SHAPE DEPENDENCE OF RÉNYI ENTROPIES

And other conformal defects

arXiv:1607.07418
w/ Lorenzo Bianchi, Shira Chapman, Xi Dong, Damian Galante, Rob Myers

Marco Meineri, EPFL

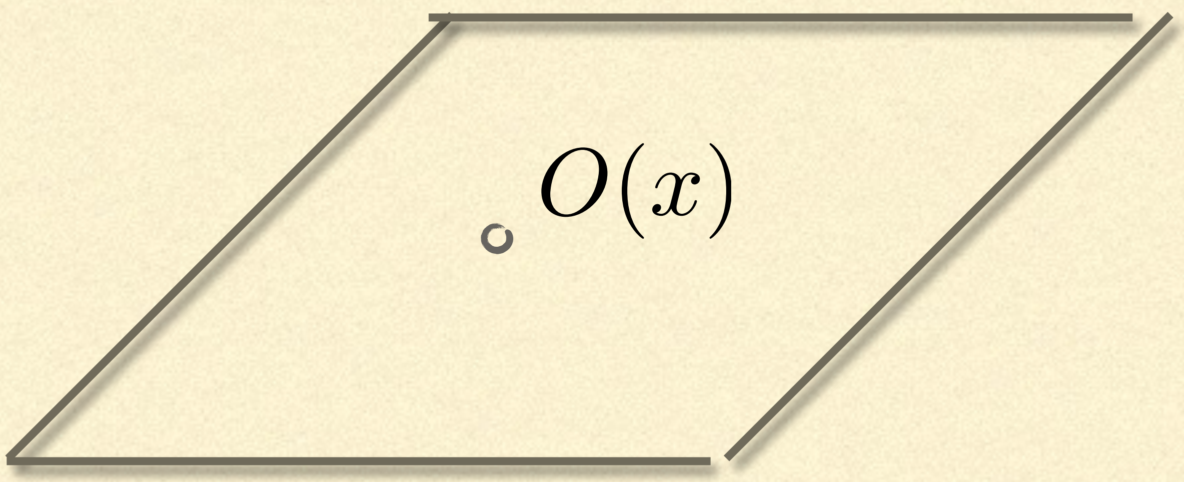
DESY theory workshop, 29/09/2016

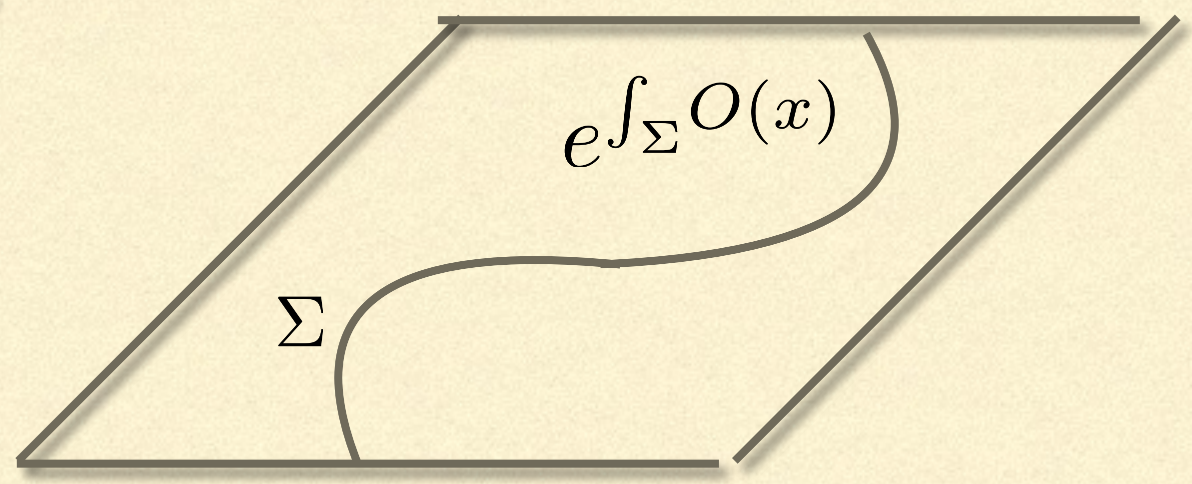
OVERVIEW

- Conformal defects and their deformation
- Entanglement in QFT and Rényi entropies
- Holographic Rényi entropies and black holes
- Away from planes and spheres

Conformal defects and their deformation (1)

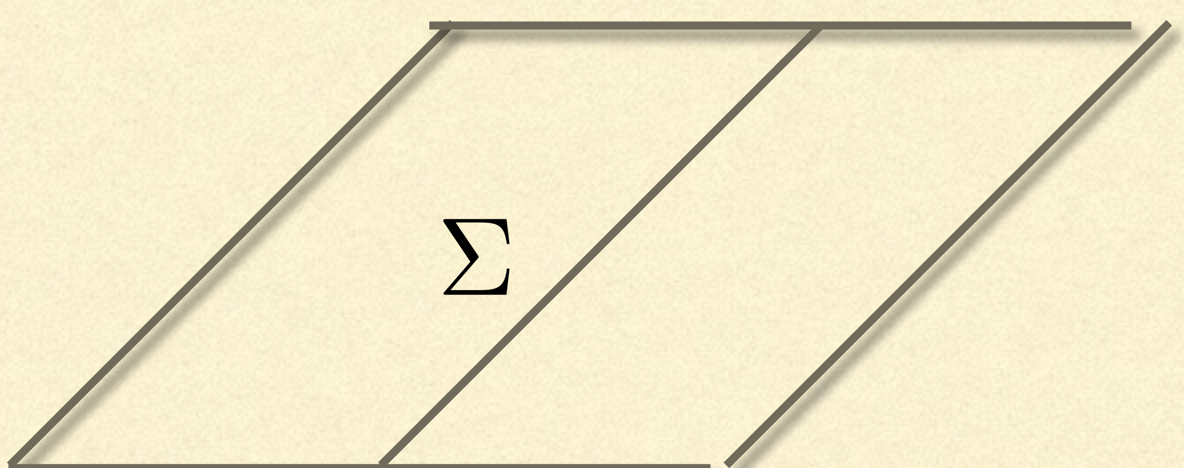
Defects are modifications of a theory localised on a hypersurface:





- We consider very symmetrical shapes: planes or spheres.
- Also, very symmetrical theories: Conformal field theories.
- Even more, the extended operator itself is very symmetric

Call this a defect CFT

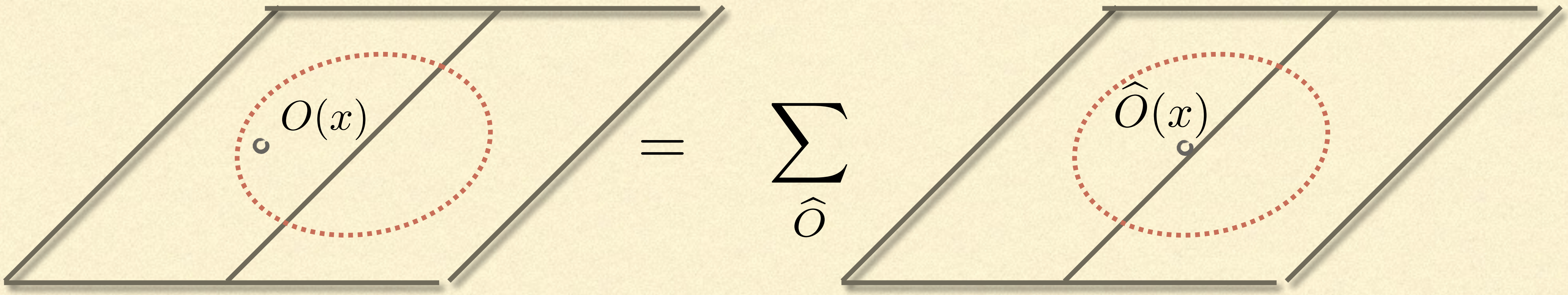


For instance

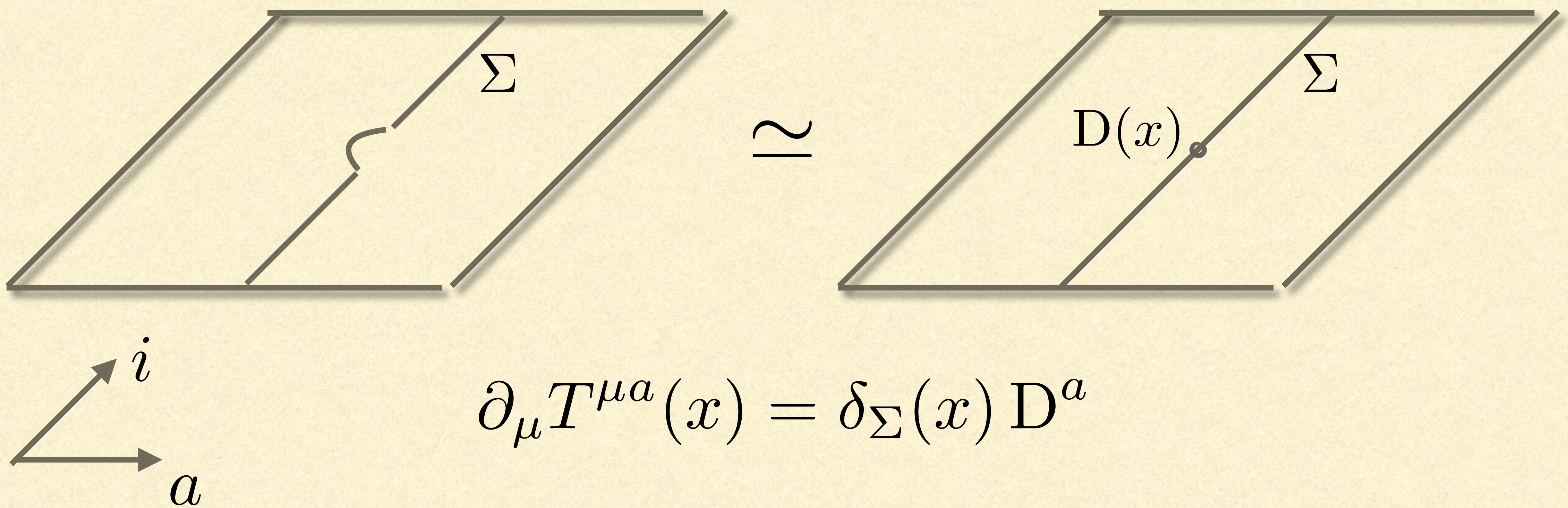
$$S = \int d^4x \, \frac{1}{2}(\partial\phi)^2 + \int_\Sigma \phi$$

