Collinear Gluon Limits from the Scattering Equations

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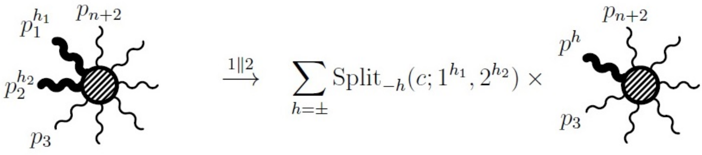
DESY Theory workshop - Rethink Quantum Field Theory

Introduction

- A collinear gluon limit is...



Momentum configuration ($\epsilon \to 0$)

$$p_1 = c^2\, p + \mathcal{O}(\epsilon) \quad , \quad p_2 = (1 - c^2)\, p + \mathcal{O}(\epsilon) \quad , \quad p^2 = 0$$

- As an equation

$$A_n(1^{h_1}, 2^{h_2}, \ldots) \xrightarrow{1\|2} \underbrace{\sum_{h=\pm} \text{Split}_{-h}(c; 1^{h_1}, 2^{h_2})}_{\propto \frac{1}{\epsilon}} \; A_{n-1}(p^h, \ldots) + \mathcal{O}(1)$$

The Cachazo-He-Yuan (CHY) Formalism

- Given n massless particles. Tree level scattering amplitude?
- Usual approach (e.g. Feynman diagrams) = Integral over n punctured $\mathbb{CP}^1$
- Constraints ("Scattering Equations"):

$$\boxed{f_i = \sum_{\substack{i=1 \\ i\neq j}}^{n} \frac{2\, p_i \cdot p_j}{\sigma_i - \sigma_j} = 0}$$

- Color-ordered amplitude

$$A_n(\{p_i, E_i\}) = \int d\mu_n\, \mathcal{I}(\{p_i, E_i, \sigma_i\})$$

[Cachazo, He, Yuan]

- Universal measure

$$d\mu_n = \prod_{i=1}^{n} d\sigma_i\, \delta(f_i)$$

The n-Gluon tree level scattering in CHY

- Pure Yang-Mills amplitude

$$A_n(\{p_i, E_i\}) = \int d\mu_n \, \mathfrak{C}_n \ \text{Pf } \Psi_n(\{p_i, E_i, \sigma_i\})$$

- Parke-Taylor factor

$$\mathfrak{C}_n = \frac{1}{\sigma_{12}\sigma_{23}\ldots\sigma_{n1}} \quad , \quad \sigma_{ij} = \sigma_i - \sigma_j$$

- Antisymmetric $2n \times 2n$ matrix

$$\Psi_n = \begin{bmatrix} A & -C^T \\ C & B \end{bmatrix}$$

$$A_{ab} = \begin{cases} \frac{2\, p_a \cdot p_b}{\sigma_a - \sigma_b} & a \neq b \\ 0 & a = b \end{cases} \qquad B_{ab} = \begin{cases} \frac{2\, E_a \cdot E_b}{\sigma_a - \sigma_b} & a \neq b \\ 0 & a = b \end{cases}$$

$$C_{ab} = \begin{cases} \frac{2\, E_a \cdot p_b}{\sigma_a - \sigma_b} & a \neq b \\ -\sum\limits_{c \neq a} \frac{2\, E_a \cdot p_c}{\sigma_a - \sigma_c} & a = b \end{cases}$$

Implementation of the collinear limit $p_1||p_2$: Kinematic part

- Starting point ($c = cos\phi$, $s = sin\phi$)

$$p_1 = c^2\, p - \epsilon\, c\, s\, q + \epsilon^2\, s^2\, r$$

$$p_2 = s^2\, p + \epsilon\, c\, s\, q + \epsilon^2\, c^2\, r$$

r-reference momentum, ϵ-perturbation parameter

- Polarizations (reference momentum $r \,\forall\, E_i$)

$$E_1^{\pm} = E_p^{\pm} - \epsilon\, \frac{s}{c}\, E_r^{\pm} \quad , \quad E_2^{\pm} = E_p^{\pm} + \epsilon\, \frac{c}{s}\, E_r^{\pm}$$

- Collinear expansion for all kinematic invariants
- What about the σ's?

