
SHAPE DEPENDENCE OF RÉNYI ENTROPIES

And other conformal defects

arXiv:1607.07418

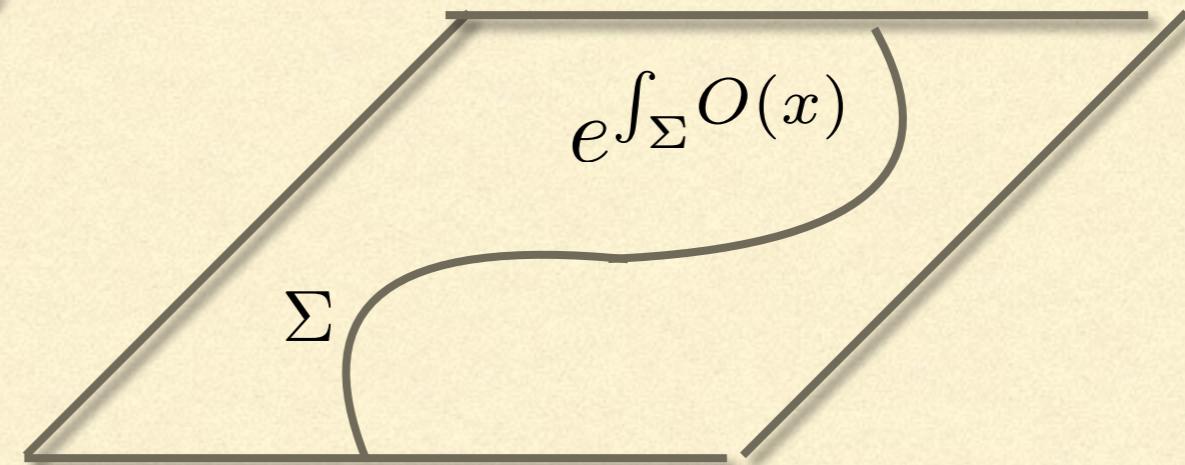
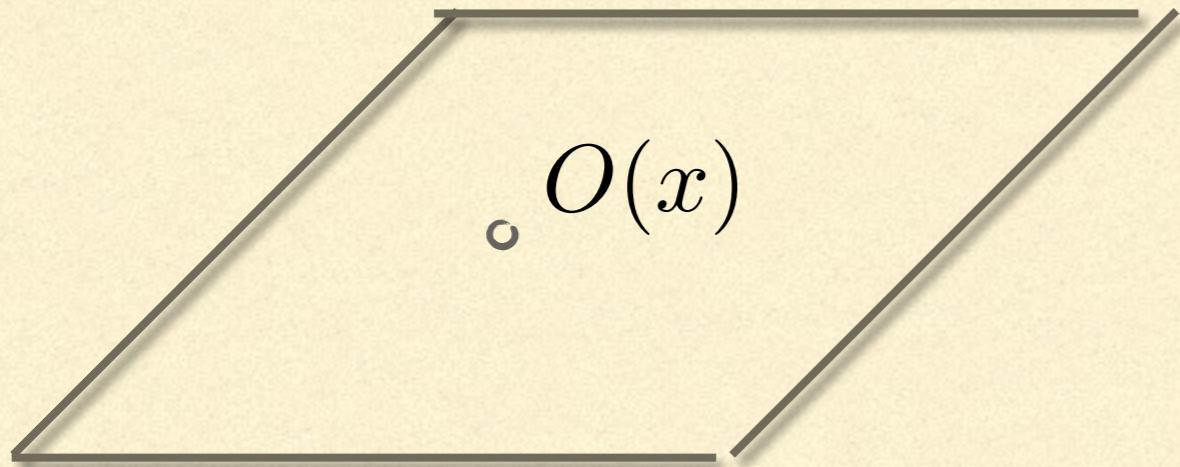
w/ Lorenzo Bianchi, Shira Chapman, Xi Dong, Damian Galante, Rob Myers

OVERVIEW

- Conformal defects and their deformation
 - Entanglement in QFT and Rényi entropies
 - Holographic Rényi entropies and black holes
 - Away from planes and spheres
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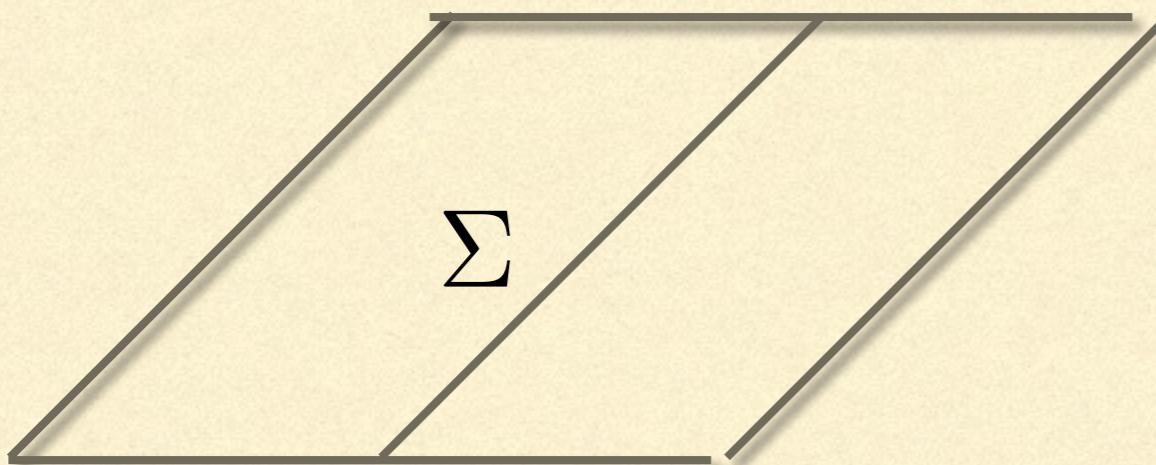
■ Conformal defects and their deformation (I)

Defects are modifications of a theory localised on a hypersurface:



- We consider very symmetrical shapes: planes or spheres.
- Also, very symmetrical theories: Conformal field theories.
- Even more, the extended operator itself is very symmetric

Call this a defect CFT



For instance

$$S = \int d^4x \frac{1}{2}(\partial\phi)^2 + \int_{\Sigma} \phi$$

■ Conformal defects and their deformation (2)

New defect OPE channel:

The diagram illustrates the decomposition of a defect operator $O(x)$ into a sum of deformed operators. On the left, a parallelogram represents a defect configuration with a circular defect at its center labeled $O(x)$. This is followed by an equals sign. To the right of the equals sign is a summation symbol $\sum_{\hat{O}}$, indicating that the original operator is equal to a sum over all possible deformed configurations. On the far right, another parallelogram shows a similar defect configuration, but the circular defect has been deformed into an oval shape labeled $\hat{O}(x)$.

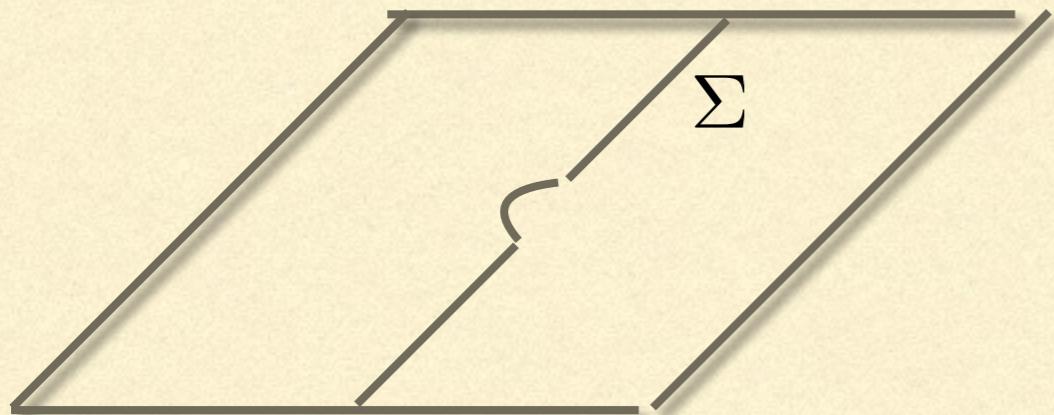
The **displacement operator**:

The diagram illustrates the displacement operator $D(x)$ and its relation to the stress-energy tensor $T^{\mu a}$. On the left, a parallelogram contains a curved line labeled Σ . This is followed by a tilde symbol ($\tilde{=}$) indicating equivalence. To the right of the tilde is another parallelogram containing a point labeled $D(x)$. Below the first parallelogram, there is a coordinate system with two axes: one horizontal axis labeled i and one vertical axis labeled a . Below the second parallelogram, the equation $\partial_\mu T^{\mu a}(x) = \delta_\Sigma(x) D^a$ is written, relating the divergence of the stress-energy tensor to the displacement operator.