Conformal defects and their deformation (2)

New defect OPE channel:



The displacement operator:

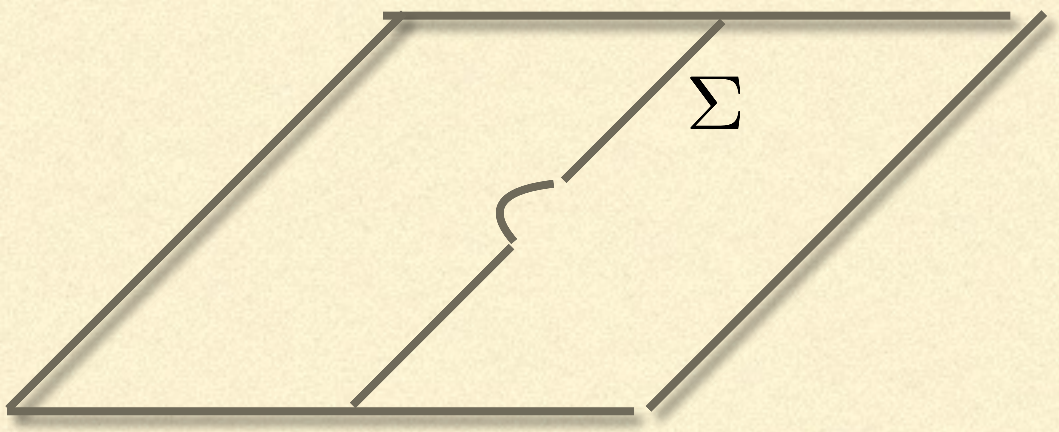


$$\partial_\mu T^{\mu a}(x) = \delta_\Sigma(x)\, \mathrm{D}^a$$

Conformal defects and their deformation (3)

The displacement operator: $\partial_\mu T^{\mu a}(x) = \delta_\Sigma(x)\, \mathrm{D}^a$

$$\langle \mathrm{D}(w)\mathrm{D}(w')\rangle = \frac{C_{\mathrm{D}}}{(w-w')^{2(d-1)}}$$



$$\delta_\epsilon \langle \cdots \rangle = \epsilon \langle \mathrm{D} \cdots \rangle$$

$$\delta \log Z \propto \epsilon^2 C_{\mathrm{D}}$$

Conformal defects and their deformation (3)

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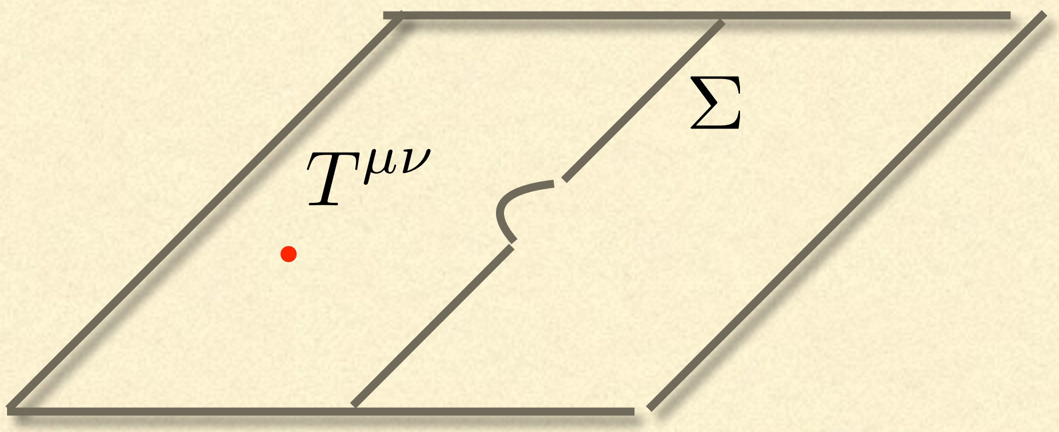

$$\delta_\epsilon \langle \cdots \rangle = \epsilon \langle \mathrm{D} \cdots \rangle$$

$$\delta \log Z \propto \epsilon^2 C_\mathrm{D}$$

$$\delta \langle T^{\mu\nu} \rangle = \epsilon \langle \mathrm{D}\, T^{\mu\nu} \rangle \sim \epsilon\, C_\mathrm{D}$$

[M. Billò, V. Gonçalves, E. Lauria, M.M.]

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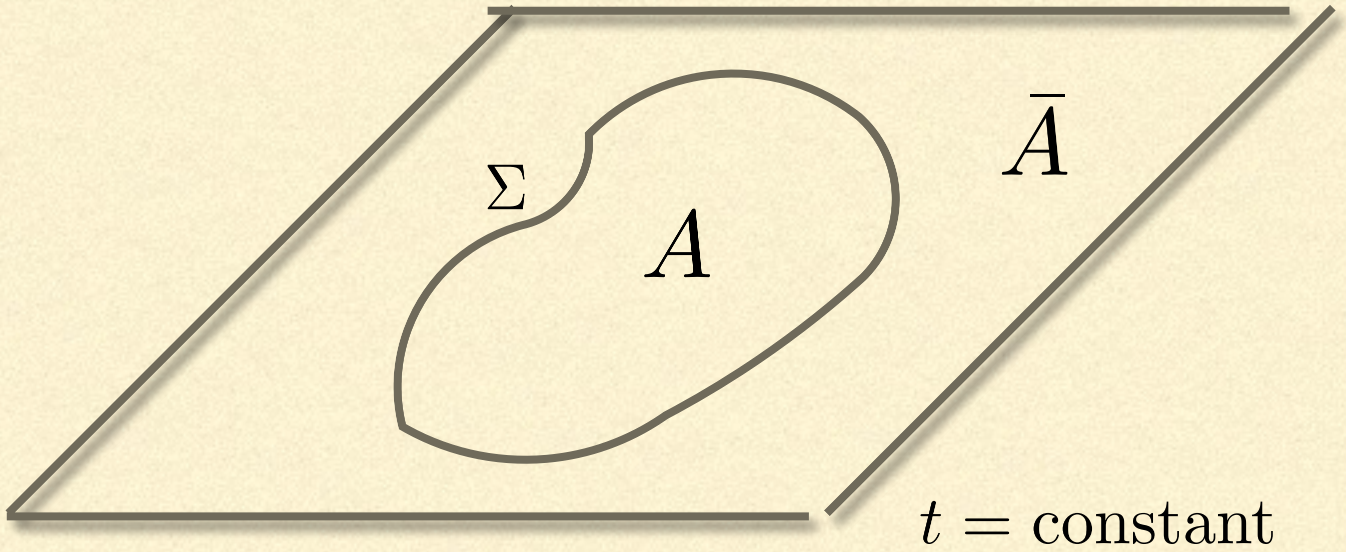
- Entanglement in QFT and Rényi entropies (I)



$$\rho_A = \mathrm{Tr}_{\bar{A}}\, \rho$$

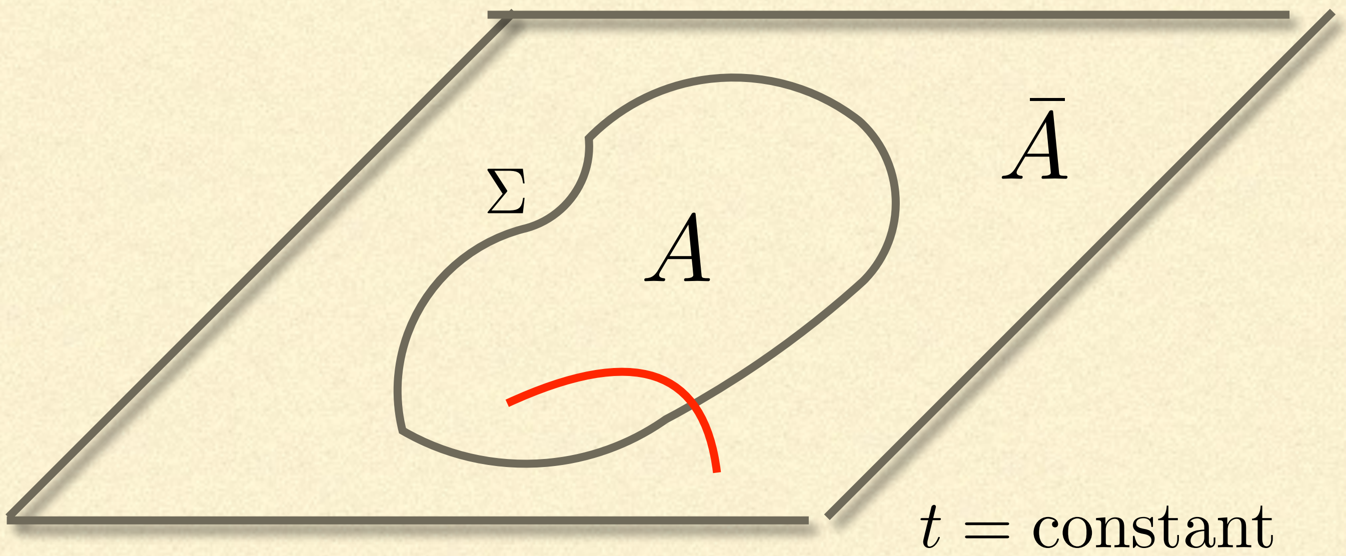
- Entanglement is about missing information

$$S_n = \frac{1}{1-n} \log \mathrm{Tr} \rho_A^n$$ Rényi entropies

$$S_{\mathrm{EE}} = -\mathrm{Tr} \rho_A \log \rho_A$$ Entanglement entropy

- Entanglement is about correlations

Entanglement in QFT and Rényi entropies (I)



