\,S_2\,S_{1,3} - 4\,S_1\,S_{1,4} - 4\,S_{1,5} + 4\,S_1\,S_{2,3} + 4\,S_{2,4} - 12\,S_{-4,1,1} \\
& + 8\,S_1\,S_{-3,1,1} - 2\,S_{-3,1,2} - 2\,S_{-3,2,1} - 24\,S_1\,S_{-2,-2,1} - 20\,S_{-2}\,S_{-2,1,1} + 16\,S_1^2\,S_{-2,1,1} \\
& - 8\,S_2\,S_{-2,1,1} + 16\,S_1\,S_{-2,1,2} - 6\,S_{-2,1,3} + 16\,S_1\,S_{-2,2,1} + 4\,S_{-2,2,2} - 6\,S_{-2,3,1} - 4\,S_1\,S_{1,1,3} \\
& - 8\,S_{1,1,4} + 8\,S_{1,3,2} - 8\,S_{-3,1,1,1} - 48\,S_1\,S_{-2,1,1,1} - 20\,S_{-2,1,1,2} - 20\,S_{-2,1,2,1} - 20\,S_{-2,2,1,1} \\
& + 16\,S_{1,1,1,3} + 64\,S_{-2,1,1,1,1}\Bigg]
\end{aligned}
$$

... and by expansion by regions at $a^4/16$: [Gonzalves (2016)], [BE, Paul (2016)]
c.f. [Kotikov, Lipatov, Rej, Staudacher, Velizhanin (2007)], [Bajnok, Janik, Lukowski (2008)].

$$\gamma_4(2) = -2496 + 576\,\zeta_3 - 1440\,\zeta_5\,,$$
$$\gamma_4(4) = -\frac{8045275}{2187} + \frac{114500}{81}\zeta_3 - \frac{25000}{9}\zeta_5\,,$$
$$\gamma_4(6) = -\frac{393946504469}{91125000} + \frac{11736088}{5625}\zeta_3 - \frac{19208}{5}\zeta_5\,,$$
$$\gamma_4(8) = -\frac{5685358151649447407}{1200725694000000} + \frac{142906863577}{54022500}\zeta_3 - \frac{1158242}{245}\zeta_5$$

$$N(2) = 9952 + 1312\,\zeta_3 + 288\,\zeta_3^2 + 3920\,\zeta_5 + 5880\,\zeta_7\,,$$
$$N(4) = \frac{1930033531879}{882165816} + \frac{15976465}{83349}\zeta_3 + \frac{1000}{21}\zeta_3^2 + \frac{795070}{1323}\zeta_5 + 700\zeta_7\,,$$
$$N(6) = \frac{357114900616418917}{1320819513750000} + \frac{15290724568}{1010728125}\zeta_3 + \frac{1372}{275}\zeta_3^2 + \frac{440377}{7425}\zeta_5 + \frac{686}{11}\zeta_7\,,$$
$$N(8) = \frac{38427134703883835503402077962 69}{1435998502695545628840000000000} + \frac{159635682336106797 01}{1791413509936050000 0}\zeta_3 + \frac{1158242}{2627625}\zeta_3^2 + \frac{11715335287}{2347663500}\zeta_5 + \frac{10654}{2145}\zeta_7\,.$$

• Matching on an ansatz in terms of harmonic sums would require many more orders in Y.