- Conformal defects and their deformation (3)

The **displacement** operator: $\partial_\mu T^{\mu a}(x) = \delta_\Sigma(x) D^a$

$$\langle D(w)D(w') \rangle = \frac{C_D}{(w - w')^{2(d-1)}}$$



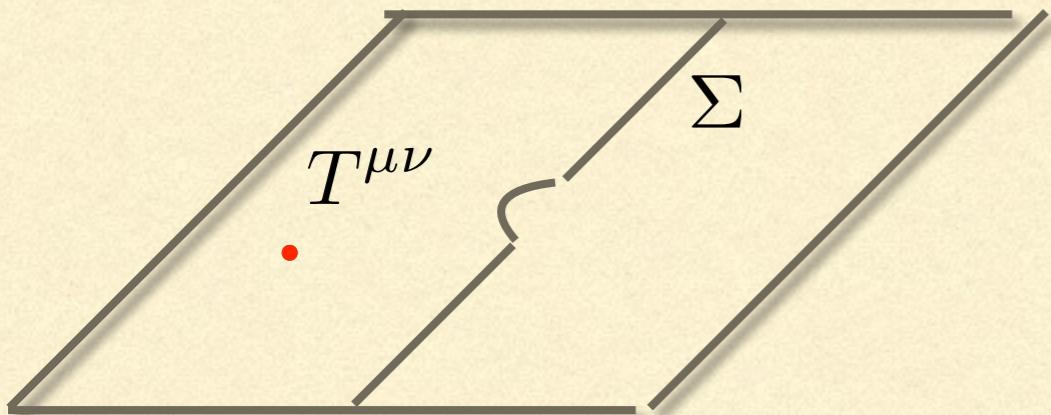
$$\delta_\epsilon \langle \dots \rangle = \epsilon \langle D \dots \rangle$$

$$\delta \log Z \propto \epsilon^2 C_D$$

■ Conformal defects and their deformation (3)

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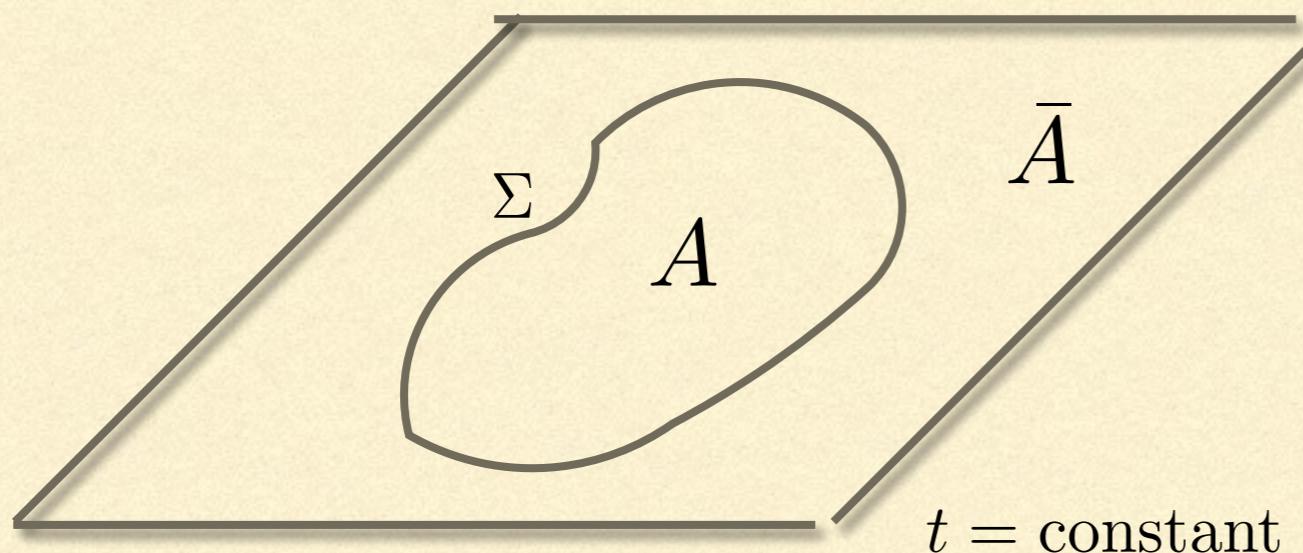
$$\delta \log Z \propto \epsilon^2 C_D$$

$$\delta \langle T^{\mu\nu} \rangle = \epsilon \langle D T^{\mu\nu} \rangle \sim \epsilon C_D$$

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- Entanglement in QFT and Rényi entropies (I)



$$\rho_A = \text{Tr}_{\bar{A}} \rho$$

- Entanglement is about missing information

$$S_n = \frac{1}{1-n} \log \text{Tr} \rho_A^n$$

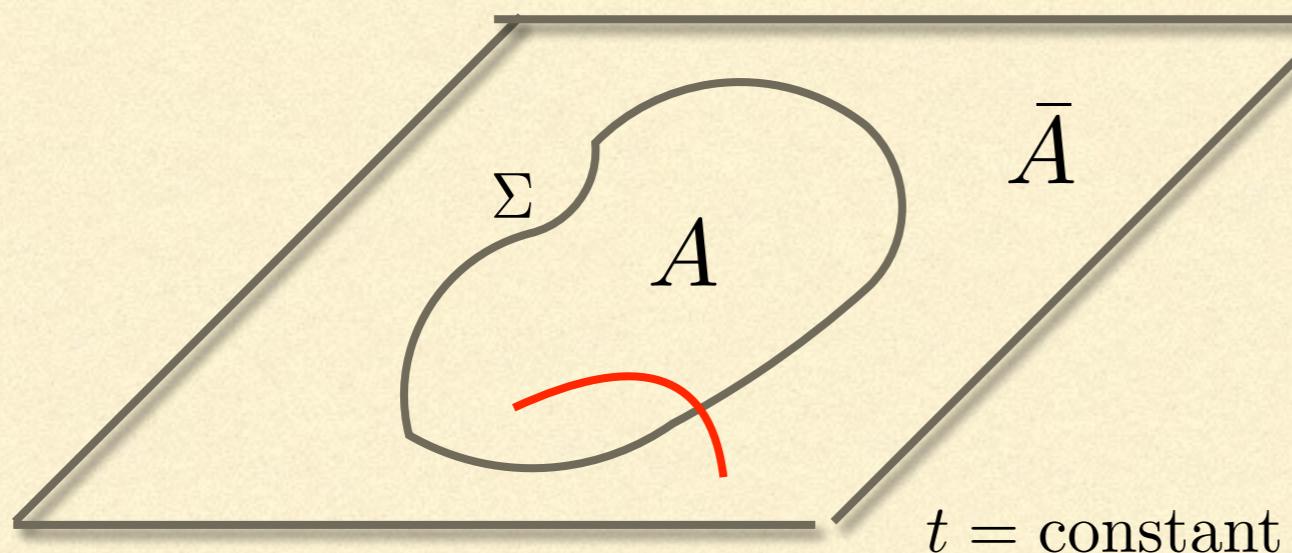
Rényi entropies

$$S_{\text{EE}} = -\text{Tr} \rho_A \log \rho_A$$

Entanglement entropy

- Entanglement is about correlations
-

- Entanglement in QFT and Rényi entropies (I)



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Rényi entropies

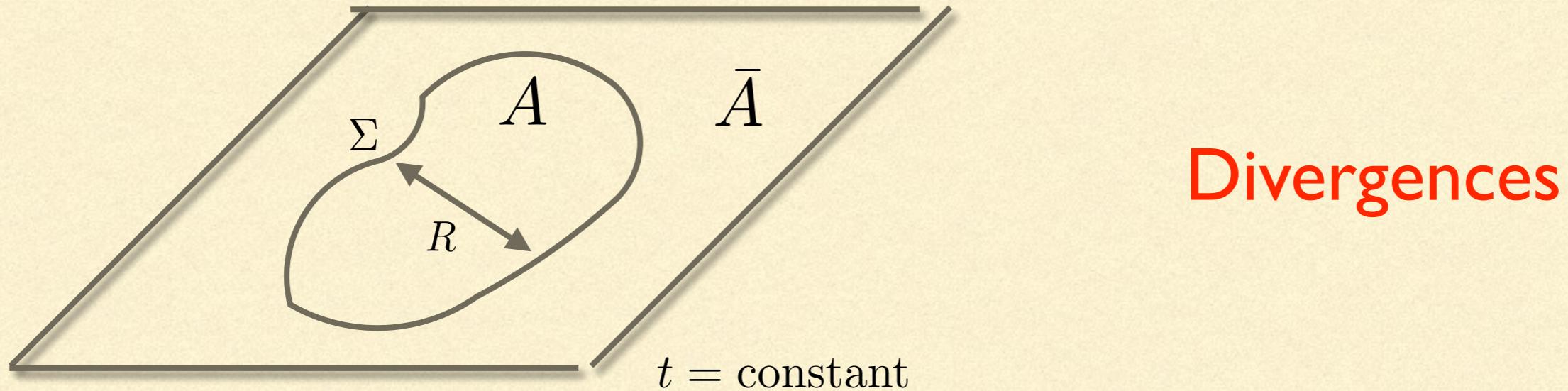
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Entanglement entropy

- Entanglement is about correlations:

Divergences

■ Entanglement in CFT and Rényi entropies (2)



Divergences

$$S_n = \begin{cases} \left(\frac{R}{\epsilon_{\text{UV}}}\right)^{d-2} + \dots + \frac{R}{\epsilon_{\text{UV}}} + s_d^{(\Sigma)} + \frac{\epsilon_{\text{UV}}}{R} + \dots & \text{odd } d \\ \left(\frac{R}{\epsilon_{\text{UV}}}\right)^{d-2} + \dots + \left(\frac{R}{\epsilon_{\text{UV}}}\right)^2 + s_d^{(\Sigma)} \log \frac{R}{\epsilon_{\text{UV}}} + \text{const} + \left(\frac{\epsilon_{\text{UV}}}{R}\right)^2 + \dots & \text{even } d \end{cases}$$

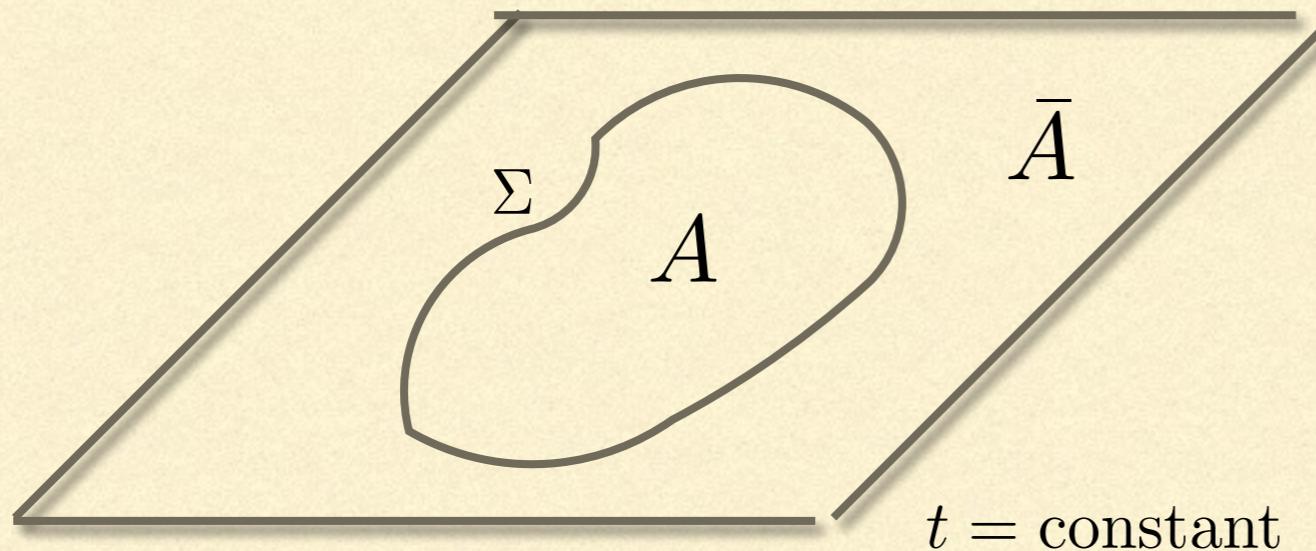
[H. Liu, M. Mezei]