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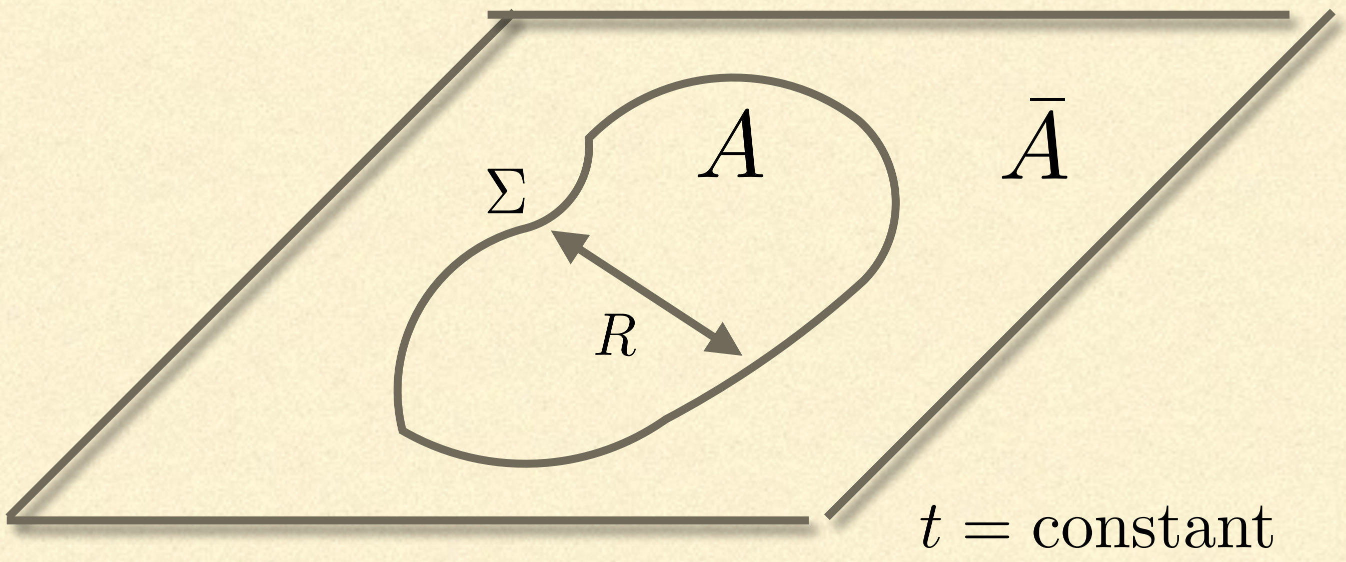
- Entanglement is about correlations: Divergences

Entanglement in CFT and Rényi entropies (2)



Divergences

$$S_n = \begin{cases} \left(\frac{R}{\epsilon_{\rm UV}}\right)^{d-2} + \cdots + \frac{R}{\epsilon_{\rm UV}} + s_d^{(\Sigma)} + \frac{\epsilon_{\rm UV}}{R} + \cdots & \text{odd } d \\ \left(\frac{R}{\epsilon_{\rm UV}}\right)^{d-2} + \cdots + \left(\frac{R}{\epsilon_{\rm UV}}\right)^2 + s_d^{(\Sigma)} \log \frac{R}{\epsilon_{\rm UV}} + \text{const} + \left(\frac{\epsilon_{\rm UV}}{R}\right)^2 + \cdots & \text{even } d \end{cases}$$

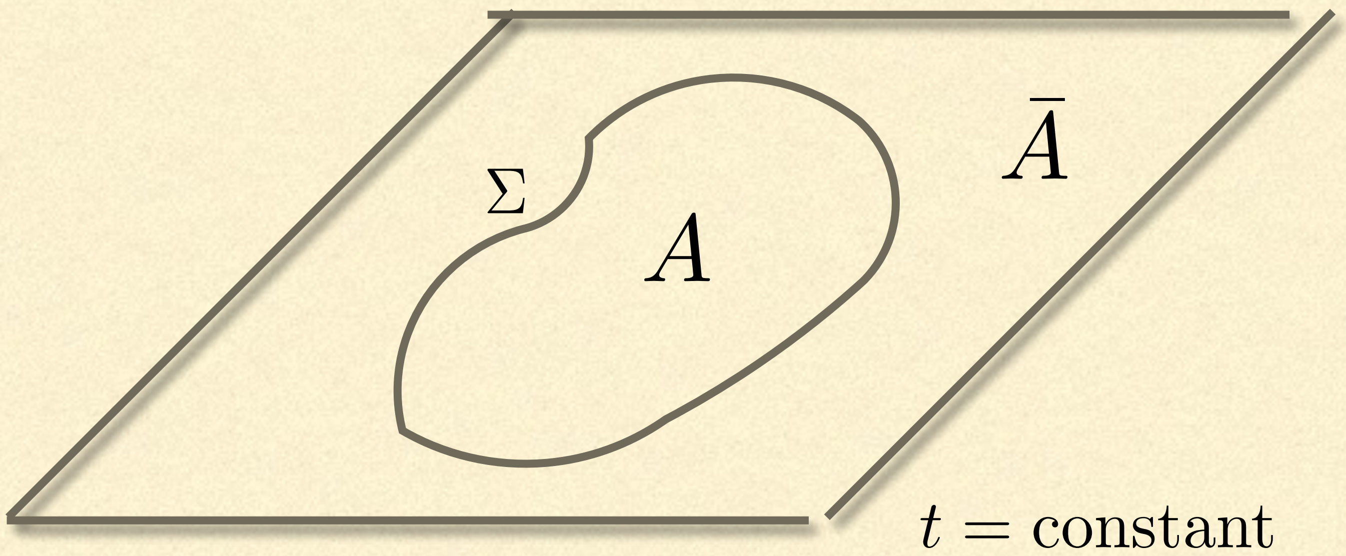
[H. Liu, M. Mezei]

For instance, in $d = 4$

$$s_4^{(\Sigma)} = -\frac{f_a(n)}{2\pi}\int_\Sigma R_\Sigma - \frac{f_b(n)}{2\pi}\int_\Sigma \tilde{K}^i_{ab}\tilde{K}^i_{ab} + \frac{f_c(n)}{2\pi}\int_\Sigma \gamma^{ab}\gamma^{cd}C_{acbd}$$

[S. N. Solodukhin]

- Entanglement in QFT and Rényi entropies



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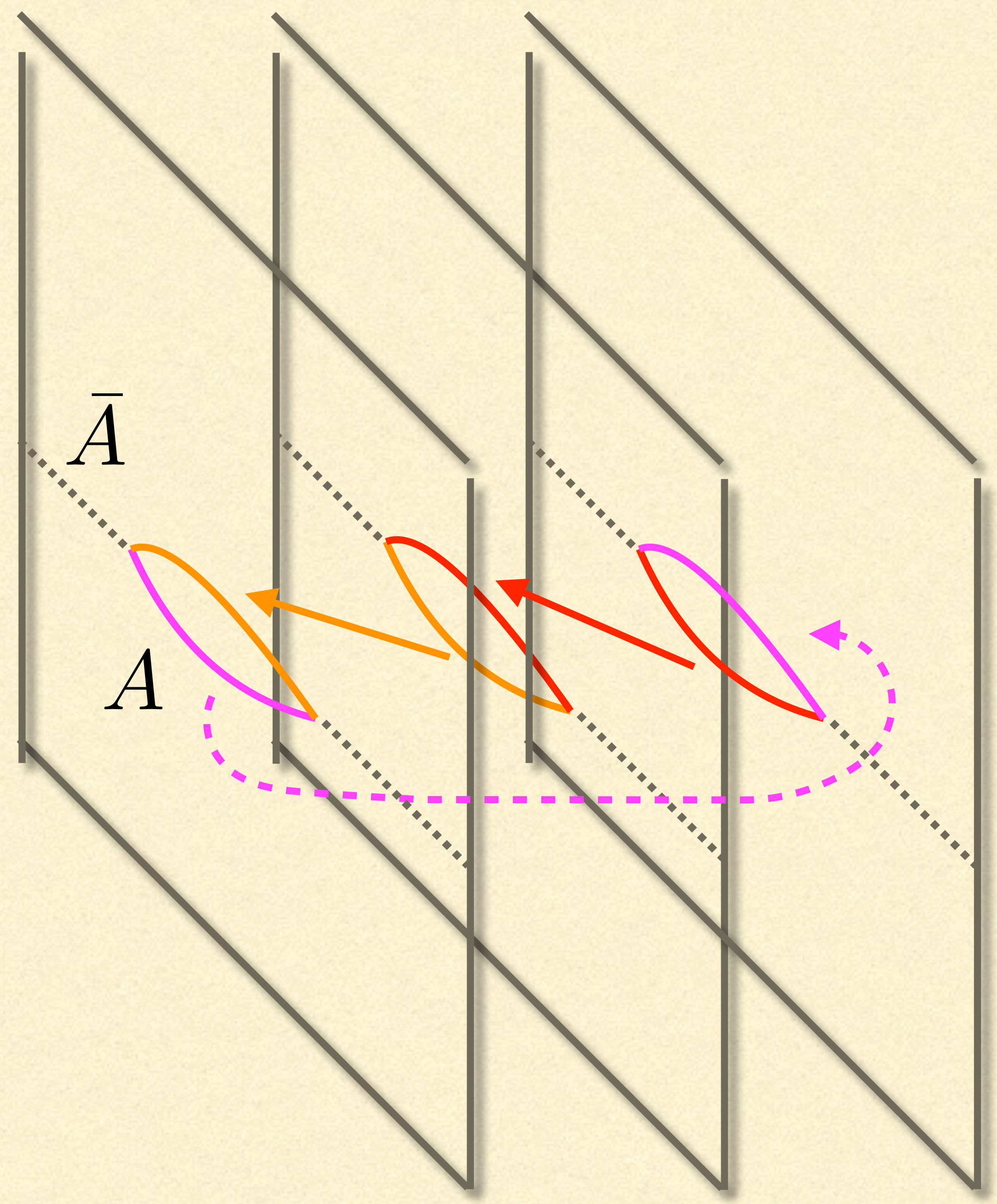
Entanglement in QFT and Rényi entropies (3)

$$S_n = \frac{1}{1-n} \log \mathrm{Tr} \rho_A^n$$ Rényi entropies

$$\rho_A(\phi_A^-, \phi_A^+) = \mathrm{Tr}_{\bar{A}}\, \Psi_0(\phi_A^-, \phi_{\bar{A}})\, \Psi_0^*(\phi_A^+, \phi_{\bar{A}})$$

$\mathrm{Tr}\rho_A^n$? Replica trick



Translations only broken at Σ

Entanglement in QFT and Rényi entropies (3)

