Mordell-Weil Torsion in the Mirror of Multi-Sections

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DESY/Hamburg University

Based on
- arXiv:1408.4808 with: D. Klevers, D. Mayorga, H. Piragua and J. Reuter
- arXiv:1604.00011 with: J. Reuter and T. Schimannek

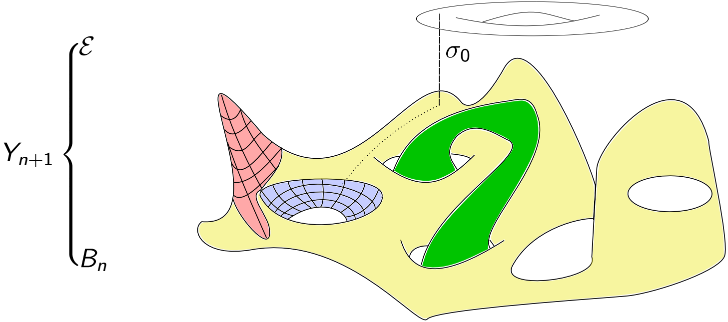
DESY Theory Workshop, Hamburg
September 28th 2016

[Morrison Vafa' 96; Anderson, Braun, Etxebarria ,Grimm, Kapfer, Keitel; Taylor Morrison, Park; Cvetič,Donagi, Klevers,Piragua, Poretschkin; Schäfer-Nameki, Lawrie, Wong; Mayrhover, Lin, Palti, Till, Weigand '12-15]

Outline

1. **Why F-theory**
2. **Conclusion**
3. **Example in 2D Ambient Spaces**
4. **Generalizations to 3D Ambient Space**
5. **Summary and Outlook**

Why F-theory



F-theory Setup

Consider a torus/elliptically fibered Calabi-Yau n+1-fold Y_{n+1} and interpret

- **Elliptic Fiber** $\mathcal{E}$: $\tau = C_0 + ig_{IIB}^{-1}$ Type IIB axio dilaton
- **Base** B_n: Physical locus D_7 branes and their intersections
- A marked **point** on $\mathcal{E}$ gives zero section: $\sigma_0 : B_n \hookrightarrow Y_{n+1}$

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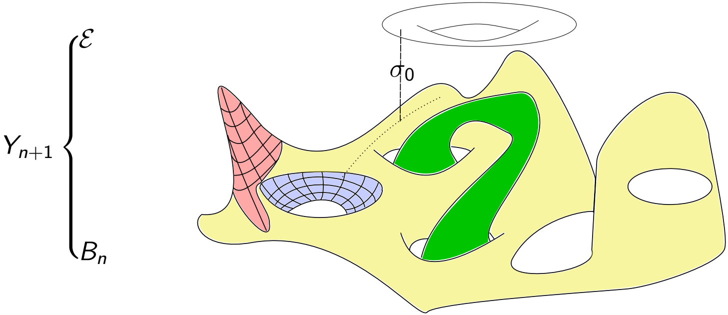

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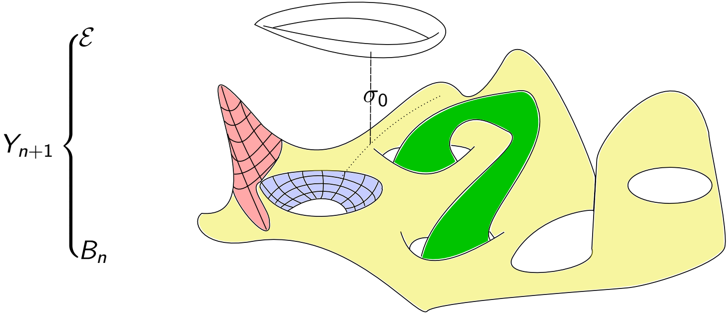

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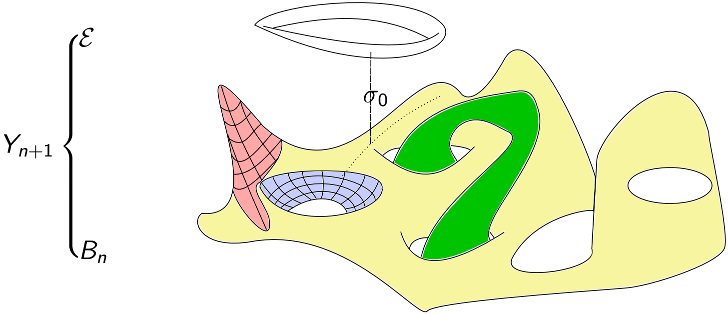