For instance, in $d = 4$

$$s_4^{(\Sigma)} = -\frac{f_a(n)}{2\pi} \int_{\Sigma} R_{\Sigma} - \frac{f_b(n)}{2\pi} \int_{\Sigma} \tilde{K}_{ab}^i \tilde{K}_{ab}^i + \frac{f_c(n)}{2\pi} \int_{\Sigma} \gamma^{ab} \gamma^{cd} C_{acbd}$$

[S. N. Solodukhin]

- Entanglement in QFT and Rényi entropies



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Rényi entropies

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Entanglement entropy

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■ Entanglement in QFT and Rényi entropies (3)

$$S_n = \frac{1}{1-n} \log \text{Tr} \rho_A^n$$

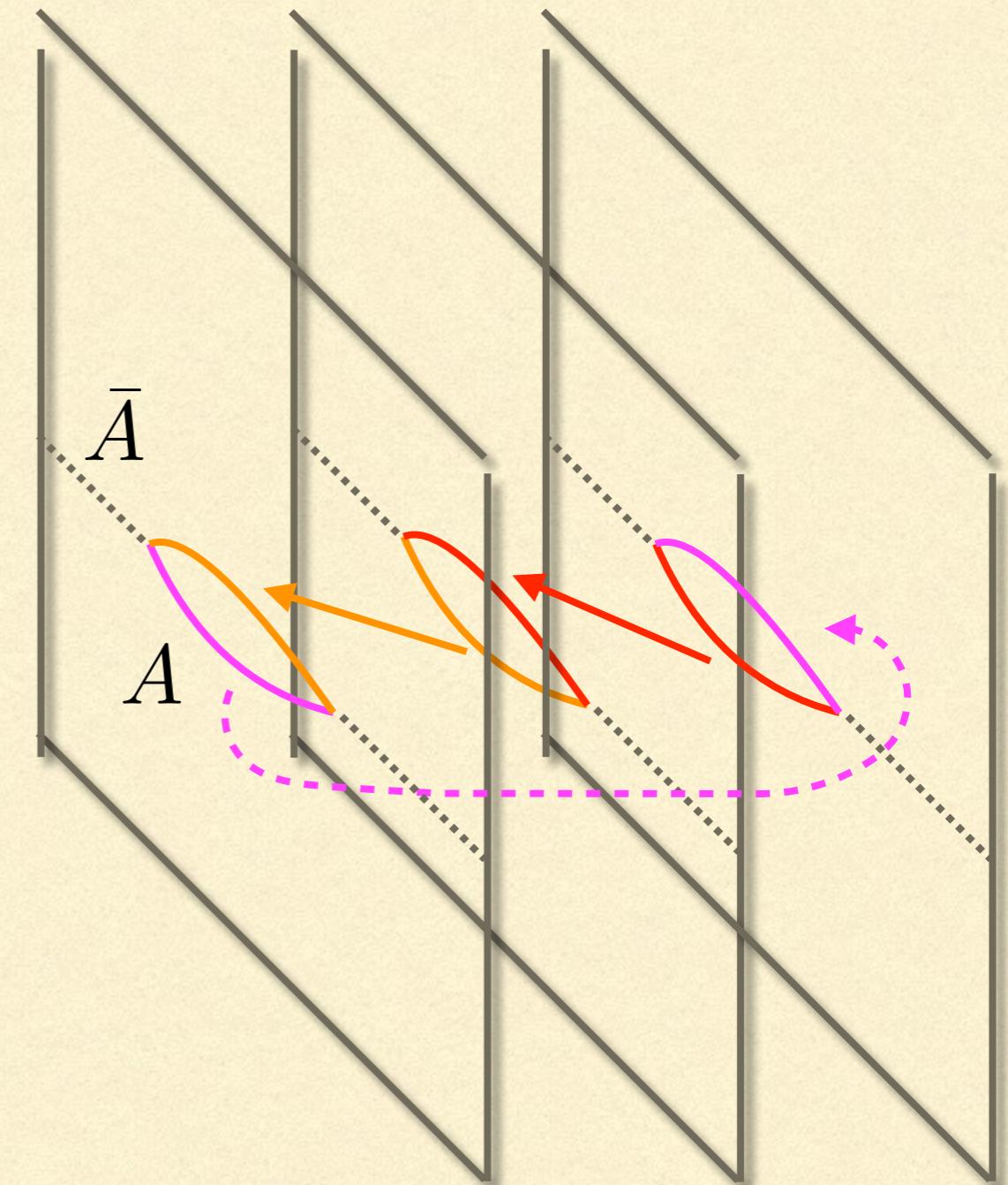
$$\rho_A(\phi_A^-, \phi_A^+) = \text{Tr}_{\bar{A}} \Psi_0(\phi_A^-, \phi_{\bar{A}}) \Psi_0^*(\phi_A^+, \phi_{\bar{A}})$$

$\text{Tr} \rho_A^n ?$

Replica trick

Translations only broken at Σ

Rényi entropies



■ Entanglement in QFT and Rényi entropies (3)

$$S_n = \frac{1}{1-n} \log \text{Tr} \rho_A^n$$

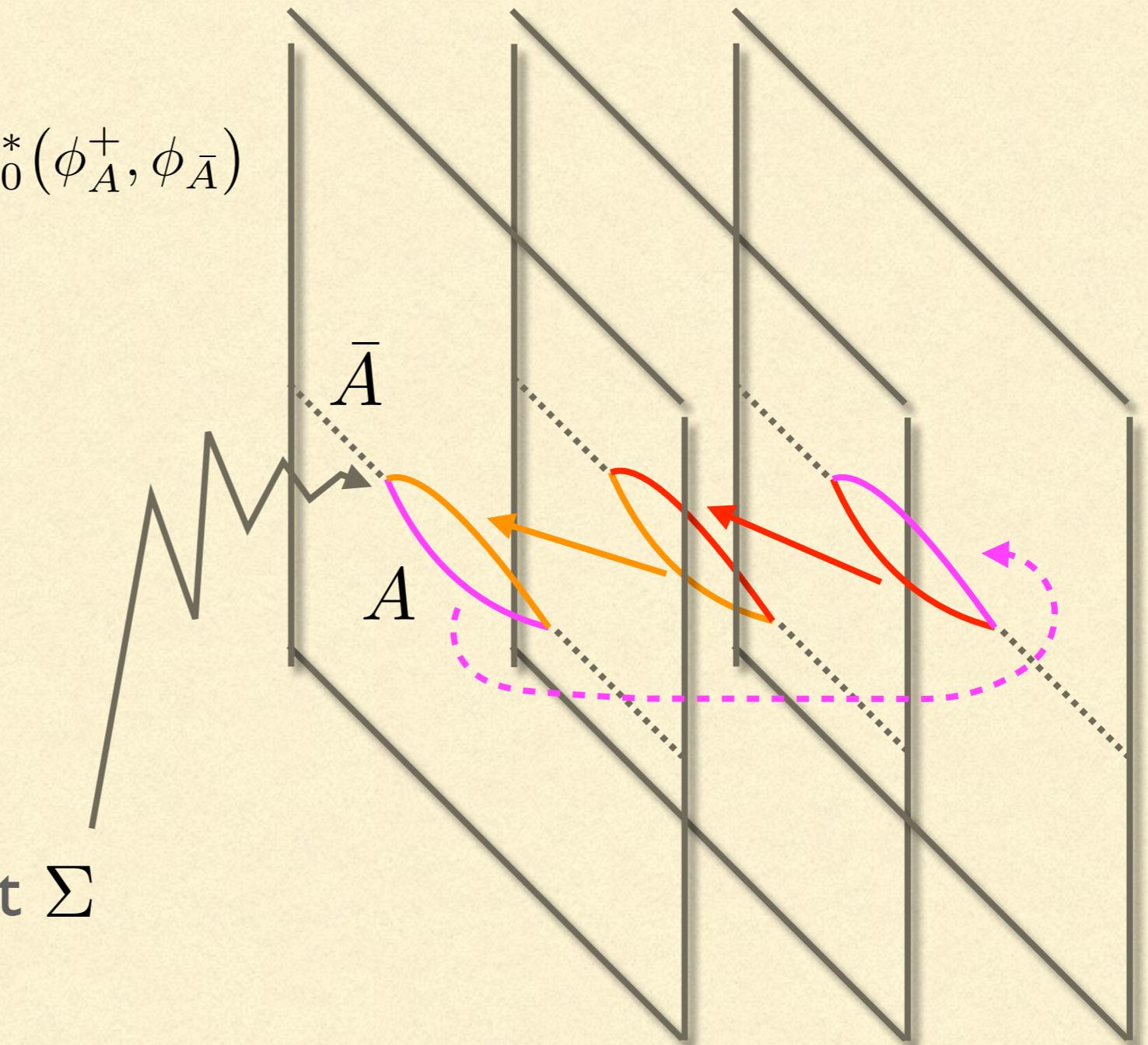
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■ Entanglement in QFT and Rényi entropies (3)