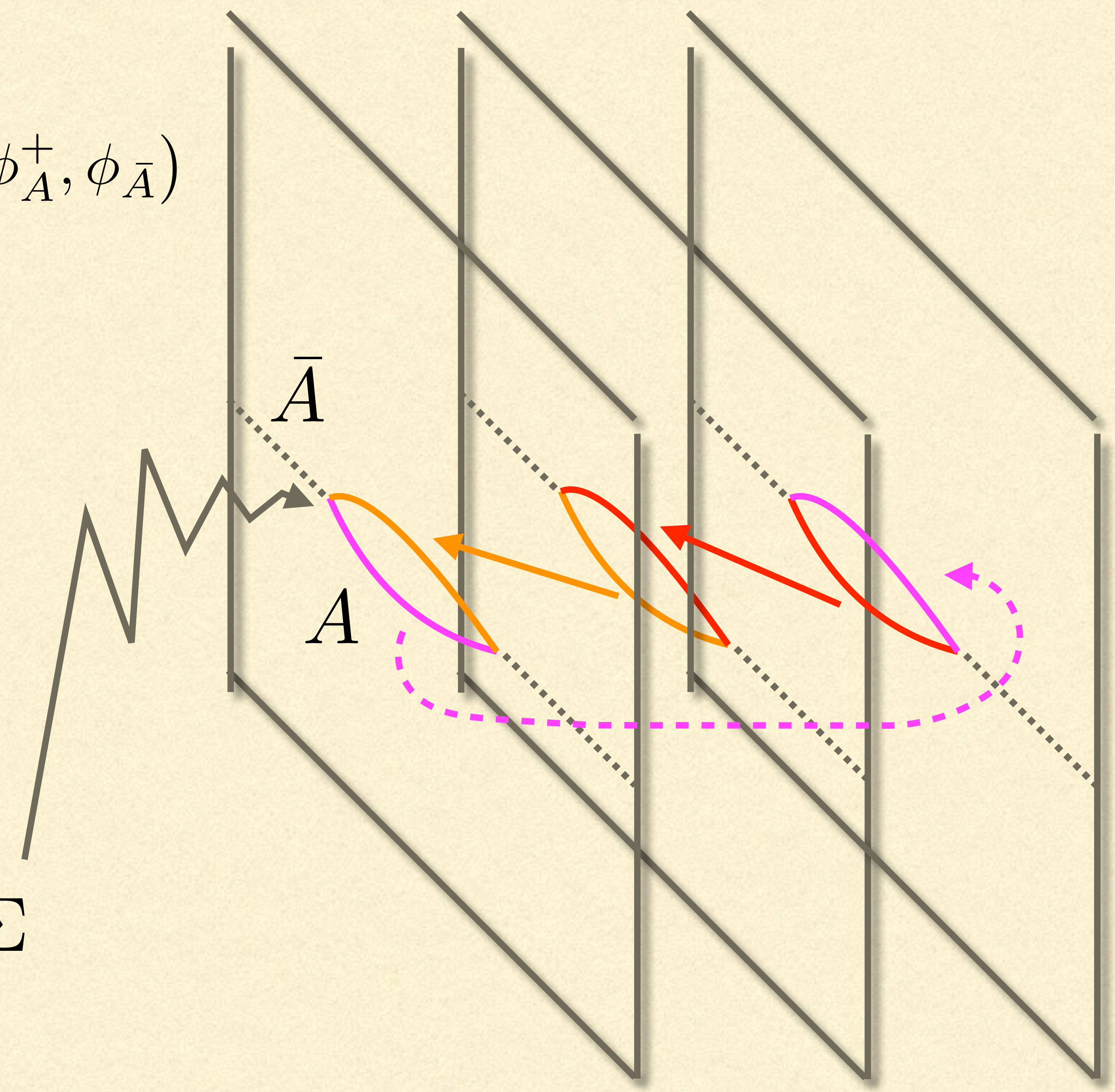
$$S_n = \frac{1}{1-n} \log \mathrm{Tr} \rho_A^n$$

Rényi entropies

$$\rho_A(\phi_A^-, \phi_A^+) = \mathrm{Tr}_{\bar{A}} \, \Psi_0(\phi_A^-, \phi_{\bar{A}}) \Psi_0^*(\phi_A^+, \phi_{\bar{A}})$$

$\mathrm{Tr}\rho_A^n$?

Replica trick



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Entanglement in QFT and Rényi entropies (3)

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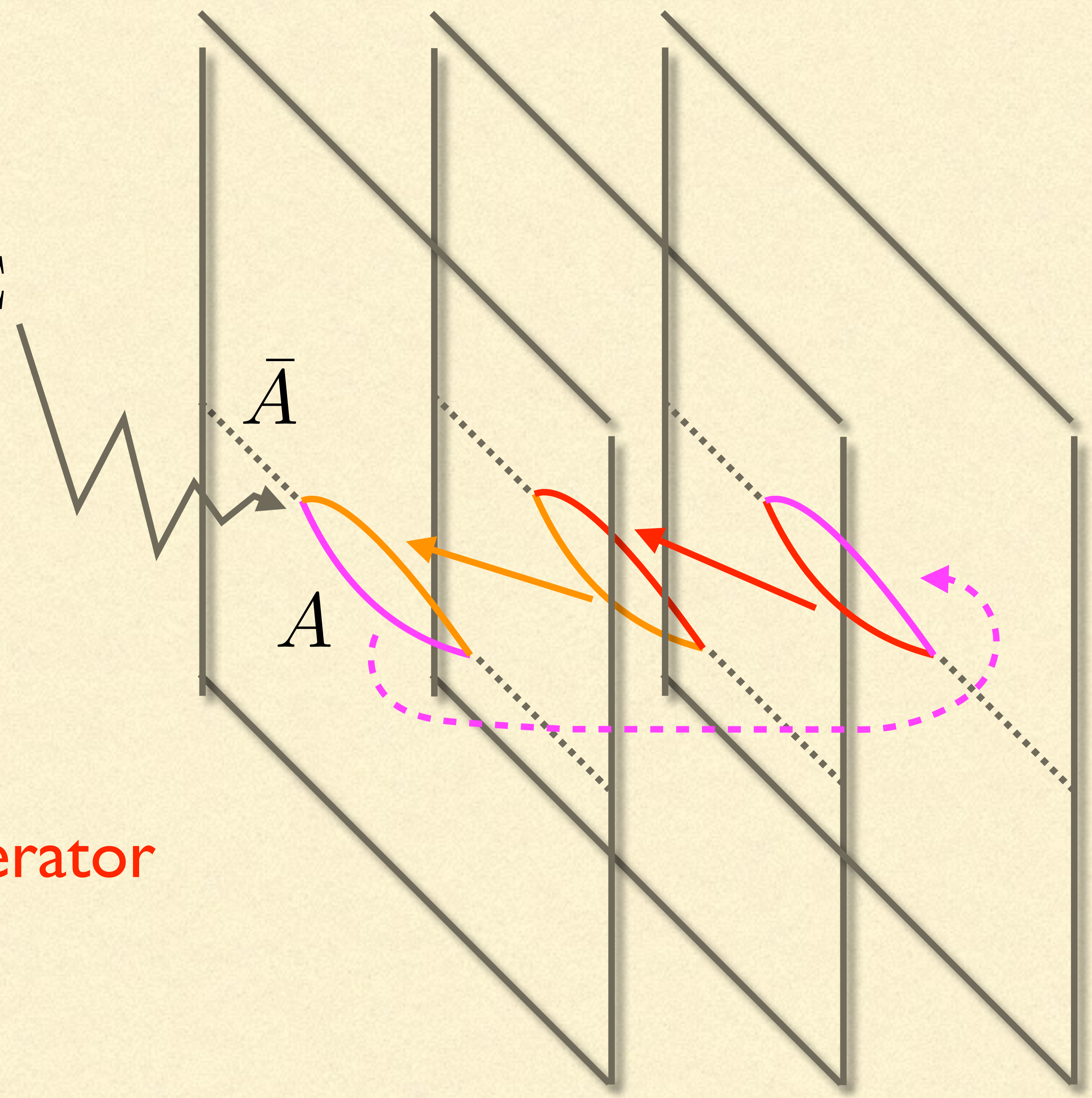

S_n is the free energy

- Of the QFT on a conical space
- Of the $(\mathrm{QFT})^n$ with a twist operator

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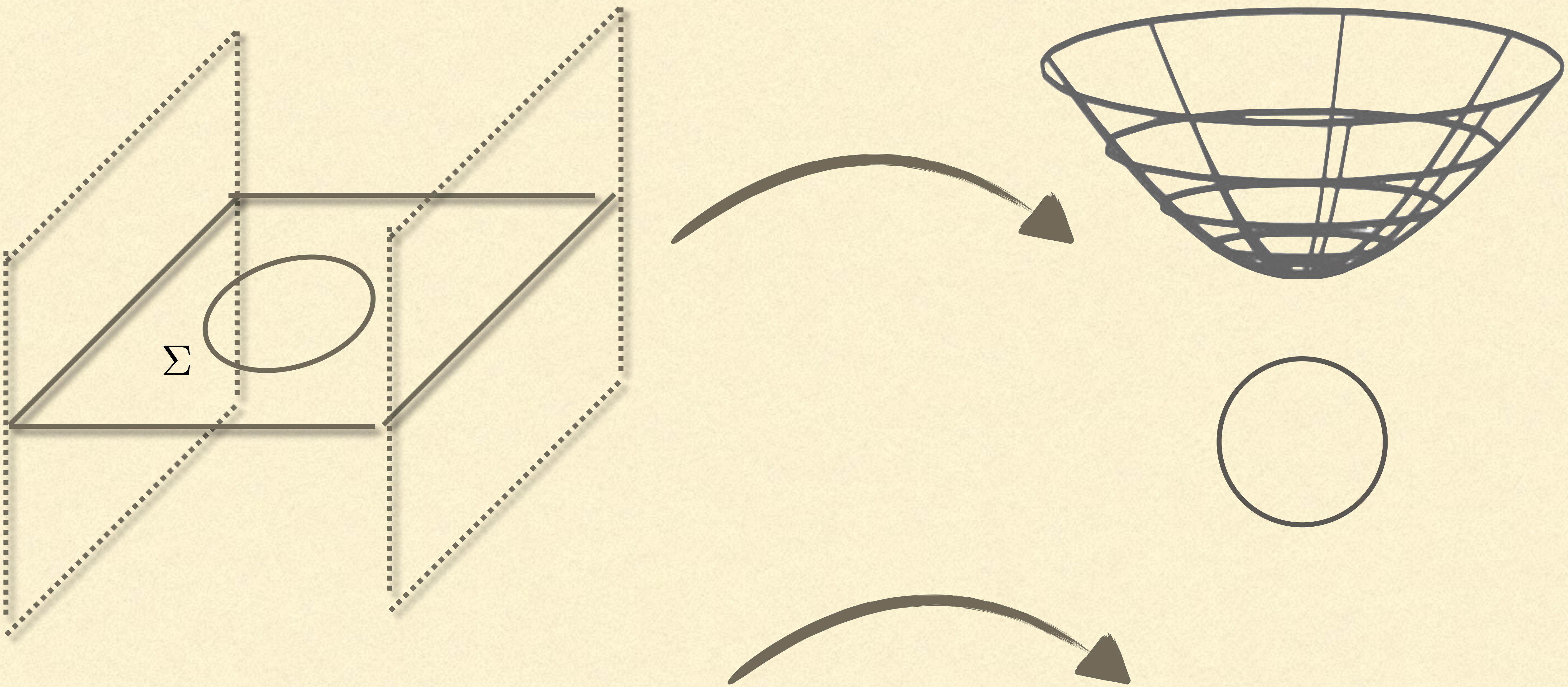
Holographic Rényi entropies and black holes (I)