The F-theory Dictionary to Physics

Gauge structure encoded in properties of the **elliptic fiber**

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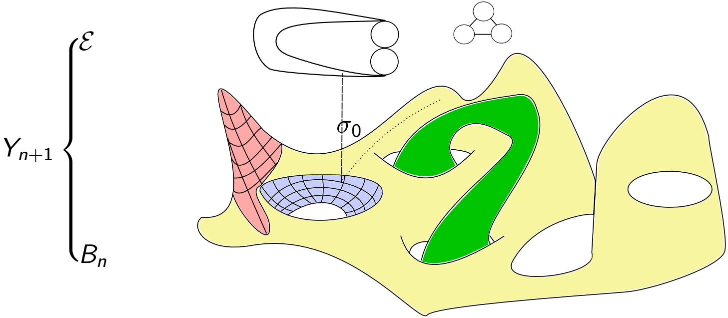

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- **Non-Abelian Algebras:** ADEFG singularities at Codimension 1

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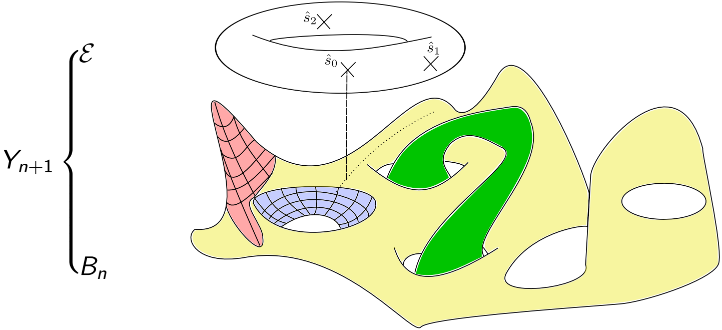

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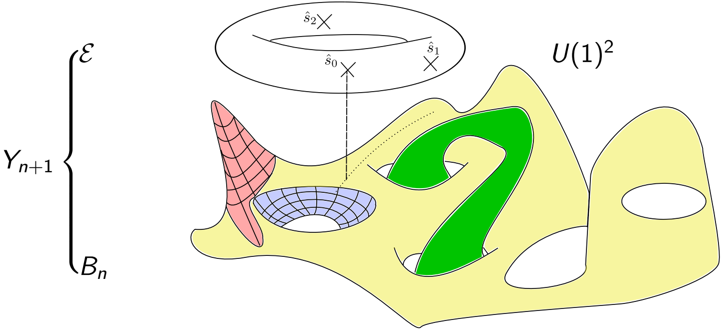

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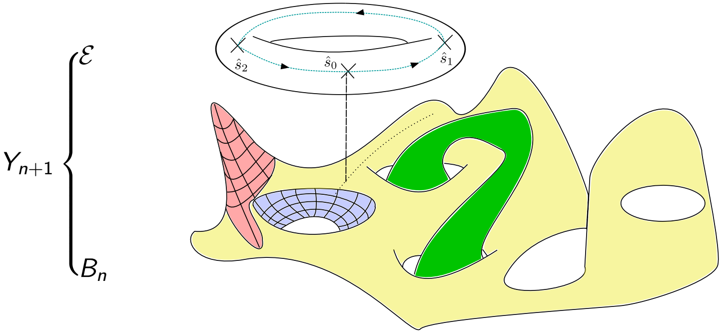

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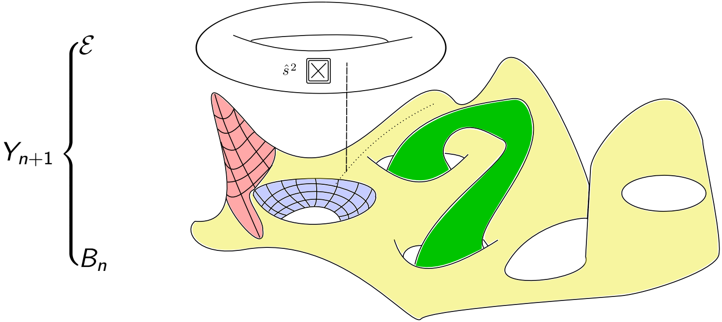

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- **Discrete $\mathbb{Z}_n$ Symmetries:** No sections at all but only n-sections $S_i \cdot \mathcal{E} = n$

Conclusion: The Mirror Fiber Conjecture

The mathematical conclusion of this talk

Given a genus-one fiber $\mathcal{E}$ for which the **Mordell-Weil group** *of the Jacobian contains* **torsion**, *the (Batyrev)* **mirror dual** *is a genus-one fiber $\mathcal{E}'$* **without a section** *and vice versa.*