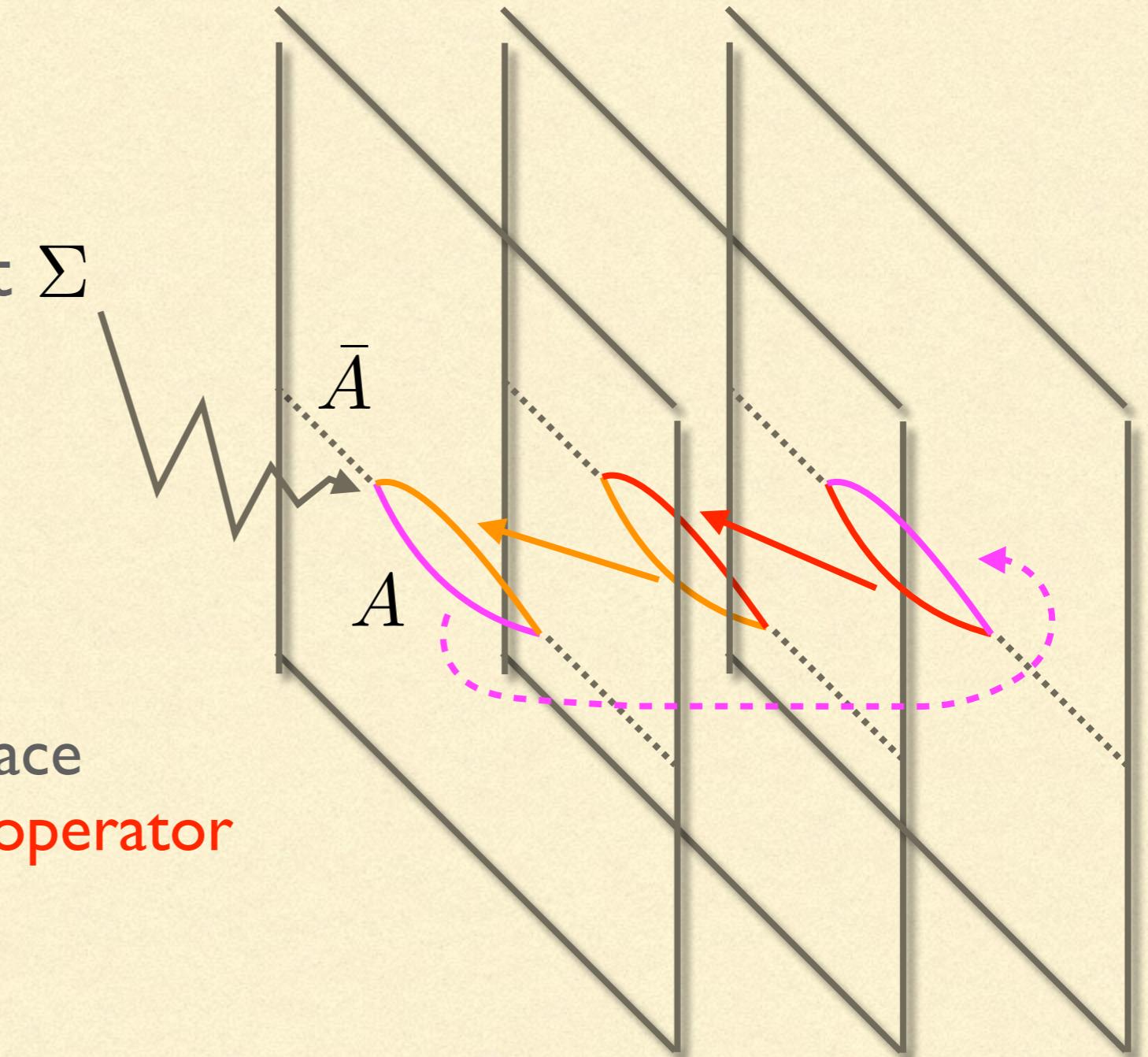
$$S_n = \frac{1}{1-n} \log \text{Tr} \rho_A^n$$

Rényi entropies

Translations only broken at Σ

S_n is the free energy

- Of the QFT on a conical space
- Of the $(\text{QFT})^n$ with a **twist operator**

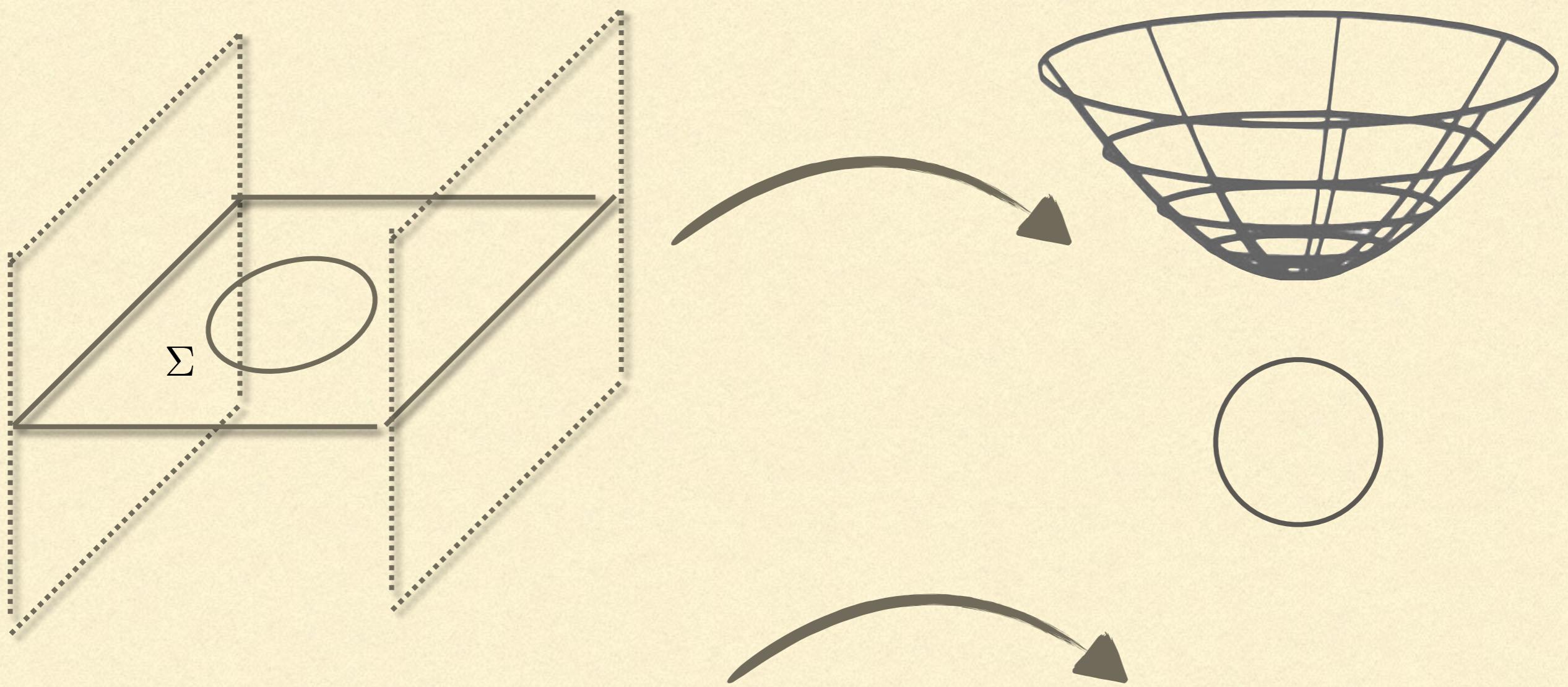


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■ Holographic Rényi entropies and black holes (I)

A useful conformal map:

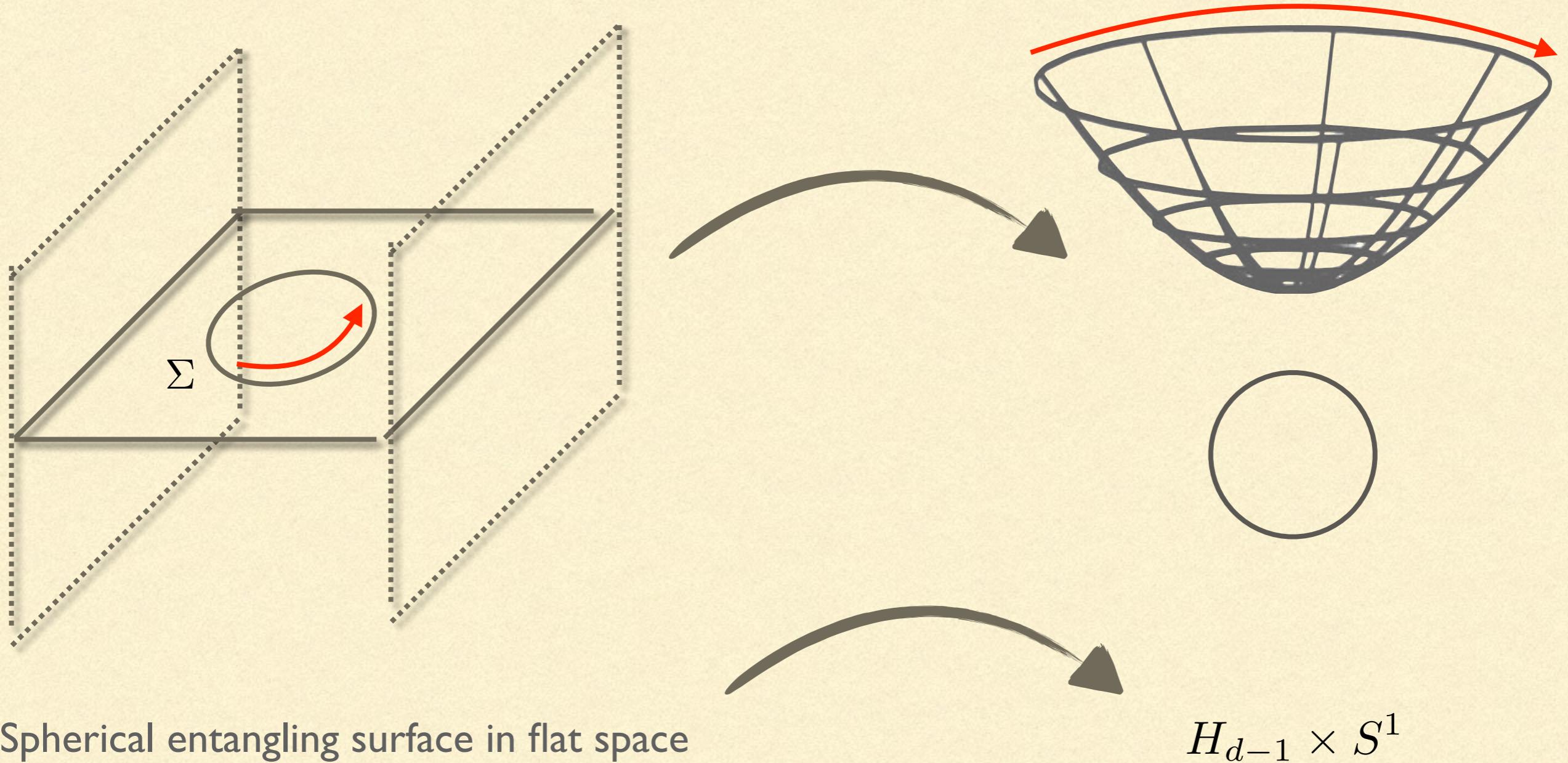


Spherical entangling surface in flat space

$H_{d-1} \times S^1$

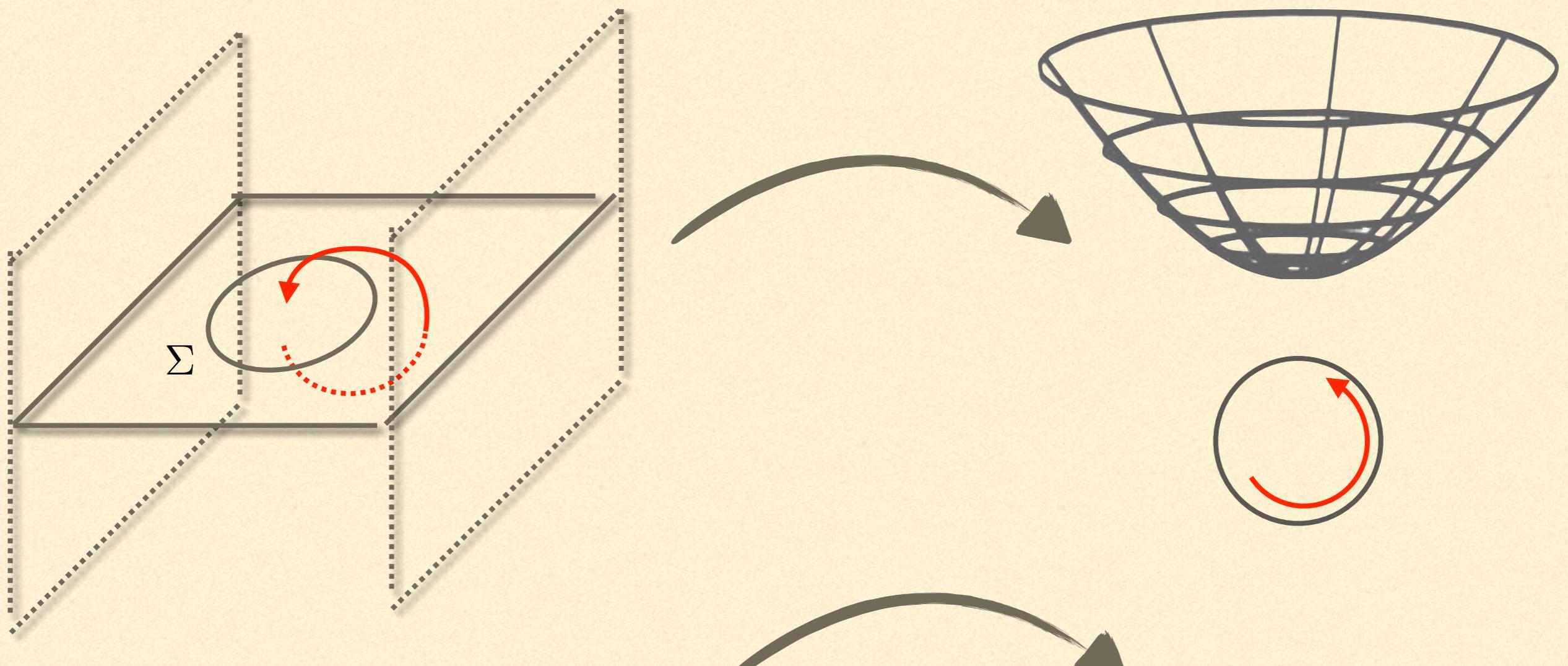
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A useful conformal map:



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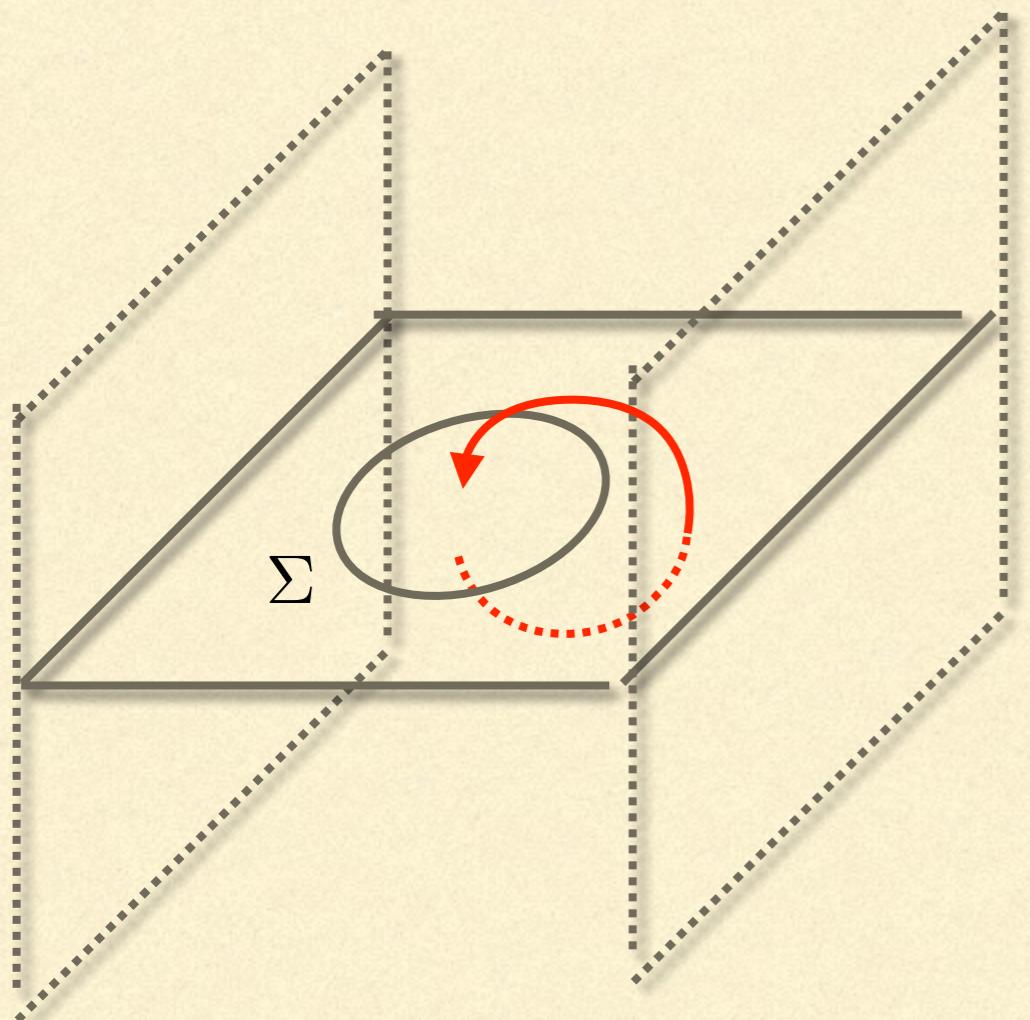
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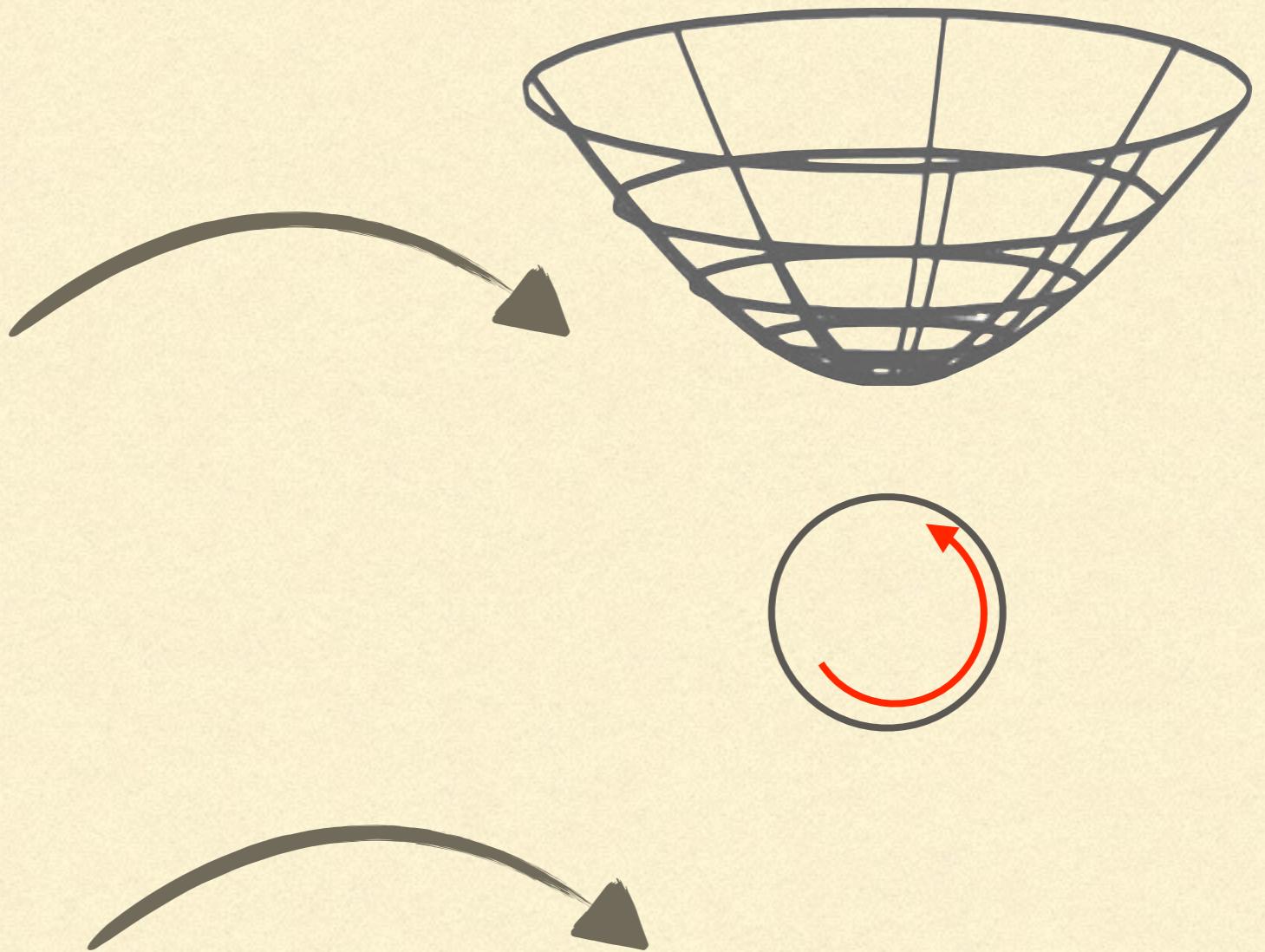
[H. Casini, M. Huerta, R. Myers]
[J. Hung, R. Myers, M. Smolkin, A. Yale]