A useful conformal map:



Spherical entangling surface in flat space

$H_{d-1} \times S^1$

- Holographic Rényi entropies and black holes (I)

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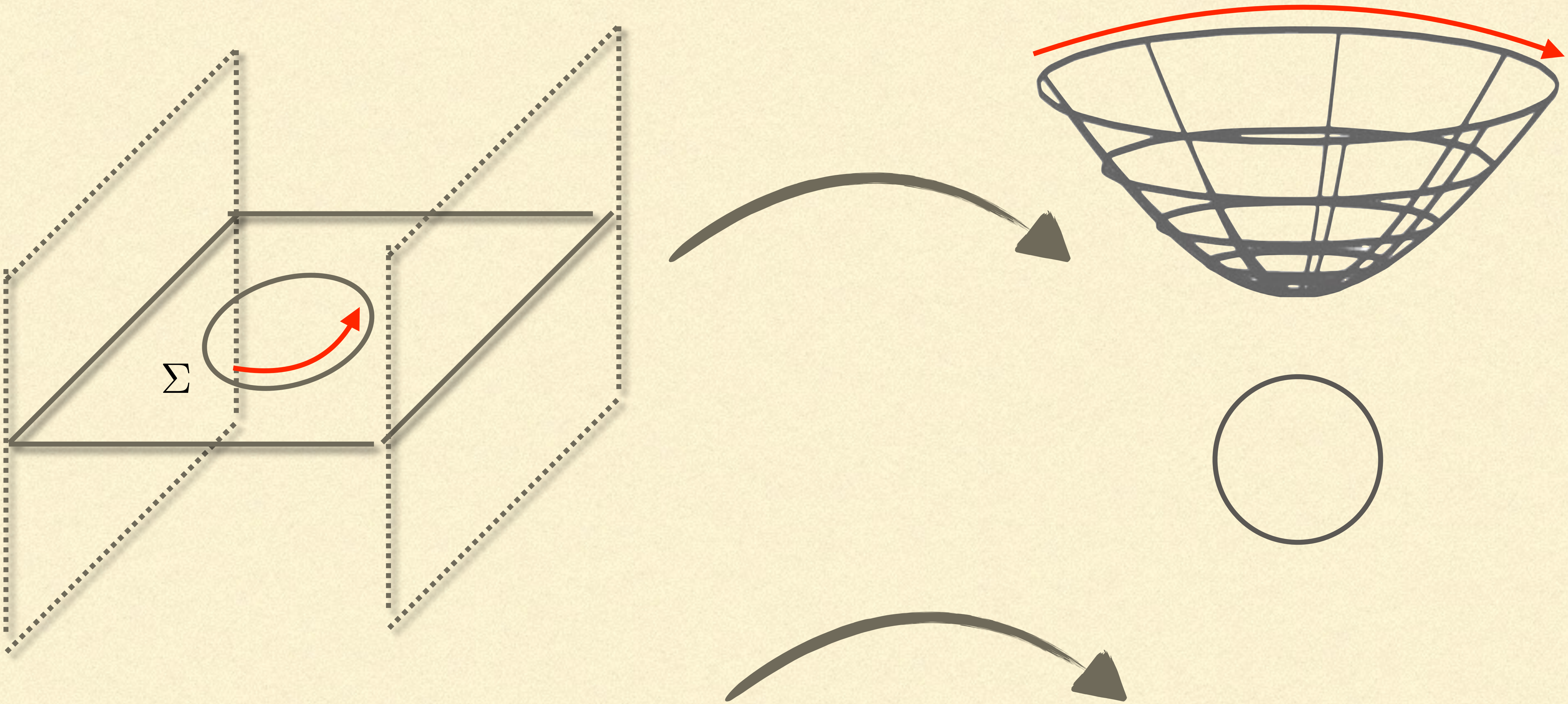

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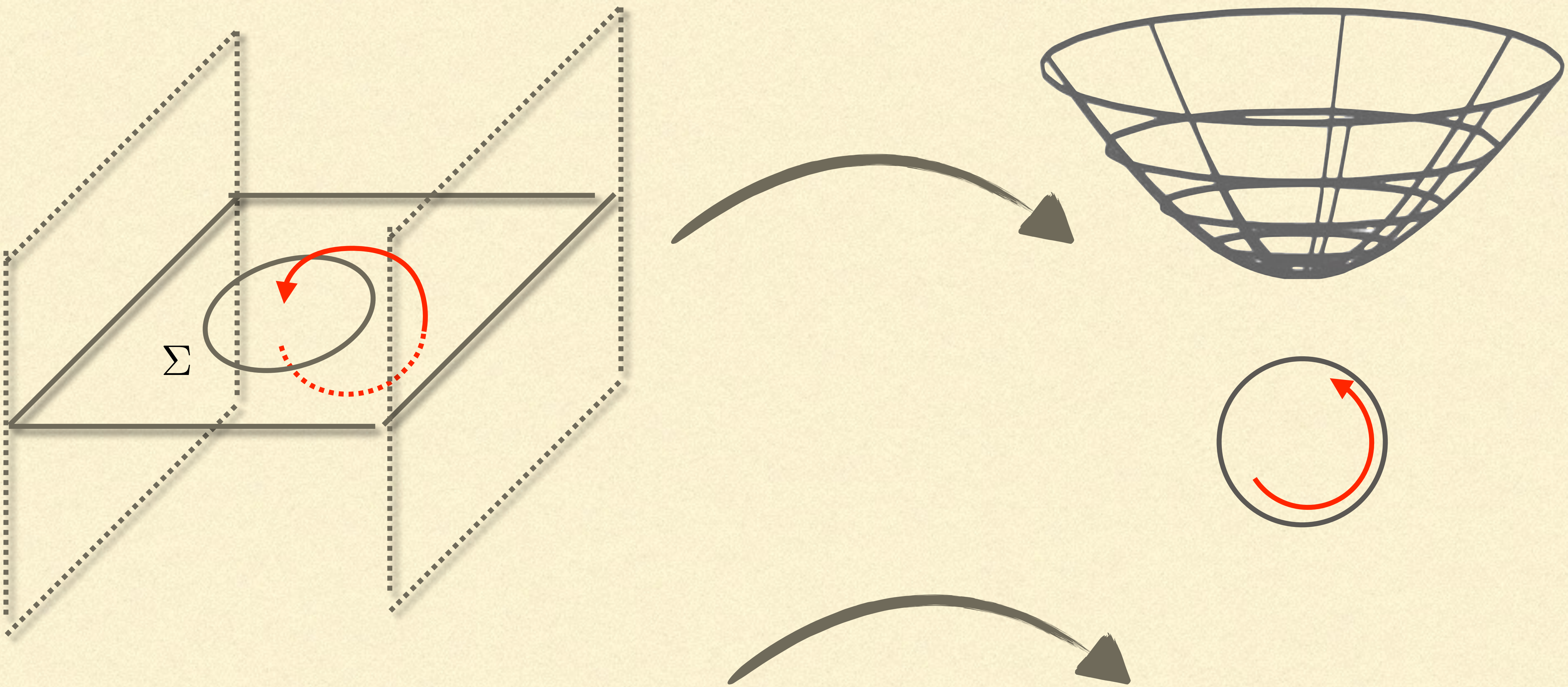

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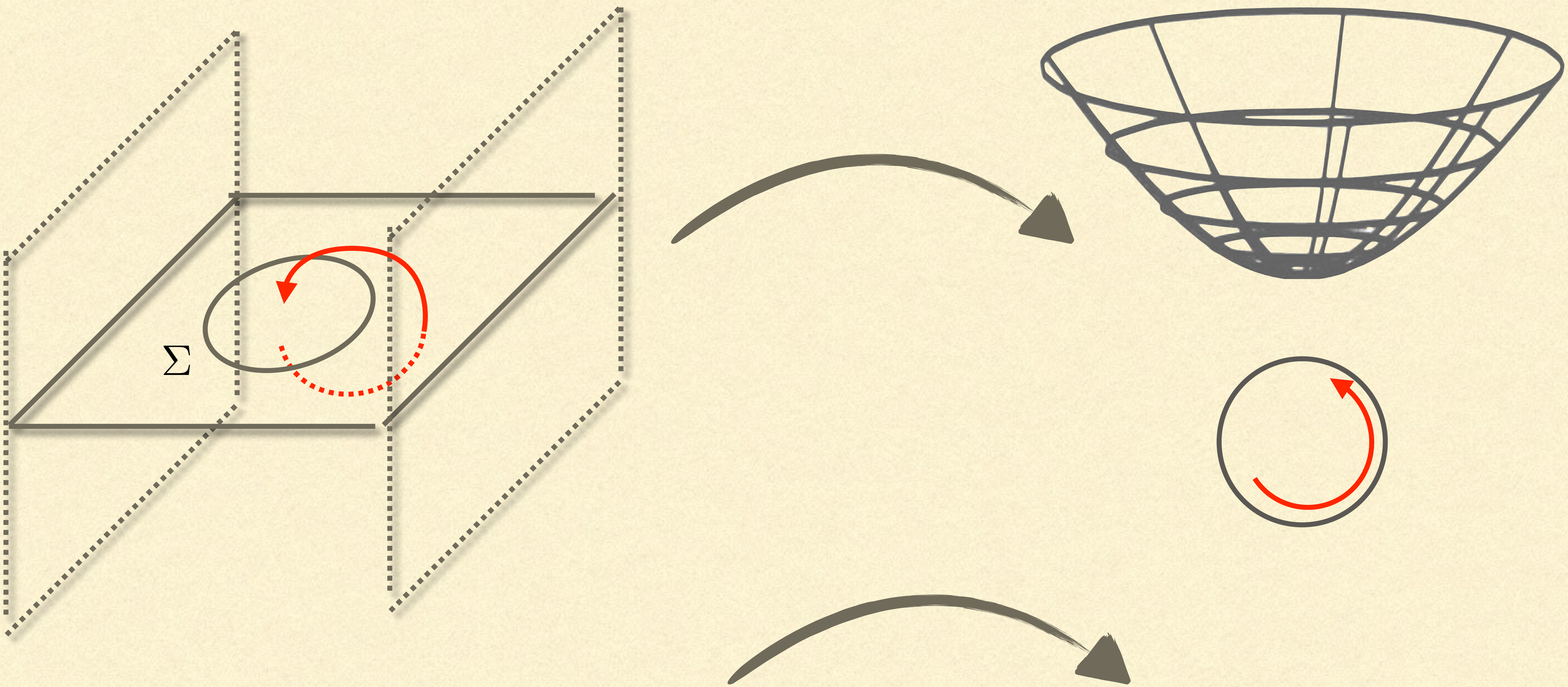
[H. Casini, M. Huerta, R. Myers]
[J. Hung, R. Myers, M. Smolkin, A. Yale]