The physical conclusion of this talk

In the Physics of F-theory this is a duality between models with **non-simply laced** gauge groups $G = \frac{\mathfrak{g}}{\mathbb{Z}_n}$ and those with **discrete gauge factors** $\hat{G} \times \mathbb{Z}_n$

$$\hat{G} \times \mathbb{Z}_n \longleftarrow \text{Mirror-Duality} \longrightarrow G = \frac{\mathfrak{g}}{\mathbb{Z}_n}$$

Recap: Complete Intersection Fibers

- A codimension n, **one-fold** is specified by a **Nef partition** of a n+1-dimensional polytope Δ [Kreutzer,Skarke]

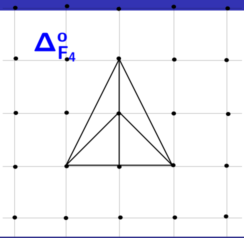
$$\begin{aligned} \Delta &= \Delta_1 + ... + \Delta_n \,, \qquad & \Delta^\circ &= \langle \nabla_1, ..., \nabla_n \rangle \,, \\ \nabla^\circ &= \langle \Delta_1, ..., \Delta_n \rangle \,, \qquad & \nabla &= \nabla_1 + ... + \nabla_n , \end{aligned}$$

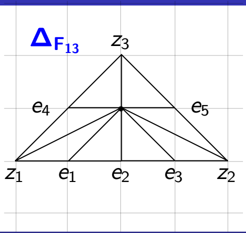
- with $a_i = (\nabla_n, \Delta_m) \geq -\delta_{m,n}$
- Then the **intersection of the n hypersurfaces** P_i in Δ specifies the one-fold

$$P_{\Delta_i} = \sum_{m \in \Delta_i} \prod_{i=1}^{n} s_m z_i^{\langle m, \rho_i \rangle + a_i} \in \mathbb{P}_{\Delta^o}$$

- The **Mirror CY** is cut out by $P_{\nabla_i} \in \mathbb{P}_{\nabla^o}$ [Batyrev, Borisov'97]
- Note :
  - There are **16 (3145)** nef partitions in **2D(3D)** ambient spaces [Kreutzer, Skarke]

Example: The Generic Fiber in F_{13}





Torus fibers from polytopes

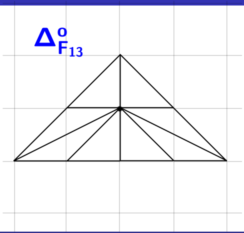
Consider 2D toric variety Δ with the one-fold hypersurface

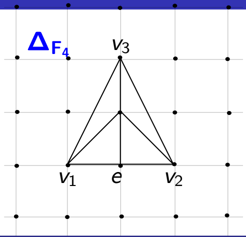
$$p_{F_{13}} = s_1 e_4^2 e_2^2 e_3 z_1^4 e_1^3 + s_2 e_4 e_2^2 e_3^2 z_2^2 z_1^2 e_1^2 e_5 + s_3 e_2^2 e_3^3 z_2^4 e_1 e_5^2 + s_6 e_4 e_2 e_3 z_2 z_1 e_1 e_5 z_3 + s_9 e_4 e_5 z_3^2$$

- Promote the **coefficients** s_i to **functions** of the base $s_i(b)$
- SU(2), SU(2), SU(4) fibers at $s_1 = 0, s_3 = 0, s_9 = 0$
- Divisors $D_{z_i} \cdot \mathcal{E}_{F_{13}} = 1$ intersect the fiber exactly once
  $\rightarrow$ There exists one **torsional relation:** $MW(P_{F_{13}}) = \mathbb{Z}_2$

$$\boxed{G_{F_{13}} = SU(2)^2 SU(4)/\mathbb{Z}_2}$$

Example: The Mirror Fiber in F_4





The generic fibration

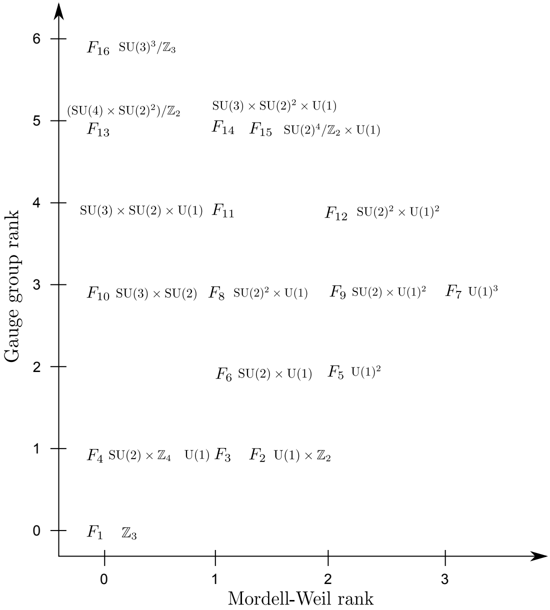
Consider the mirror curve by changing the role of $\Delta \leftrightarrow \Delta^o$, $P^{1,1,2}[4]$:

$$p_{F_4} = d_1 e^2 v_1^4 + d_2 e^2 v_1^3 v_2 + d_3 e^2 v_1^2 v_2^2 + d_4 e^2 v_1 v_2^3 + d_5 e^2 v_2^4 + d_6 e v_1^2 v_3 + d_7 e X v_2 v_3 + d_8 e v_2^2 v_3 + d_9 v_3^2 \,,$$

- Promote the **coefficients** d_i to **functions** of the base $d_i(b)$
- At $d_9 = 0$ the fibers split into an SU(2) fiber
- Toric divisors $D_{v_i} \cdot \mathcal{E}_{F_4} = 2$ intersect the fiber **only multiple** times
  $\rightarrow$ A **genus-one fibration** only with two-sections

$$\boxed{G_{F_4} = SU(2) \times \mathbb{Z}_2}$$

Fibrations as hypersurface in 2D Varieties



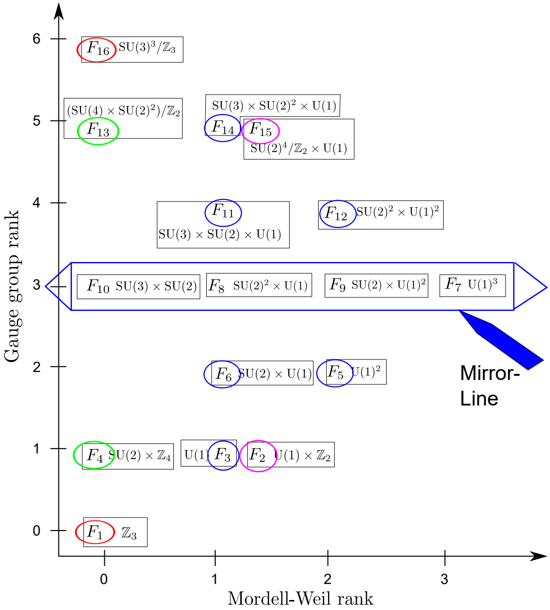
Mordell-Weil Torsion

- $\mathbb{Z}_2, \mathbb{Z}_3$ **torsion** in the upper theories

Multi Sections

- **Genus one** curves with two-and three-sections

Fibrations as hypersurface in 2D Varieties



Mordell-Weil Torsion

- $\mathbb{Z}_2, \mathbb{Z}_3$ **torsion** in the upper theories

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- $MW(P_\Delta) = MW(P_{\Delta^\circ})$
- TorMW $\leftrightarrow$ n-Sections
- $Rank(G_\Delta + G_{\Delta_i^\circ}) = 6$

Multi Sections

- **Genus one** curves with two-and three-sections

The mirror Conjecture

Is the observation general?

The **fiber embedding space** plays a central role in the observation:

- 2D Ambient spaces are **particularly simple**

Generalization to 3D Ambient spaces

Does the observation **generalize** to **higher dimensional** ambient spaces of the fiber

The conjecture holds also in all 3145 cases of codimension 2 fibers

Mirror Duality in CICY Fibers

- **Test for** the presence of a section by the **mirror-dual torsion**
  $\rightarrow$ Construction of **non-toric** rational sections
- Only **1024**/3145 Nef partitions have a **inequivalent** Weierstrass forms
  $\rightarrow$ Different **non-toric** realizations of sections and resolution divisors
- **New class of models: Self-Mirror** Genus-one curves
  $\rightarrow$ Gauge groups must have quotient and discrete symmetries $\frac{G\times\mathbb{Z}_n}{\mathbb{Z}_n}$

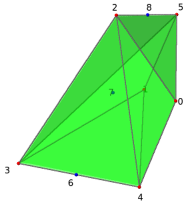
Genus-one curves with Torsion

- **No Section** $\rightarrow$ **no** generator of the **Mordell-Weil group**
- Can not apply the **geometric group law**
- Can **not test for torsion**, still we expect it to be present

What to do?