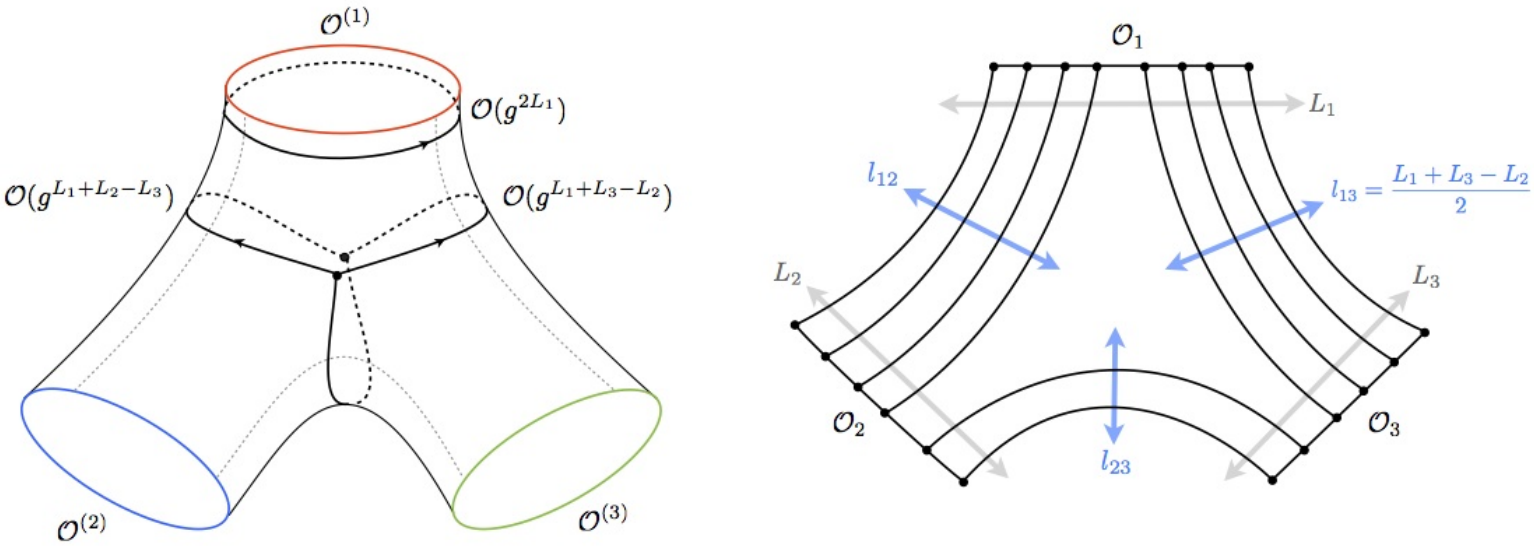
Structure constants from integrability

Operators as **spin chains**: [Minahan, Zarembo, Beisert, Staudacher, ...]

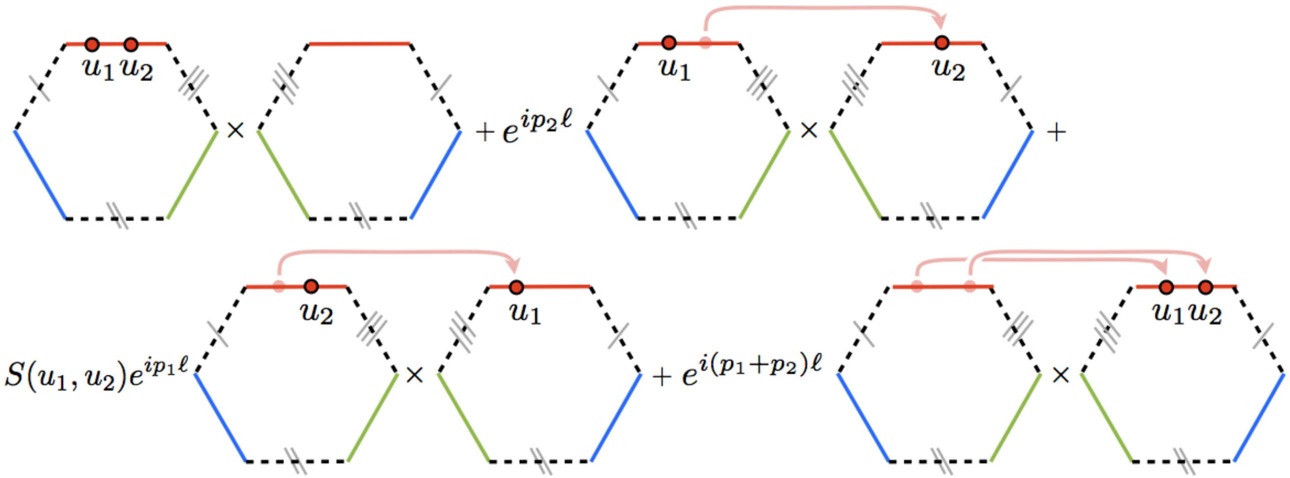
- **sl(2)** sector: $\mathcal{O} = \mathrm{Tr}\Big((DZ)\, Z \ldots (D^3 Z)\, Z \ldots \Big)$ where D's are covariant derivatives.
- **Z** is a **site**, **D** an **excitation** with rapidity $\mathbf{u} = \frac{1}{2} \cot\left(\frac{p}{2}\right)$
- **Bethe equations** for L sites, s excitations:

$$\left(\frac{u_i + \frac{i}{2}}{u_i - \frac{i}{2}}\right)^L = \prod_{j \neq i} \frac{u_i - u_j - i}{u_i - u_j + i}, \qquad \prod_j \left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right) = 1$$

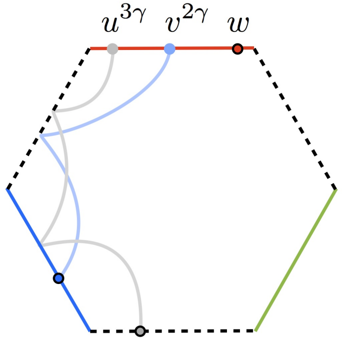
Three-point function: [Basso, Komatsu, Vieira (2015)]



- **Input**: **Bethe roots** characterising the three **operators**.
- Split into **top** and **bottom hexagon. Entangled state:**



- Physical **excitations** on three edges $\Rightarrow$ **asymptotic** piece [Gromov, Vieira, . . .]
- **Finite size** effects starting at g^4 for $L = 2$ $\Rightarrow$ **mirror particles** on the others
- **Move** excitations to **one edge** by **crossing** (2γ) and **mirror** (γ) transformations



$0\,\gamma$:	x^+	x^-
$1\,\gamma$:	$\frac{1}{x^+}$	x^-
$2\,\gamma$:	$\frac{1}{x^+}$	$\frac{1}{x^-}$
$3\,\gamma$:	x^+	$\frac{1}{x^-}$
$4\,\gamma$:	x^+	x^-

$$\mathcal{A} = \sum_{\alpha\cup\bar\alpha=\{u_i\}} w(\alpha,\bar\alpha)(-1)^{|\alpha|}\sum_X(-1)^{f_X}\,\mathfrak{h}_{D\ldots D}(\alpha)\,\mathfrak{h}_{D\ldots D}(\bar\alpha)$$

$$\mathfrak{h}_{D_1\ldots D_n} = \Big(\prod_{i<j} h_{ij}\Big)\,\langle\,\psi^1_n\ldots\psi^1_1\,|\,S\,|\,\bar\psi^{\dot 2}_1\ldots\bar\psi^{\dot 2}_n\,\rangle\,,\qquad h_{12} = \frac{x_1^- - x_2^-}{x_1^- - x_2^+}\,\frac{1-1/(x_1^- x_2^+)}{1-1/(x_1^+ x_2^+)}\,\frac{1}{\sigma_{12}}\,,$$

$$x^\pm = x(u\pm i/2),\quad x(u) = \frac{u}{2}\big(1+\sqrt{1-2\,g^2/u^2}\big)\,.$$

- Use **S** on **left or right**, e.g. $S\,|\bar\psi^{\dot 2}_1\bar\psi^{\dot 2}_2\bar\psi^{\dot 2}_3\rangle = D_{12}D_{13}D_{23}\,|\bar\psi^{\dot 2}_3\bar\psi^{\dot 2}_2\bar\psi^{\dot 2}_1\rangle$. [Beisert (2005)]
- Contract using $\langle\psi^\alpha|\bar\psi^{\dot\beta}\rangle = \epsilon^{\alpha\dot\beta}$, $\langle\phi^a|\bar\phi^{b'}\rangle = \epsilon^{ab'}$.