A useful conformal map:

Thermal space



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[H. Casini, M. Huerta, R. Myers]
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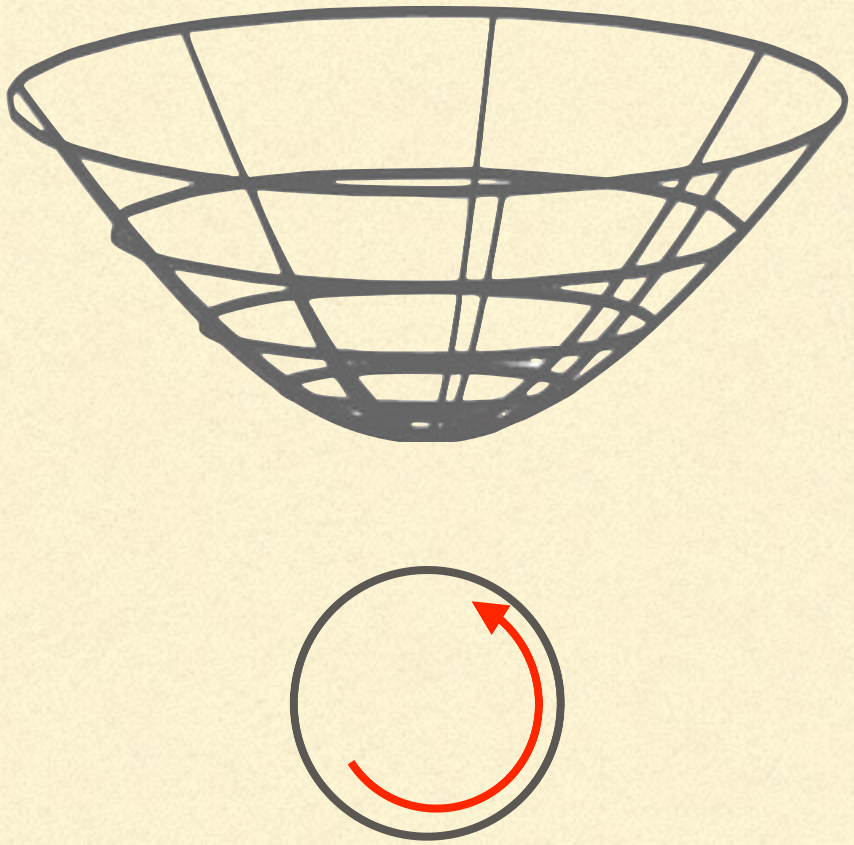
Thermal space

Rényi entropy across a sphere of radius R

$\simeq$

Thermal free energy on hyperbolic space



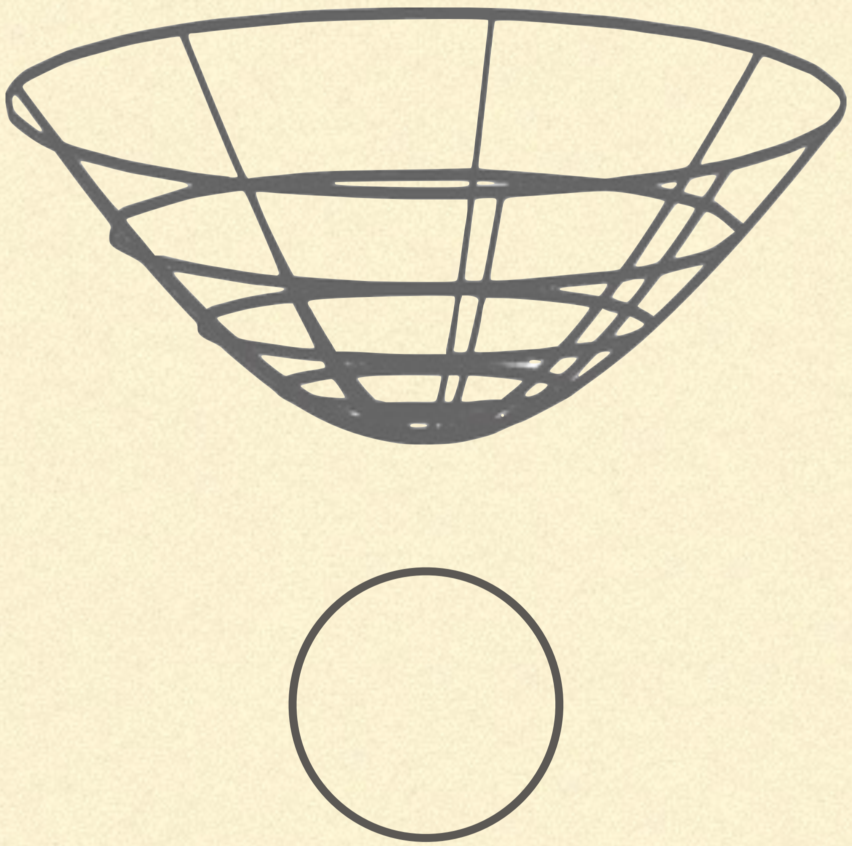
$$\text{Temperature} = \frac{1}{2\pi n R}$$

$H_{d-1} \times S^1$

- Holographic Rényi entropies and black holes (2)

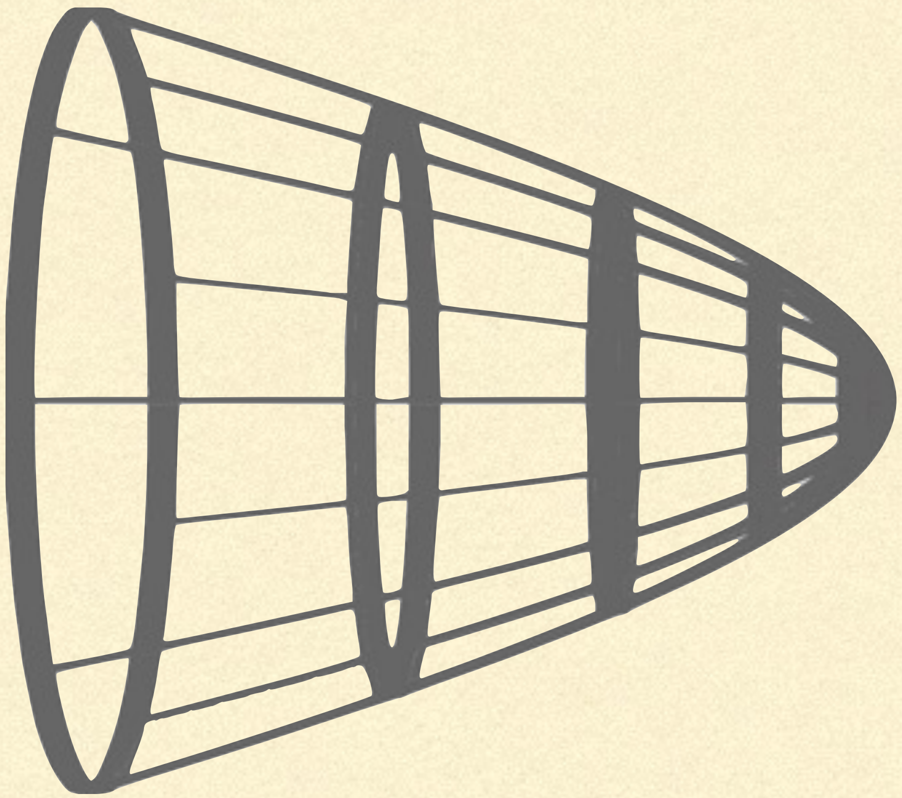
[H. Casini, M. Huerta, R. Myers]
[J. Hung, R. Myers, M. Smolkin, A. Yale]

Boundary



$$H_{d-1} \times S^1$$

Bulk



$$ds^2 = \frac{1}{\left(\frac{r^2}{R^2} f(r) - 1\right)} dr^2 + \left(\frac{r^2}{R^2} f(r) - 1\right) d\tau^2 + r^2 dH_{d-1}^2$$

Holographic Rényi entropies and black holes (2)