A useful conformal map:



Spherical entangling surface in flat space

Thermal space



■ Holographic Rényi entropies and black holes (I)

[H. Casini, M. Huerta, R. Myers]
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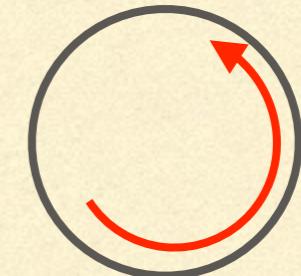
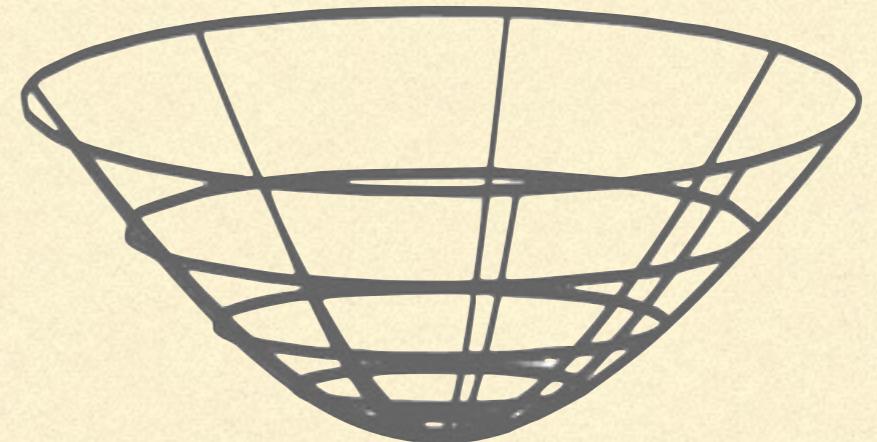
Rényi entropy across a sphere of radius R

\simeq

Thermal free energy on hyperbolic space

$$\text{Temperature} = \frac{1}{2\pi nR}$$

Thermal space

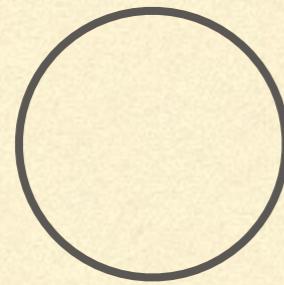
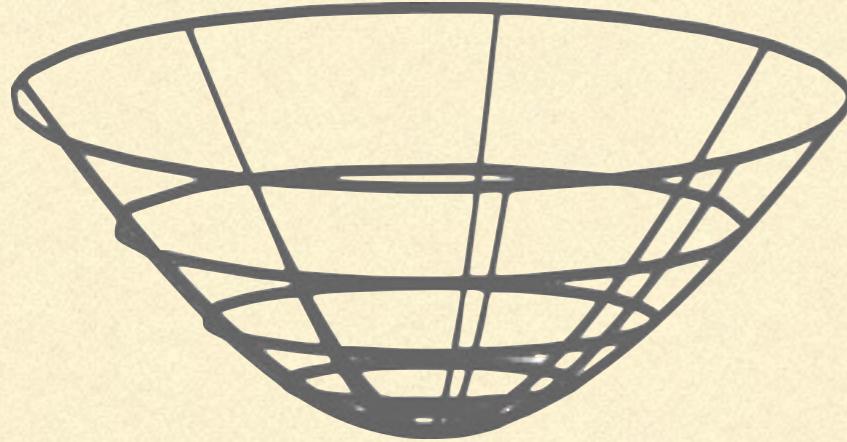


$$H_{d-1} \times S^1$$

■ Holographic Rényi entropies and black holes (2)

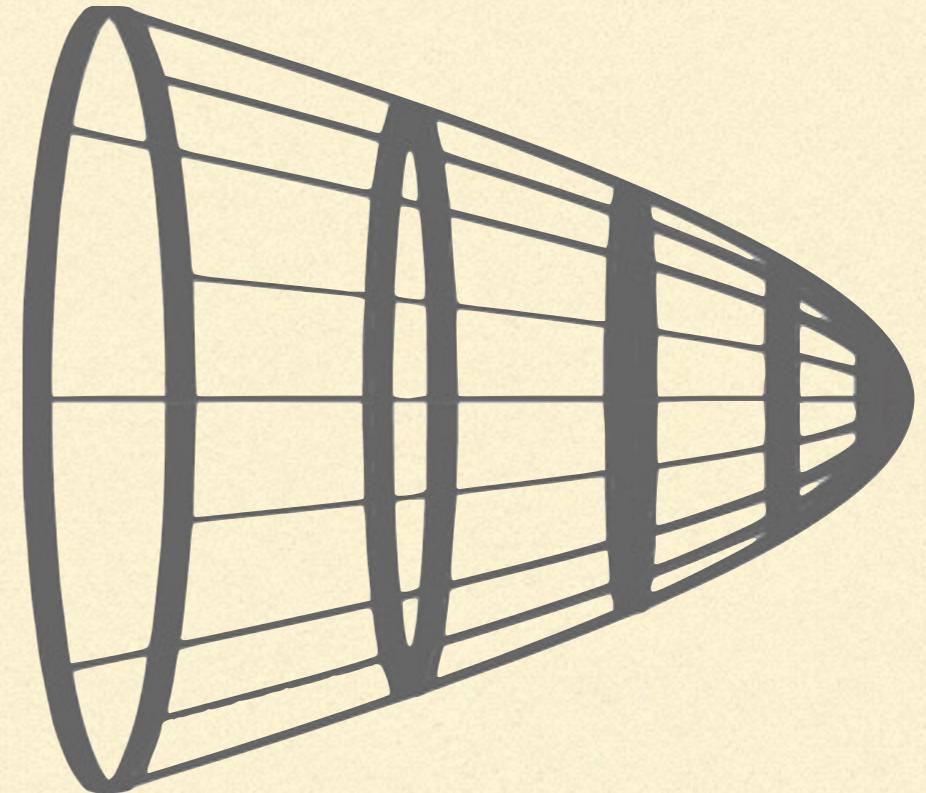
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Boundary