- **Lüscher** corrections: X_a **bound state** of $D^{1\dot 2}, \bar D^{2\dot 1}, Y^{12'}, \bar Y^{21'}$ (scalars) in **mirror kinematics**.

$$\delta\mathcal{A} = \sum_{a>0}\int\frac{du}{2\pi}\,\mu(u)\;\omega(\alpha,\bar\alpha)\;\mathfrak{h}_{XD\ldots D}(u^{n\gamma},\alpha)\;\mathfrak{h}_{D\ldots D\bar X}(\bar\alpha,u^{-n\gamma})\,,\qquad \mu(u) = \frac{a\,(g^2)^{l+1}}{(2\,u^{+a}\,u^{-a})^{l+2}}+\ldots$$

- Mirror measure: **bridge length** as exponent of g^2.
- Sum over **bound states** $\Rightarrow$ $\zeta-$values

Results & predictions

S	$\left(\frac{C^{\bullet\circ\circ}}{C^{\circ\circ\circ}}\right)^2$ for twist $L=2$, bridge $l=1$ and spin S
2	$\frac{1}{6} - g^2 + (7+3\zeta_3)g^4 - (48+8\zeta_3+25\zeta_5)g^6 + \dots$
4	$\frac{1}{70} - \frac{205}{1764}g^2 + \left(\frac{76393}{74088} + \frac{5}{14}\zeta_3\right)g^4 - \left(\frac{242613655}{28005264} + \frac{1315}{1323}\zeta_3 + \frac{125}{42}\zeta_5\right)g^6 + \dots$
6	$\frac{1}{924} - \frac{553}{54450}g^2 + \left(\frac{880821373}{8624880000} + \frac{7}{220}\zeta_3\right)g^4 - \left(\frac{1364275757197}{1423105200000} + \frac{520093}{6534000}\zeta_3 + \frac{35}{132}\zeta_5\right)g^6 + \dots$
8	$\frac{1}{12870} - \frac{14380057}{18036018000}g^2 + \left(\frac{5944825782678337}{682443241880400000} + \frac{761}{300300}\zeta_3\right)g^4$ $- \left(\frac{758072803634287465765957}{8607383632540733040000000} + \frac{15248925343}{2840672835000}\zeta_3 + \frac{761}{36036}\zeta_5\right)g^6 + \dots$
10	$\frac{1}{184756} - \frac{3313402433}{559838594952000}g^2 + \left(\frac{171050793565932326659}{2488046776199329363200000} + \frac{671}{3527160}\zeta_3\right)g^4$ $- \left(\frac{9135036882706194334305789554347}{12439610127669853644128645760000000} + \frac{11482697774339}{352698314819760000}\zeta_3 + \frac{3355}{2116296}\zeta_5\right)g^6 + \dots$

In agreement with [BE (2012)].

S	$\left(\frac{C^{\bullet\circ\circ}}{C^{\circ\circ\circ}}\right)^2$ for twist $L=2$, bridge $l_{12}=l_{31}=1$, $l_{23}>1$ and spin S
2	$\frac{1}{6} - g^2 + 7g^4 + (10\zeta_5\eta - 10\zeta_5 + 7\zeta_3 - 48)g^6 + \dots$
4	$\frac{1}{70} - \frac{205}{1764}g^2 + \frac{36653}{37044}g^4 + \left(\left(\frac{1}{6}\zeta_3 + \frac{25}{21}\zeta_5\right)\eta - \frac{25}{21}\zeta_5 + \frac{193}{216}\zeta_3 - \frac{442765625}{56010528}\right)g^6 + \dots$
6	$\frac{1}{924} - \frac{553}{54450}g^2 + \frac{826643623}{8624880000}g^4$ $+ \left(\left(-\frac{1}{1440} + \frac{7}{264}\zeta_3 + \frac{7}{66}\zeta_5\right)\eta - \frac{7}{66}\zeta_5 + \frac{24143}{297000}\zeta_3 - \frac{1183056555847}{1423105200000}\right)g^6 + \dots$
8	$\frac{1}{12870} - \frac{14380057}{18036018000}g^2 + \frac{2748342985341731}{341221620940200000}g^4 + \left(\left(-\frac{79}{604800} + \frac{3}{1040}\zeta_3 + \frac{761}{90090}\zeta_5\right)\eta\right.$ $\left.- \frac{761}{90090}\zeta_5 + \frac{1039202363}{158918760000}\zeta_3 - \frac{1270649655622342732745039}{17214767265081466080000000}\right)g^6 + \dots$
10	$\frac{1}{184756} - \frac{3313402433}{559838594952000}g^2 + \frac{156422034186391633909}{2488046776199329363200000}g^4 + \left(\left(-\frac{45071}{2813045760} + \frac{781}{2930256}\zeta_3 +\right.\right.$ $\left.\left.\frac{671}{1058148}\zeta_5\right)\eta - \frac{671}{1058148}\zeta_5 + \frac{8295615163}{167991576480000}\zeta_3 - \frac{7465848687069712820911408164847}{12439610127669853644128645760000000}\right)g^6 + \dots$