[H. Casini, M. Huerta, R. Myers]
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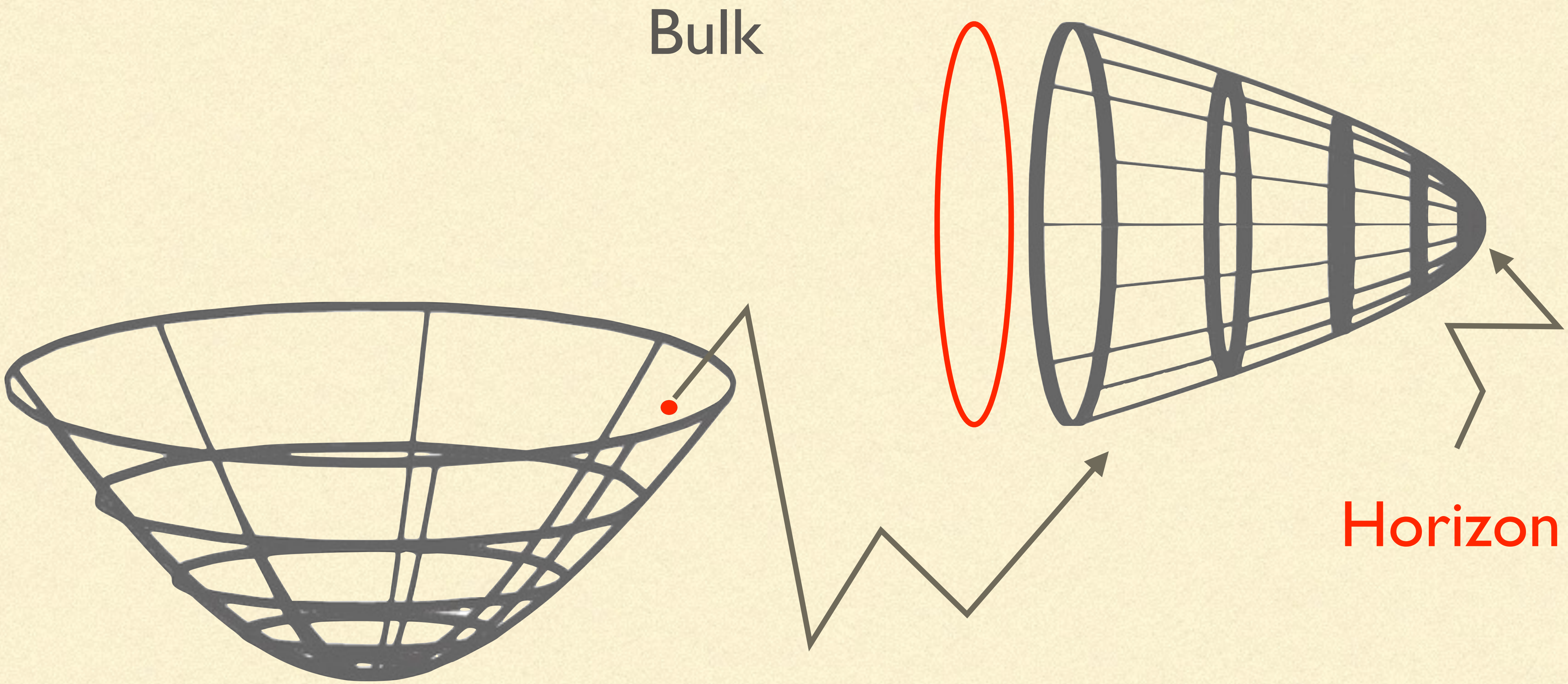

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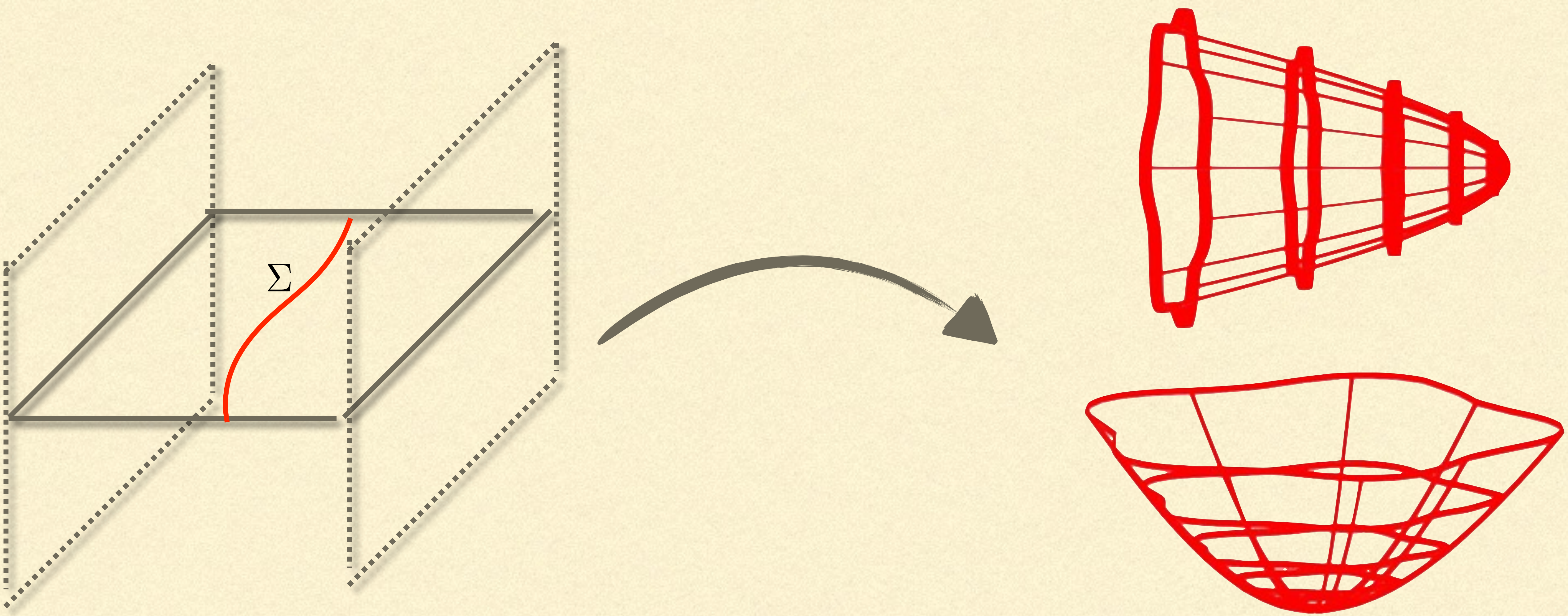
- Away from planes and spheres (1)



$$\delta S_n \propto \delta \log Z \propto \epsilon^2 C_{\mathrm{D}}$$

Conformal defects and their deformation (3)

The displacement operator: $\partial_\mu T^{\mu a}(x) = \delta_\Sigma(x)\, \mathrm{D}^a$

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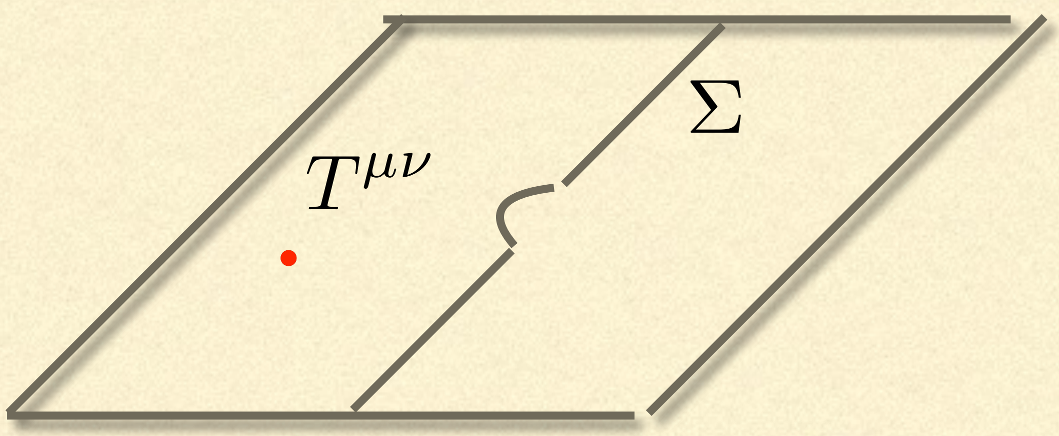

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Away from planes and spheres (2)

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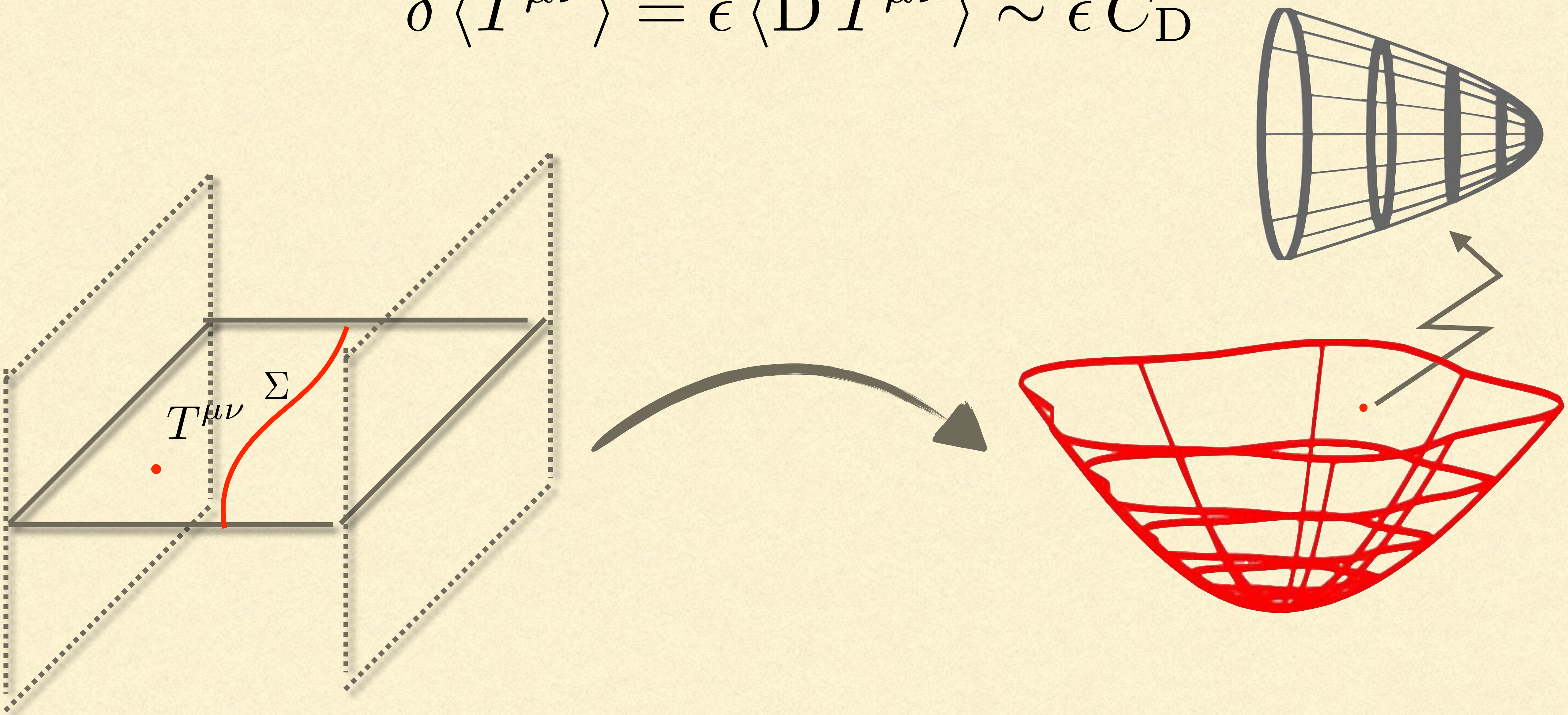
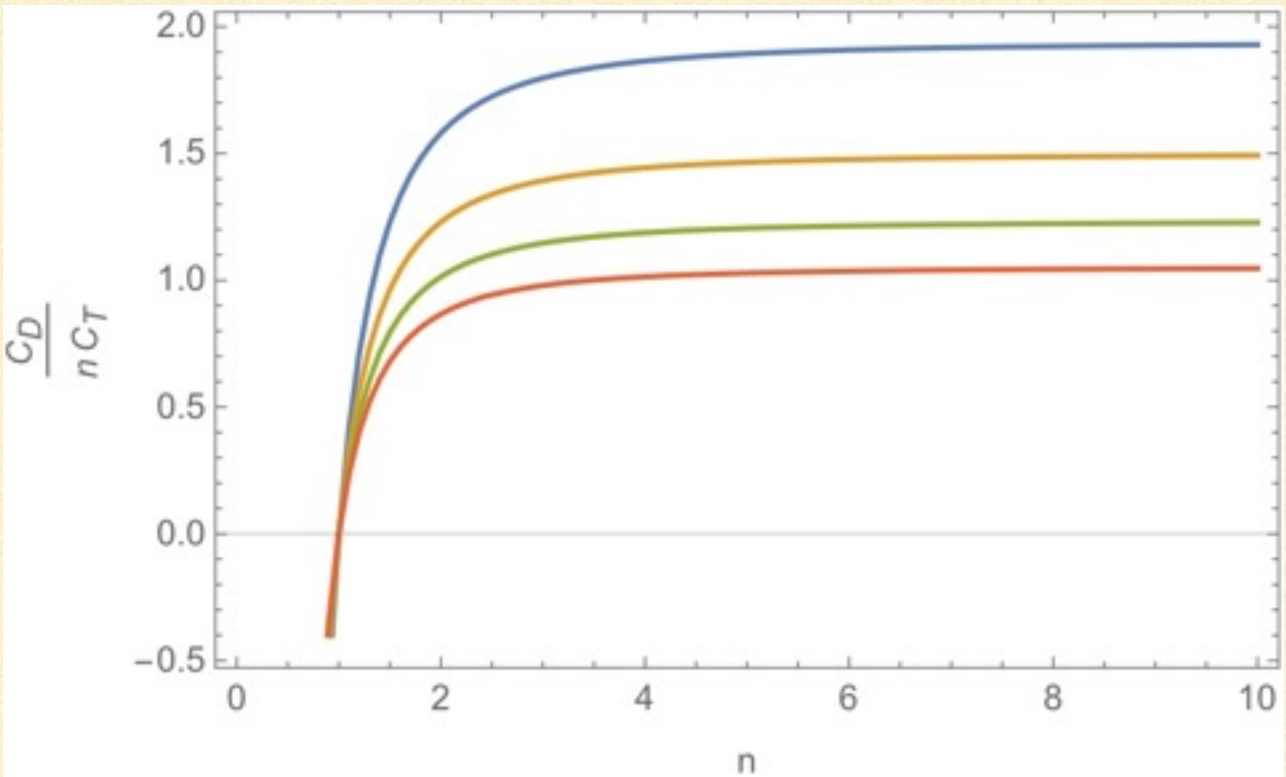