$H_{d-1} \times S^1$

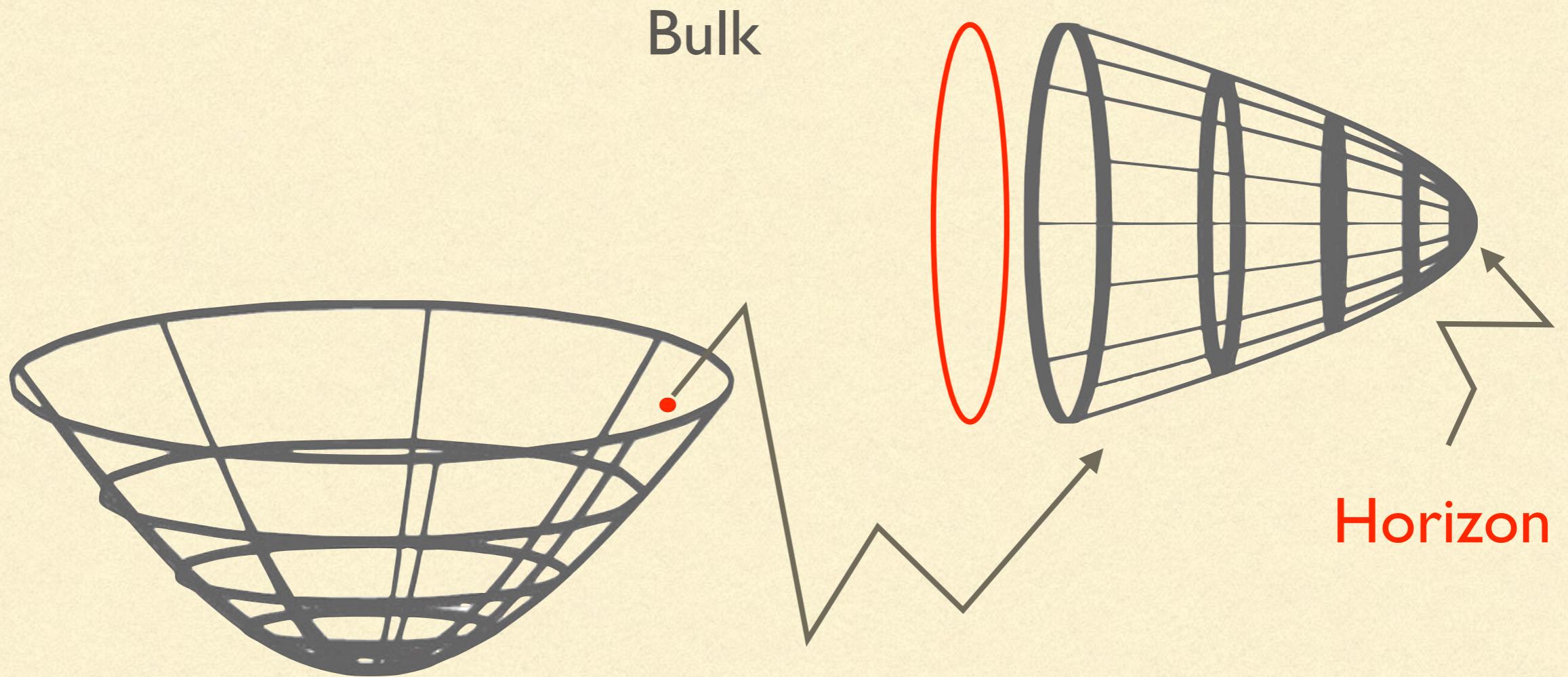
Bulk



$$ds^2 = \frac{1}{\left(\frac{r^2}{R^2} f(r) - 1\right)} dr^2 + \left(\frac{r^2}{R^2} f(r) - 1\right) d\tau^2 + r^2 dH_{d-1}^2$$

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[H. Casini, M. Huerta, R. Myers]
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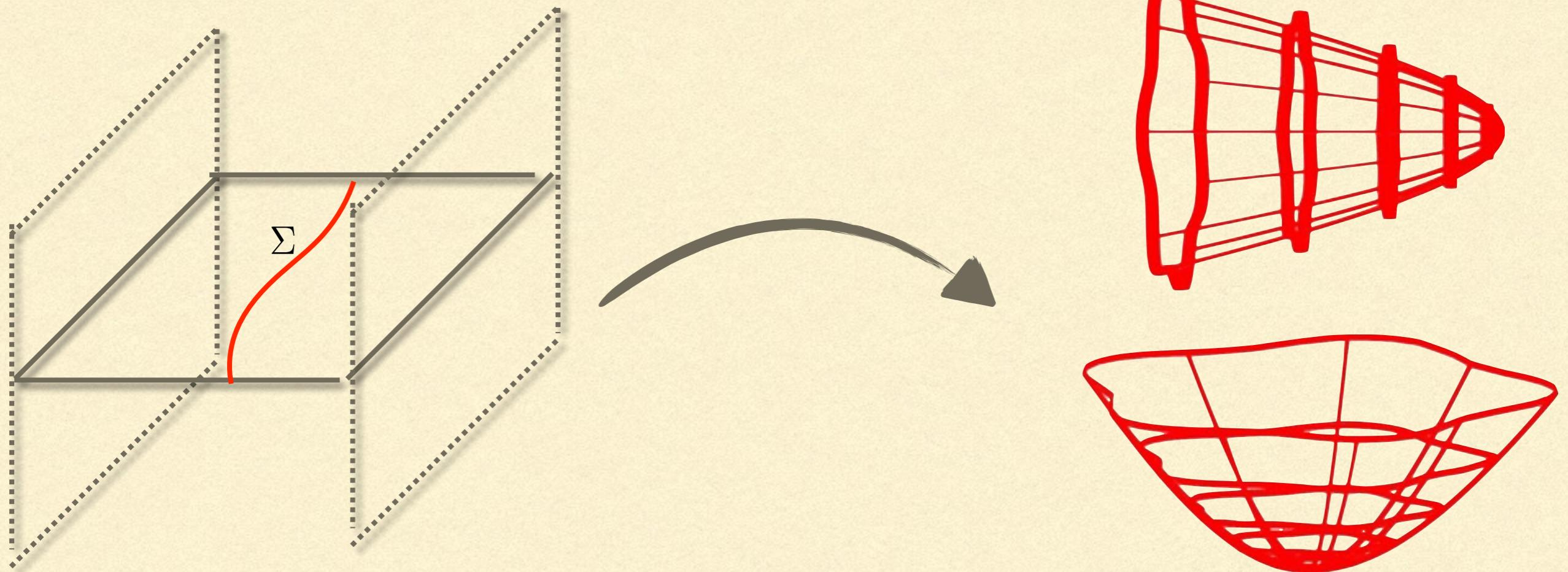


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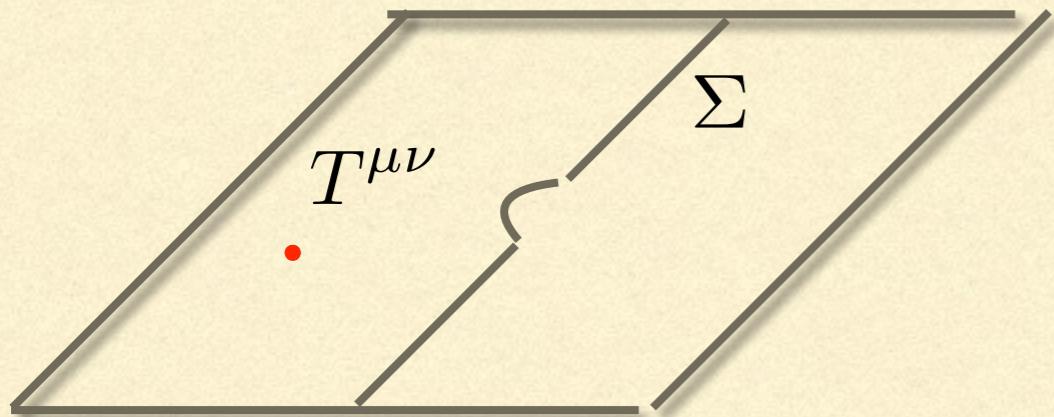


$$\delta S_n \propto \delta \log Z \propto \epsilon^2 C_D$$

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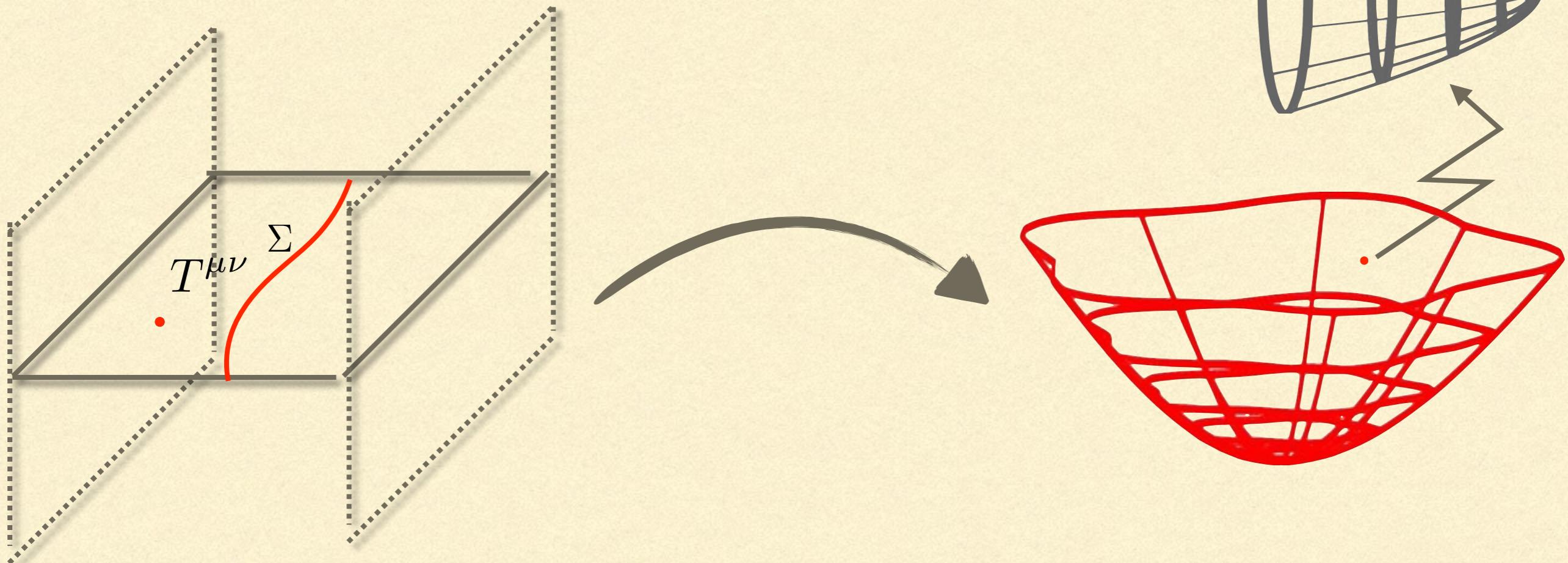
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■ Away from planes and spheres (2)

$$\delta \langle T^{\mu\nu} \rangle = \epsilon \langle D T^{\mu\nu} \rangle \sim \epsilon C_D$$