$$f_i = \sum_{\substack{i=1 \\ i\neq j}}^{n} \frac{2\, p_i \cdot p_j}{\sigma_i - \sigma_j} = 0$$

Implementation of the collinear limit $p_1 || p_2$: Puncture part

- Change of variables

$$\sigma_1 = \sigma_p - \frac{\xi}{2} \quad , \quad \sigma_2 = \sigma_p + \frac{\xi}{2} \; .$$

- Use

$$d\mu_n = d\mu_{n-2}\, d\sigma_1\, d\sigma_2\, \delta(f_1)\, \delta(f_2) = d\mu_{n-2}\, 2\, d\sigma_p\, d\xi\, \delta(f_+)\, \delta(f_-)$$

$$f_+ = (f_1 + f_2) \quad , \quad f_- = (f_1 - f_2) - (c^2 - s^2)(f_1 + f_2)$$

- Integrate out ξ, using

$$\delta(f_-) = \sum_{\xi_{sol}} \frac{\delta(\xi - \xi_{sol})}{\left| \frac{\partial f_-}{\partial \xi} \right|_{\xi = \xi_{sol}}} = \sum_{\xi_{sol}} \delta(\xi - \xi_{sol})\, \mathcal{J}$$

- End up with

$$A_n^{1||2} = 2 \sum_{\xi_{sol}} \int d\mu_{n-2}\, d\sigma_p\, \delta(f_+)\, \mathfrak{C}_n\, \mathcal{J}\; \mathrm{Pf}\, \Psi_n \; ,$$

- Everything depends on ξ_{sol}

Implementation of the collinear limit $p_1 || p_2$: Puncture part

- Obtain ξ_{sol} from $f_- = 0 \to$ two classes of solutions!
  1. Non-degenerate solutions $\xi_{sol} := \xi_{n-d} =$ finite
  2. Degenerate solutions $\xi_{sol} := \xi_d =$ vanishing
- Degenerate solutions vanish as $(\epsilon \to 0)$

$$\xi_d = \epsilon\,\xi_1 + \epsilon^2\,\xi_2 + \mathcal{O}(\epsilon^3)$$

same ϵ as in

$$p_1 = c^2\,p - \epsilon\,c\,s\,q + \epsilon^2\,s^2\,r$$

- ξ_1 , ξ_2 , ... obtained from

$$f_- = \epsilon\,a(\xi_1) + \epsilon^2\,b(\xi_1, \xi_2) + \ldots,$$

where $a(\xi_1) = b(\xi_1, \xi_2) = \ldots = 0$.

Implementaton of the collinear limit $p_1||p_2$: Amplitude

- Numerical evidence

$$A_n^{1||2} = \underbrace{\mathcal{O}(\epsilon^{-1}) + \mathcal{O}(1)}_{\xi_d} + \underbrace{\mathcal{O}(\epsilon)}_{\xi_d\,,\,\xi_{n-d}}$$

- Hence

$$A_n^{1||2} = 2\sum_{\xi_d} \int d\mu_{n-2}(\epsilon)\, d\sigma_p\, \delta(f_+(\epsilon))\, \mathfrak{C}_n(\epsilon)\, \mathcal{J}(\epsilon)\ \mathrm{Pf}\, \Psi_n(\epsilon)\ ,$$

- Perturbation series in ϵ, divergence sits in ($\xi_d = \epsilon\, \xi_1 + \ldots$)

$$\mathfrak{C}_n(\epsilon) = \frac{-1}{\underbrace{\xi_d}_{\sigma_{12}}}\ \frac{1}{\underbrace{\sigma_p + \frac{\xi_d}{2} - \sigma_3}_{\sigma_{23}}}\ \frac{1}{\underbrace{\sigma_n - \sigma_p + \frac{\xi_d}{2}}_{\sigma_{n1}}}\ \frac{1}{\sigma_{34}\cdots\sigma_{n-1,n}}$$