Here $\eta = \frac{1}{2}$ for $L=3$ and $\eta = 0$ for $L \geq 4$.

Conclusions ...

- The hexagon approach crucially depends on the **bridge length**, in particular the order at which finite size effects come in. We **tested** $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_2^s \rangle$ and **predicted** $\langle \mathcal{O}_3 \mathcal{O}_3 \mathcal{O}_2^s \rangle$, $\langle \mathcal{O}_L \mathcal{O}_L \mathcal{O}_2^s \rangle$, $L \geq 4$.
  The same numbers were found in [Basso, Gonzalves, Komatsu, Vieira (2015)].
- The **same methods** as in the stress-tensor four-point function apply for correlators of $\mathcal{O}^L$.
  The $L \geq 3$ predictions were confirmed in [Chicherin, Drummond, Heslop, Sokatchev (2015)].

... and outlook

- The **integrability** scenario is **intricate at four-loops**, e.g. with two mirror magnons there is a **double pole**. **Compensated by norm**?
- The norm could be the hexagon with the unit operator at one point.
  We need methods to work the **hexagon with more than one non-trivial operator**.
- Planar **four-loop stress-tensor correlator**: ideally solve integrals. Beginnings: [BE, Smirnov (2016)]
  Structure constants by expansion by regions [Gonzalves (2016)], [BE, Paul (2016)]
- Can one develop an **octagon operator** for four-point functions?
- Can we obtain correlators in terms of **explicit functions** of two variables **from integrability**?

Opposite channel: $3\gamma,\ -3\gamma$

$$\mathfrak{h}\ \mathfrak{h} \;=\; \Big(\prod_{i<j\in\alpha} h_{ij}D_{ij}\Big)\Big(\prod_{i<j\in\bar\alpha} h_{ij}D_{ij}\Big)\Big(\prod_{i\in\alpha} h(u^{3\gamma},u_i)R(u^{3\gamma},u_i)\Big)\Big(\prod_{i\in\bar\alpha} h(u_i,u^{-3\gamma})S(u_i,u^{-3\gamma})\Big)$$

For $X = D^{1\dot 2}, \bar D^{2\dot 1}, Y^{12'}, \bar Y^{21'}$, respectively: ($D_{ij}, E_{ij}, G_{ij}, L_{ij}$ depend on $x^{\pm}$, similar to h_{ij})

$$\{R,S\} \;=\; \{D, \frac{1}{2}(D-E)\}\,,\quad \{\frac{1}{2}(D-E), D\}\,,\quad \{G,L\}\,,\quad \{G,L\}\,.$$

From the definitions:

$$h(u_i,u^{-3\gamma})\,S(u_i,u^{-3\gamma}) \;=\; h(u^{3\gamma},u_i)\,R(u^{3\gamma},u_i) \qquad (*)$$

- **Factorisation** $\mathcal{A}_{\text{Tailoring}} * T_1(u^\gamma)$, **generalise** to $\mathbf{T_a}(\mathbf{u}^\gamma)$.

$$T_a(u^\gamma) \;\propto\; \left(1+g^2\sum_i u_i^{(1)}\partial_{u_i}\right)\left(Q(a+1)+Q(-a-1)-Q(a-1)-Q(-a+1)\right)$$
$$+\;\; g^2(-2iS_1(s))\left[\frac{Q(-a-1)}{u^{-a}}-\frac{Q(a+1)}{u^{+a}}+\sum_{k=1}^{a-1}\frac{Q(-a+2k-1)-Q(-a+2k+1)}{u^{-a+2k}}\right]+\ldots$$

with the **Baxter polynomial** $Q(u) \;=\; \prod_i (u-u_i)\,,\ \ Q(a) \;:=\; Q(u^{+a})\,,\ \ u^{+a} \;=\; u+\frac{i}{2}\,a.$