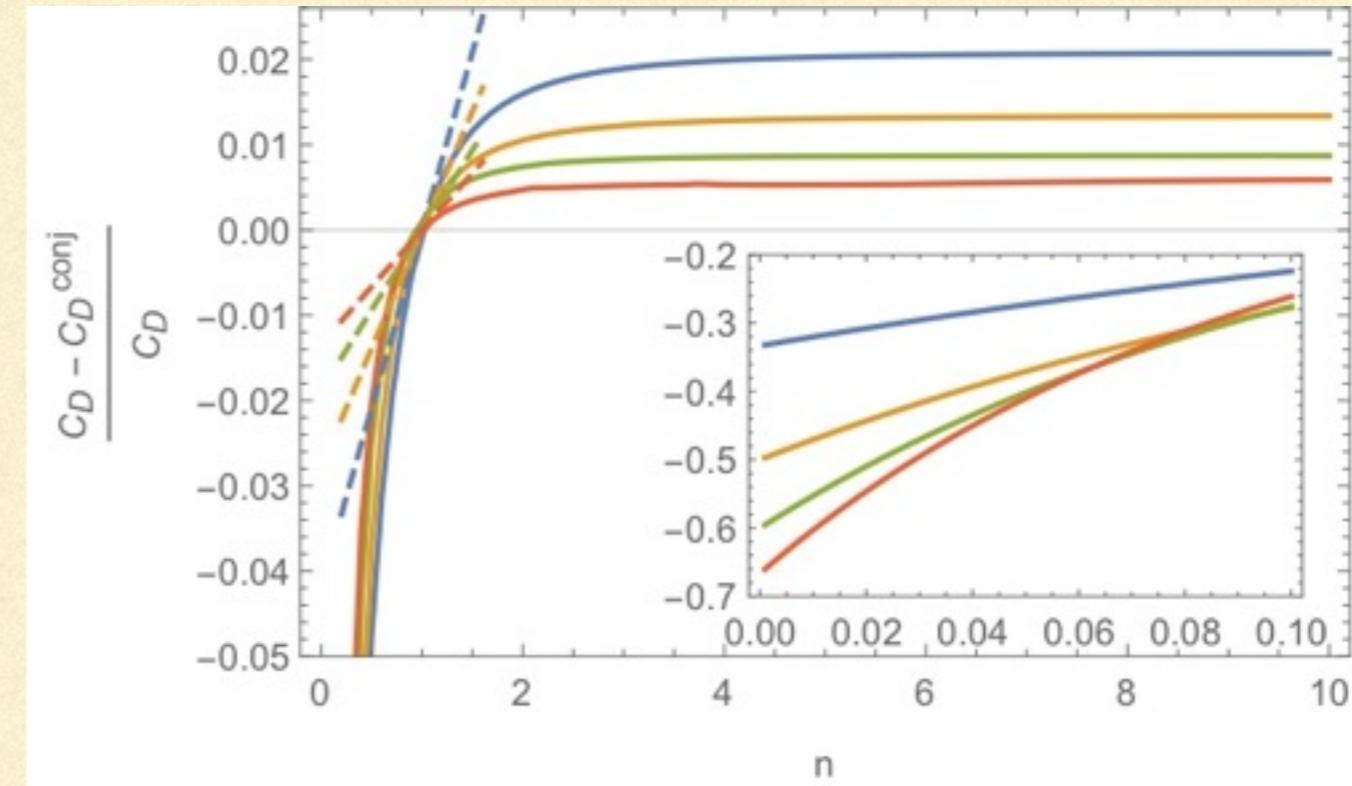
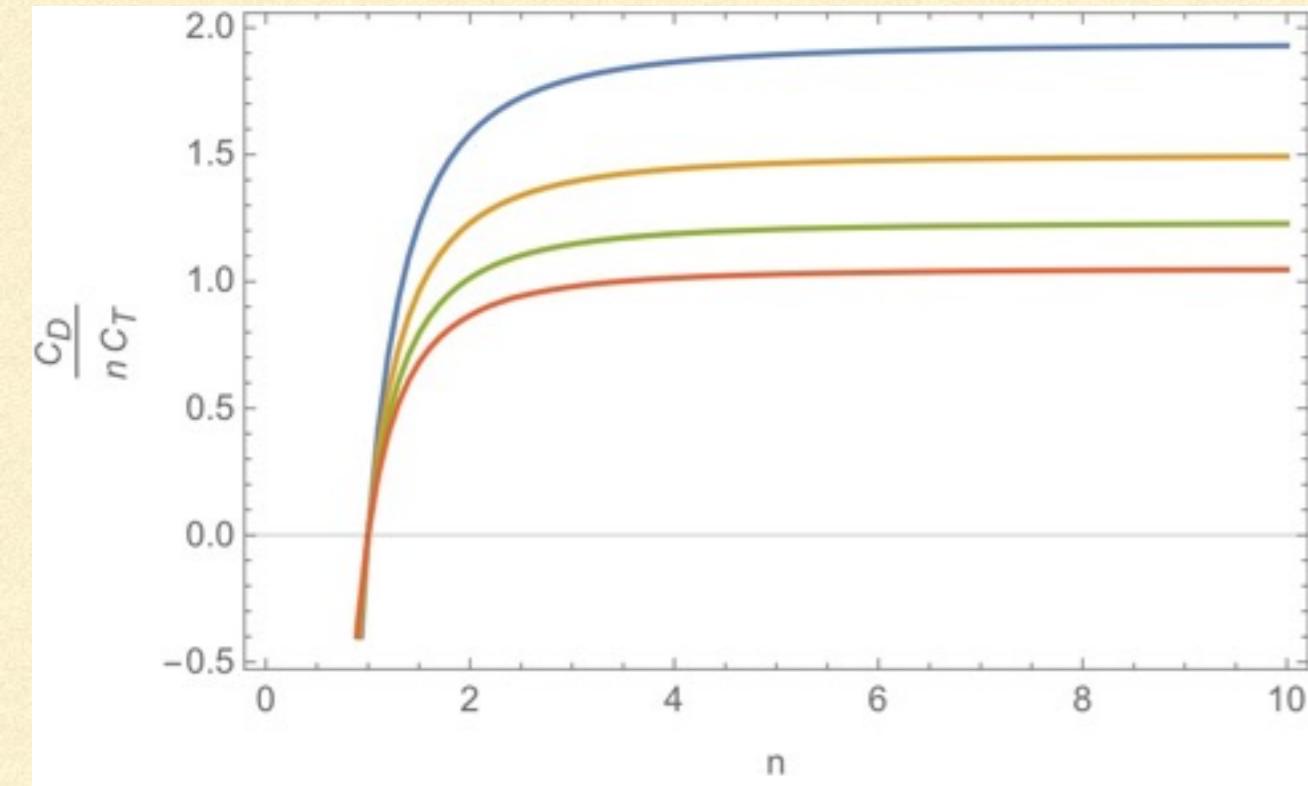
$$ds^2 = \frac{dr^2}{\frac{r^2}{R^2} f(r) - 1} + \left(\frac{r^2}{R^2} f(r) - 1 \right) d\tau^2 + \frac{r^2}{\rho^2} \left(d\rho^2 + [\delta_{ij} + 2 k(r) \tilde{K}_{ij}^a x_a] dy^i dy^j + \frac{4}{d-2} v(r) \partial_i K^b x_b \rho d\rho dy^i \right)$$

$+ \dots$

[X. Dong]

- Away from planes and spheres (2)

Einstein gravity (can do for Einstein-Gauss-Bonnet)



$$C_D^{\text{conj}}(n) = d \Gamma\left(\frac{d+1}{2}\right) \left(\frac{2}{\sqrt{\pi}}\right)^{d-1} h_n$$

$$\langle T^{\mu\nu} \rangle \propto h_n$$

[L. Bianchi, M. M., R. Myers, M. Smolkin]

$$\lim_{n \rightarrow 0} \frac{C_D - C_D^{\text{conj}}}{C_D} = -\frac{d-2}{d}$$

Universal?

■ Away from planes and spheres (2)

Analytics close to $n = 1$. λ is the Gauss-Bonnet coupling

$$\frac{C_D^{(d=4)} - C_D^{\text{conj}(d=4)}}{C_D^{(d=4)}} = \left(\frac{1}{9\sqrt{1-4\lambda}} - \frac{1}{12} \right) (n-1) + O(n-1)^2,$$

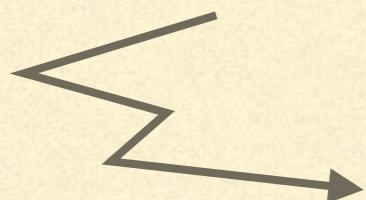
$$\frac{C_D^{(d=5)} - C_D^{\text{conj}(d=5)}}{C_D^{(d=5)}} = \left(\frac{1}{16\sqrt{1-4\lambda}} - \frac{7}{160} \right) (n-1) + O(n-1)^2,$$

$$\frac{C_D^{(d=6)} - C_D^{\text{conj}(d=6)}}{C_D^{(d=6)}} = \frac{1}{75} \left(\frac{3}{\sqrt{1-4\lambda}} - 2 \right) (n-1) + O(n-1)^2.$$

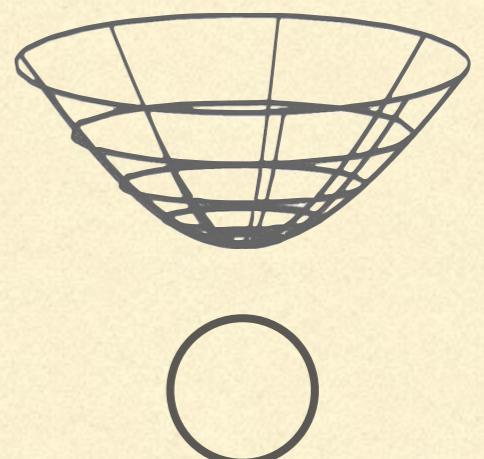
Conjecture is true for entanglement entropy

[T. Faulkner, R. Leigh, O. Parrikar]

Next order vanishes for $\lambda = \lambda_{\min}$



Determined by $\langle T^{\mu\nu} T^{\rho\sigma} T^{\lambda\tau} \rangle$



■ Conclusions

- Rényi entropies across slightly deformed planes and spheres can be computed. Actually, in even dimensions C_D controls (part of) the result for finite deformations too.
 - A conjecture, true for entanglement entropy, is disproven for Rényi entropies (but only mildly violated, and seems to be valid for free theories).
 - Universal behaviour as $n \rightarrow 0$ for holographic CFTs, but violated by free theories!
 - Thinking of Rényi entropies as conformal defects seems to be useful.
 - Method is much more general: would work as well for any defect whose holographic dual is known.
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Thank you!