Results

- Up to subleading order

$$A_n^{1||2} = \int d\mu_{n-1}\, \mathrm{Coll}^{\mathrm{gluon}}\, \mathfrak{C}_{n-1}\, \mathrm{Pf}\,\Psi_{n-1} + \mathcal{O}(\epsilon)$$

$$\mathrm{Coll}^{\mathrm{gluon}} = \mathrm{Split}_{-h}(c; 1^{h_1}, 2^{h_2}) + \mathcal{K}^{\mathrm{gluon}}_{\mathrm{coll}}(E_p, \{p_i\}, \{\sigma_i\})$$

- Reproduction of all known results for the Splitting function
- Result is universal
- Factorization on amplitude level (leading order) and integrand level (subleading order)
- Intriguing property

$$p \cdot \frac{\partial}{\partial E_p} A_n^{1||2}(p, 3, \ldots, n) = \frac{c^2 - s^2}{c^2\, s^2} A_{n-1}(p, 3, \ldots, n)$$

[D.Nandan, J.Plefka, WW, unpublished]

Outlook

- Techniques applicable to any theory of massless particles with CHY description
- Gravity: Leading order [Bern,Dixon,Perelstein,Rozowsky]

$$\mathcal{A}_n^{1||2} = \frac{[p\,r]}{c^2\,s^2\,\langle r\,p\rangle}\mathcal{A}_{n-1} + \frac{1}{c^2\,s^2}\int d\mu_{n-1}\,\frac{C_{pp}^2}{\mathcal{P}_2}\,\mathrm{Pf}(\Psi_{n-1})\,\mathrm{Pf}(\Psi_{n-1})$$

- Bi-adjoint scalars: Subleading order vanishes
- All results universal
- Connection to Gravity manifest
- Amplitude factorization at subleading order? Let $c^2 = s^2$

$$\mathcal{K}_{\text{coll}}^{\text{gluon}}(E_p^h,\{p_i\},\{\sigma_i\}) = -\frac{1}{2}\,\frac{C_{pp}}{\mathcal{P}_2}\left(\frac{1}{\sigma_n-\sigma_p}+\frac{1}{\sigma_p-\sigma_3}\right)$$

$$C_{pp} = -\sum_{a=3}^{n}\frac{2\,E_p\cdot p_a}{\sigma_p-\sigma_a}\quad,\quad \mathcal{P}_2 = \sum_{a=3}^{n}\frac{2\,p\cdot p_a}{(\sigma_p-\sigma_a)^2}$$

Bonus

- Kernel

$$\mathcal{K}^{\text{gluon}}_{\text{coll}}(E_p, \{p_i\}, \{\sigma_i\}) = \frac{C_{pp}}{\mathcal{P}_2}\left(\frac{1}{c^2}\,\frac{1}{\sigma_{np}} + \frac{1}{s^2}\,\frac{1}{\sigma_{p3}}\right) + \frac{c^2 - s^2}{c^2\, s^2\, \mathcal{P}_2}\left(C^{(2)}_{pp} - \frac{C_{pp}\mathcal{P}_3}{\mathcal{P}_2}\right)$$

$$C^{(i)}_{pp} = -\sum_{a=3}^{n} \frac{2\, E_p \cdot p_a}{(\sigma_p - \sigma_a)^i} \quad , \quad \mathcal{P}_i = \sum_{a=3}^{n} \frac{2\, p \cdot p_a}{(\sigma_p - \sigma_a)^i}$$

- Connection to Gravity (n=5)

$$s_{5p}\, \mathcal{A}^{1||2}(1^h, 2^h, 3, 4, 5) - s_{4p}\, \mathcal{A}^{1||2}(1^h, 2^h, 3, 5, 4) = \frac{1}{c^2} A(P^{hh}, 3, 4, 5)$$

[Stieberger, Taylor]

- Measure

$$d\,\mu_n = (\sigma_{ij}\,\sigma_{jk}\,\sigma_{ki})\,(\sigma_{pq}\,\sigma_{qr}\,\sigma_{rp}) \prod_{\substack{a=1\\ a\neq i,j,k}}^{n} d\sigma_a \prod_{\substack{b=1\\ b\neq p,q,r}}^{n} \delta(f_b)$$

- Pfaffian

$$\text{Pf}'\ \Psi_n = \frac{(-1)^{i+j}}{\sigma_{ij}}\,\text{Pf}\,\Psi_n^{i,j}$$