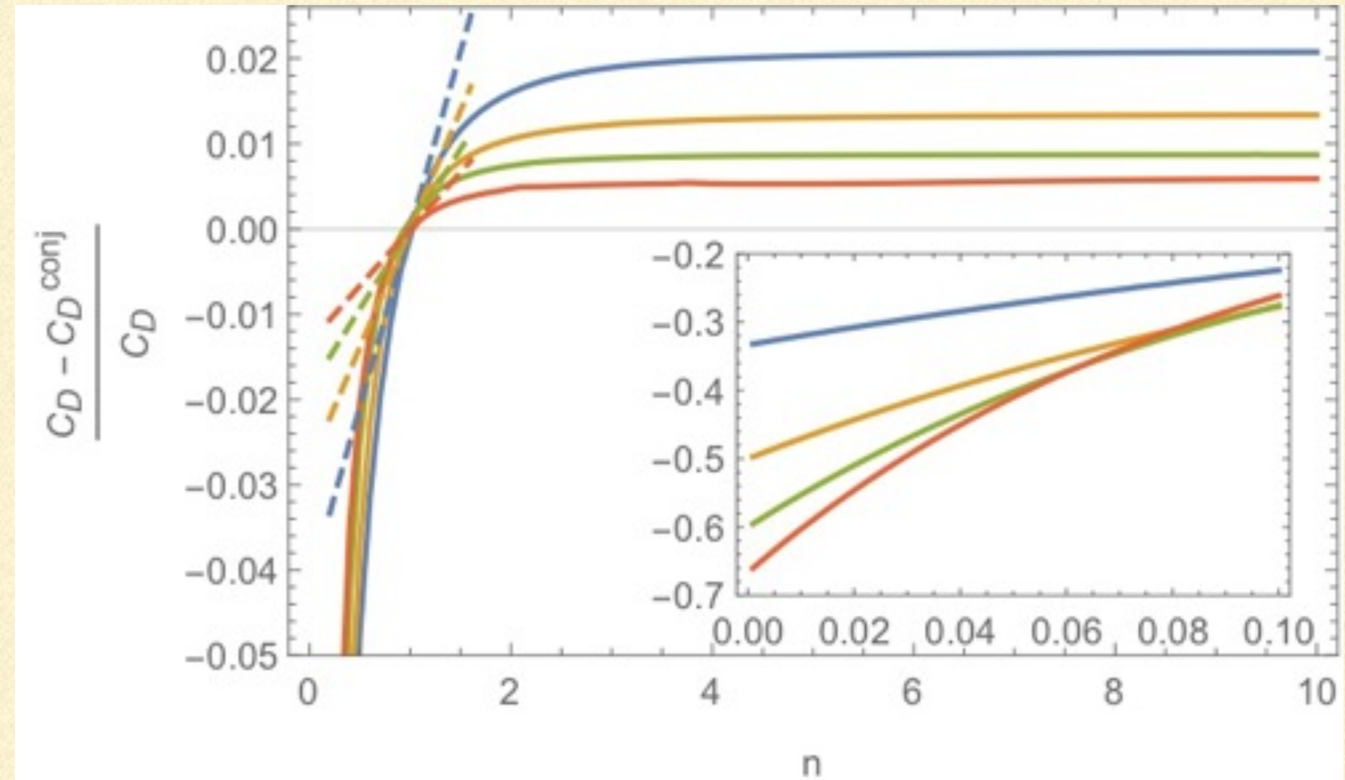
$$ds^2 = \frac{dr^2}{\frac{r^2}{R^2} f(r) - 1} + \left(\frac{r^2}{R^2} f(r) - 1 \right) d\tau^2 + \frac{r^2}{\rho^2} \left(d\rho^2 + [\delta_{ij} + 2\, k(r) \tilde{K}^a_{ij} x_a] dy^i dy^j + \frac{4}{d-2} v(r) \partial_i\, K^b x_b \rho d\rho dy^i \right) + \cdots$$

[X. Dong]

Away from planes and spheres (2)

Einstein gravity (can do for Einstein-Gauss-Bonnet)





$$C_{\rm D}^{\rm conj}(n) = d\,\Gamma\left(\tfrac{d+1}{2}\right)\,\left(\frac{2}{\sqrt{\pi}}\right)^{d-1}\,h_n \qquad \langle T^{\mu\nu}\rangle \propto h_n$$

[L. Bianchi, M. M. , R. Myers, M. Smolkin]

$$\lim_{n\to 0}\frac{C_{\rm D}-C_{\rm D}^{\rm conj}}{C_{\rm D}} = -\frac{d-2}{d}$$

Universal?

Away from planes and spheres (2)

Analytics close to $n = 1$. λ is the Gauss-Bonnet coupling

$$\frac{C_{\mathrm{D}}^{(d=4)} - C_{\mathrm{D}}^{\mathrm{conj}\,(d=4)}}{C_{\mathrm{D}}^{(d=4)}} = \left(\frac{1}{9\sqrt{1-4\lambda}} - \frac{1}{12}\right)(n-1) + O\,(n-1)^2\,,$$

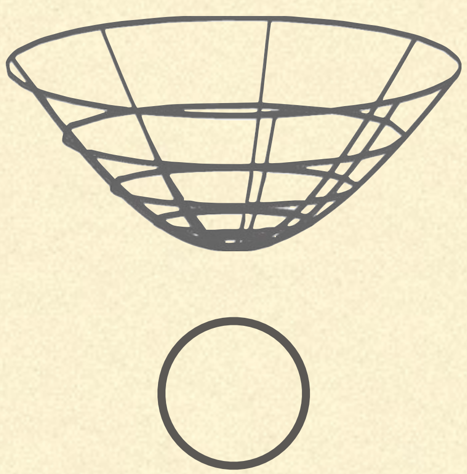
$$\frac{C_{\mathrm{D}}^{(d=5)} - C_{\mathrm{D}}^{\mathrm{conj}\,(d=5)}}{C_{\mathrm{D}}^{(d=5)}} = \left(\frac{1}{16\sqrt{1-4\lambda}} - \frac{7}{160}\right)(n-1) + O(n-1)^2,$$

$$\frac{C_{\mathrm{D}}^{(d=6)} - C_{\mathrm{D}}^{\mathrm{conj}\,(d=6)}}{C_{\mathrm{D}}^{(d=6)}} = \frac{1}{75}\left(\frac{3}{\sqrt{1-4\lambda}} - 2\right)(n-1) + O\,(n-1)^2\,.$$

Conjecture is true for entanglement entropy [T. Faulkner, R. Leigh, O. Parrikar]

Next order vanishes for $\lambda = \lambda_{\mathrm{min}}$

Determined by $\langle T^{\mu\nu} T^{\rho\sigma} T^{\lambda\tau} \rangle$



Conclusions

- Rényi entropies across slightly deformed planes and spheres can be computed. Actually, in even dimensions $C_{\rm D}$ controls (part of) the result for finite deformations too.
- A conjecture, true for entanglement entropy, is disproven for Rényi entropies (but only mildly violated, and seems to be valid for free theories).
- Universal behaviour as $n \to 0$ for holographic CFTs, but violated by free theories!
- Thinking of Rényi entropies as conformal defects seems to be useful.
- Method is much more general: would work as well for any defect whose holographic dual is known.

Thank you!