Self-Dual Genus-one Example

- Consider the 3D ambient space with vertices that span Δ:



$$
\begin{array}{cccc}
z_0 & z_1 & z_2 & z_3 \\
\hline
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}
\end{array}
$$

$$
\begin{array}{cccc}
z_4 & z_5 & z_6 & z_8 \\
\hline
\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\end{array}
$$

- And **Self-Mirror** nef partition

$$\Delta_1 = \langle z_0,\, z_3,\, z_4,\, z_6 \rangle \,, \quad \Delta_2 = \langle z_1,\, z_2,\, z_5,\, z_8 \rangle \,,$$

- With complete intersection equation

$$p_1 = a_2 z_0^2 z_4^2 z_6 + a_4 z_0 z_1 z_5^2 z_8 + a_1 z_0 z_3 z_4 z_6 + a_3 z_0 z_2 z_5 z_8 + a_0 z_3^2 z_6 \,,$$
$$p_2 = a_9 z_0 z_1 z_4^2 z_6 + a_8 z_1^2 z_5^2 z_8 + a_6 z_1 z_3 z_4 z_6 + a_7 z_1 z_2 z_5 z_8 + a_5 z_2^2 z_8 \,.$$

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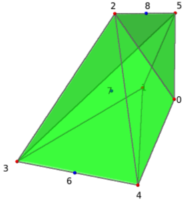

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- And **Self-Mirror** nef partition

$$\Delta_1 = \langle z_0,\, z_3,\, z_4,\, z_6 \rangle\,, \quad \Delta_2 = \langle z_1,\, z_2,\, z_5,\, z_8 \rangle\,,$$

- **Intersections** of the ambient divisors **with** the **fiber** $\mathcal{E}$: $[D_{z_0}, D_{z_1}, D_{z_2}, D_{z_3}, D_{z_4}, D_{z_5}, D_{z_6}, D_{z_8}] \cdot \mathcal{E} = [0, 0, 2, 2, 2, 2, 0, 0]$
- This is a **genus-one** curve only with **two-sections** $\rightarrow \mathbb{Z}_2$ Symmetry
- $D_{z_0}, D_{z_1}, D_{z_6}, D_{z_8}$ are $SU(2)^4$ **resolution divisors**

Self-Dual Genus-one Example

- We can map the CICY equations:

$$p_1 = a_2 z_0^2 z_4^2 z_6 + a_4 z_0 z_1 z_5^2 z_8 + a_1 z_0 z_3 z_4 z_6 + a_3 z_0 z_2 z_5 z_8 + a_0 z_3^2 z_6 \,,$$
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- into the **Weierstrass form** (Jacobian) with (f, g, Δ) [Braun, Grimm, Keitel'15]

$$f = A_4 - \frac{1}{3}{A_2}^2 \,, \quad g = \frac{1}{27} A_2 (2{A_2}^2 - 9A_4) \,, \qquad (1)$$
$$\Delta = {A_4}^2 (4A_4 - {A_2}^2) \,,$$

- with the **birational** replacement

$$A_2 \rightarrow 4a_1 a_4 a_5 a_6 + a_3^2 a_6^2 - 2a_1 a_3 a_6 a_7 + {a_1}^2 {a_7}^2 - 4a_0 a_2 {a_7}^2 - 4{a_1}^2 a_5 a_8$$
$$+16a_0 a_2 a_5 a_8 - 8a_0 a_4 a_5 a_9 + 4a_0 a_3 a_7 a_9 \,, \qquad (2)$$
$$A_4 \rightarrow 16a_0 a_5 ({a_4}^2 a_5 - a_3 a_4 a_7 + {a_3}^2 a_8) \cdot (a_2 {a_6}^2 - a_1 a_6 a_9 + a_0 {a_9}^2) \,.$$

- Eqn. (1) is the general Weierstrass Form admitting $\mathbb{Z}_2$ torsion [Aspinwall,Morrison'98]

$$\boxed{G_{(122,0)} = \frac{SU(2)^4}{\mathbb{Z}_2} \times \mathbb{Z}_2}$$

Summary and Outlook

Conjecture 1. *Given a genus-one fiber $\mathcal{E}$ for which the Mordell-Weil group of the Jacobian contains torsion, the mirror dual is a genus-one fiber $\mathcal{E}'$ without a section and vice versa.*

Physics: Duality between discrete-and quotient symmetries in the low energy

Summary and Outlook

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Proof/Check of the Conjecture

- **Explicitly checked** for all 3145 codimension-two genus 1 curves
  - **Equivalent realizations** of the same elliptic curve
  - **Genus-one fibers with torsion** in the Jacobian (+ **Self-Mirrors**)

Outlook

- **Proof** the conjecture **in general?**
- What is the **physical explanation?**
- **Use classification of torsion for** classification of **discrete symmetries**?