Wedge local fields in integrable models with bound states

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Joint work with Yoh Tanimoto

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- Local observables in interacting QFTs have an extremely complicated structure. In perturbative QFT they are given in terms of infinite series expansion whose convergence is hard to control.

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→ Wedge-locality

- Observables localized in bounded regions are then obtained as intersections of von Neumann algebras

$$\mathcal{A}(\mathcal{O}) := \mathcal{A}(W_L + x) \cap \mathcal{A}(W_R - y) \quad \text{where} \quad \mathcal{O} = W_L + x \cap W_R - y$$

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- Here: extend the construction to models with bound states.

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- multi-particle scattering matrix – product of two-particle scattering matrices ("factorizing S matrix").

Task: Given a function S, construct a corresponding quantum field theory.

Examples

- $S(\zeta)$ is a complex-valued meromorphic function on $\mathbb{C}$ with a number of properties, e.g (in the case of one particle species):

$$\text{crossing symmetry:} \quad S(i\pi - \zeta) = S(\zeta),$$
$$\text{Bootstrap equation:} \quad S(\zeta) = S(\zeta + \frac{i\pi}{3}) S(\zeta - \frac{i\pi}{3}).$$

- Example for such a function S: the Bullough-Dodd model

$$S(\zeta, B) = f_{\frac{2}{3}}(\zeta) f_{\frac{B}{3} - \frac{2}{3}}(\zeta) f_{-\frac{B}{3}}(\zeta),$$

where

$$f_a(\zeta) := \frac{\tanh \frac{1}{2}(\zeta + i\pi a)}{\tanh \frac{1}{2}(\zeta - i\pi a)}, \quad 0 < B < 1.$$

- There are also models with several particle species, e.g. the $Z(N)$-Ising model, the affine Toda field theories and the sine-Gordon model.

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$\rightarrow$ In the example, they are at $\zeta = \frac{2i\pi}{3}, \frac{i\pi}{3}$.

Mathematical framework

The theory is constructed as a deformation of a free field:

- Zamolodchikov-Faddeev algebra (elements $z(\theta)$, $z^\dagger(\theta)$):

$$
\begin{aligned}
z(\theta_1)z(\theta_2) &= S(\theta_1 - \theta_2)\, z(\theta_2)z(\theta_1)\,, \\
z^\dagger(\theta_1)z^\dagger(\theta_2) &= S(\theta_1 - \theta_2)\, z^\dagger(\theta_2)z^\dagger(\theta_1)\,, \\
z(\theta_1)z^\dagger(\theta_2) &= S(\theta_2 - \theta_1)\, z^\dagger(\theta_2)z(\theta_1) + \delta(\theta_1 - \theta_2)\cdot \mathbf{1}.
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  These act on an "S-symmetric" Fock space.

- Representation of the Poincaré group, including the space-time reflections J.
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\phi(x) := \int d\theta \left(e^{ip(\theta)\cdot x} z^\dagger(\theta) + e^{-ip(\theta)\cdot x} z(\theta) \right).
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$$\phi(x) := \int d\theta \left(e^{ip(\theta)\cdot x} z^\dagger(\theta) + e^{-ip(\theta)\cdot x} z(\theta) \right).$$

- This field is not local:

$$[\phi(x), \phi(y)] \neq 0 \text{ even if } x \text{ spacelike separated from } y.$$

Deformed fields

- For models without poles in the physical strip (e.g. sinh-Gordon), the field is wedge-local: with $\phi'(x) := U(j)\phi(-x)U(j)$,

$$[\phi(x), \phi'(y)] = 0 \text{ if } x \text{ spacelike separated to the left of } y.$$

Deformed fields

- In the presence of bound states, not even wedge-local:

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This because S is not analytic in the physical strip $0 < \operatorname{Im}\zeta < \pi$. By shifting an integral contour from $\mathbb{R}$ to $\mathbb{R} + i\pi$ the commutator gives the residues of S at the poles $\frac{2\pi i}{3}$ and $\frac{\pi i}{3}$.

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- We need to modify ϕ to get a wedge-local field.

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- On the single-particle Hilbert space and on a suitable domain* it acts as

$$(\chi_1(f)\xi)(\theta) := \sqrt{2\pi|R|}f^+\left(\theta + \frac{i\pi}{3}\right)\xi\left(\theta - \frac{i\pi}{3}\right),$$

where $R := \mathrm{res}_{\zeta=\frac{2\pi i}{3}} S(\zeta)$.
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  (Note: $\chi_1(f)$ realizes the idea that the state of one elementary particle ξ is fused with f^+ into the same species of particle.)
- We define a new field $\tilde{\phi}(f) = \phi(f) + \chi(f)$.

Weak wedge commutativity

Theorem

Let f and g be real test functions supported in W_L and W_R, respectively. Then, for each Φ, Ψ in a suitable linear space*, it holds that

$$\langle \tilde{\phi}(f)\Phi, \tilde{\phi}'(g)\Psi \rangle = \langle \tilde{\phi}'(g)\Phi, \tilde{\phi}(f)\Psi \rangle .$$

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- The construction of the corresponding wedge-algebras is a non-trivial step due to the subtle domain properties of $\tilde{\phi}$.
- The local algebras can be constructed and showed to be non-trivial under certain assumptions on $\tilde{\phi}$.