Evaluate by **shift trick**:

$$\delta\mathcal{A} = \sum_a \int \frac{du}{2\pi}\,\mu(u)\,\frac{(Q(a+1)+Q(-a-1)-Q(a-1)-Q(-a+1))}{Q(i/2)} + \dots$$

- $\langle \mathcal{O}_L \mathcal{O}_L \mathcal{O}_2^s \rangle$: $l = L-1$.

  $L=2 \;\Rightarrow\; O(g^4)+O(g^6)+\dots \qquad L=3 \;\Rightarrow\; O(g^6)+\dots \qquad L \geq 4 \;\Rightarrow\; 0$ up to three loops.

Adjacent channels γ, $-\gamma$ and $5\,\gamma$, $-5\,\gamma$:

- Equation $(*)$ is only true for a $6\,\gamma$ shift between $X, \bar{X}$.

  Partition dependent factors $\prod_i \sigma(u^\gamma, u_i)/\sigma(u^{5\gamma}, u_i)\,,\quad i \in \alpha$ or $i \in \bar{\alpha}$, respectively.

Evaluate by **residues**:

$$\delta\mathcal{A} = 2\sum_a \int \frac{du}{2\pi}\,\mu(u)\,\frac{Q(i/2)\,\tilde{Q}(u,a)\,T_a(u^{-\gamma})}{Q(a+1)Q(-a-1)Q(a-1)Q(-a+1)}$$

$$T_a(u^{-\gamma}) = g^2(-2i\,S_1(s))\left[-\frac{Q(-a+1)}{u^{-a}} + \frac{Q(a-1)}{u^{+a}} + \sum_{k=1}^{a-1} \frac{Q(-a+2k-1)-Q(-a+2k+1)}{u^{-a+2k}}\right]$$

- $\langle \mathcal{O}_L \mathcal{O}_L \mathcal{O}_2^s \rangle$: $\quad O(g^6)$ for all L.

Henn's method for conformal four-loop integrals [BE, Smirnov (2016)]

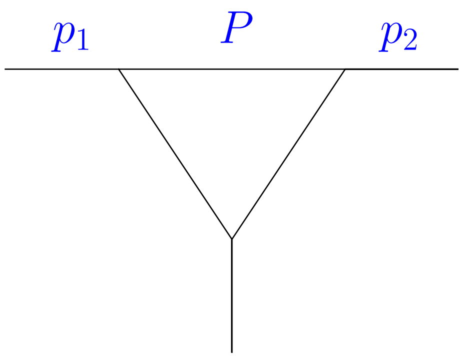
The **triangle rule** is perhaps the original example of an **IBP relation**:

$$I(\alpha_0,\beta_1,\beta_2,\alpha_1,\alpha_2) = \int d^D P \frac{1}{(P^2)^{\alpha_0}((P+p_1)^2)^{\beta_1}(p_1^2)^{\alpha_1}((P+p_2)^2)^{\beta_2}(p_2^2)^{\alpha_2}}$$

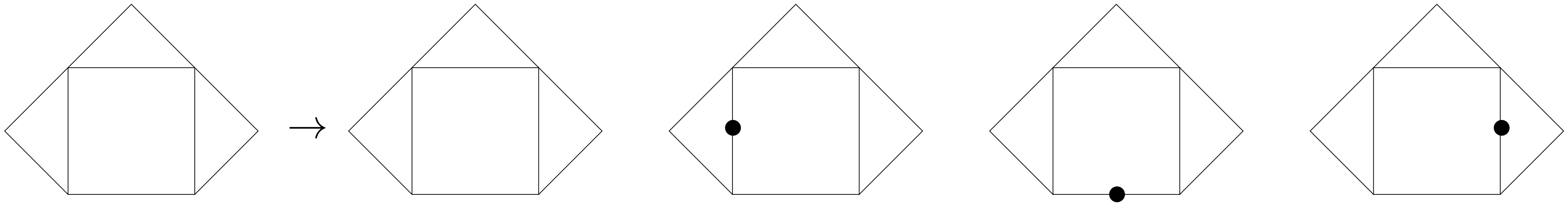
Introduce $\frac{\partial}{\partial P^\mu} P^\mu$ under the integral. Completing squares:

$$\begin{aligned} & (D-2\alpha_0-\beta_1-\beta_2)\, I(\alpha_0,\beta_1,\beta_2,\alpha_1,\alpha_2) \\ = \;& \beta_1 \left(I(\alpha_0-1,\beta_1+1,\beta_2,\alpha_1,\alpha_2) - I(\alpha_0,\beta_1+1,\beta_2,\alpha_1-1,\alpha_2)\right) \\ + \;& \beta_2 \left(I(\alpha_0-1,\beta_1,\beta_2+1,\alpha_1,\alpha_2) - I(\alpha_0,\beta_1,\beta_2+1,\alpha_1,\alpha_2-1)\right) \end{aligned}$$

- **Iterate** till one of the α_i vanishes in every term $\Rightarrow$ **lower sector**. **Many terms** for high indices!
- `LiteRed` looks for recursion relations like the triangle rule.
- If no closed form recursion is found, **Laporta**: large linear system and Gauss elimination by `FIRE`. Can express all integrals in terms of a set of **masters**.



- Want (point 4 at infinity):



- Including lower sectors: **213 primary masters**
- Express ∂_x, $\partial_{\bar{x}}$ as derivatives w.r.t. outer points. **IBP reduce derivatives** to masters:

$$\frac{\partial}{\partial x} f \;=\; A_1(x,\bar{x},\epsilon)\, f\,, \qquad \frac{\partial}{\partial \bar{x}} f \;=\; A_2(x,\bar{x},\epsilon)\, f\,, \qquad f \;=\; \{I_1,\ldots,I_{213}\}$$

- The `FIRE` output for A_1, A_2 is **nearly Fuchsian**. Denominators define the **letters**.
- **Switch** to masters of **uniform transcendentality** $\Rightarrow$ **canonical form**:

$$A_1 = \epsilon\left(\frac{M_0}{x} + \frac{M_1}{x-1} + \frac{M_2}{x-\bar{x}} + \frac{M_3}{x-(1-\bar{x})} + \frac{M_4}{x-\frac{1}{\bar{x}}} + \frac{M_5}{x-\frac{\bar{x}}{\bar{x}-1}}\right)\,, \qquad M_i \text{ const}$$

and similar for A_2. Similarity transform $\Rightarrow$ non-linear!

- Bubble subintegral: G_{11} is not UT, G_{21} is!

$$\epsilon\, G(1,1,0,0) \;=\; 1+2\epsilon+\left(4-\frac{1}{2}\zeta_2\right)\epsilon^2+\left(8-\zeta_2-\frac{7}{3}\zeta_3\right)\epsilon^3+\ldots\,,$$
$$\epsilon\, G(2,1,0,0) \;=\; -1+\frac{1}{2}\zeta_2\,\epsilon^2+\frac{7}{3}\zeta_3\,\epsilon^3+\ldots\,.$$

- **UT = pseudo-conformality**

(1) Index 2 on one line in a bubble (can insert numerator, iterate),

(2) Index 2 on one line of a three vertex,

(3) Numerator for a four-vertex.

$$I=\frac{4}{(x-\bar{x})^2}\left(-\mathcal{L}_{\{3,5\}}+\mathcal{L}_{\{5,3\}}+\mathcal{L}_{\{2,5,0\}}-\mathcal{L}_{\{4,3,0\}}-\mathcal{L}_{\{1,5,0,0\}}+\mathcal{L}_{\{3,3,0,0\}}-\mathcal{L}_{\{2,3,0,0,0\}}+\mathcal{L}_{\{1,3,0,0,0,0\}}\right)$$

in terms of Brown SVHPLs (and 212 other results).