Green-Schwarz superstring on the lattice

based on 1601.04670,1605.01726 and in progress
with M. Bianchi, V. Forini, B. Leder, P. Töpfer, E. Vescovi.

Lorenzo Bianchi

Universität Hamburg

September 29^{th}, 2016.
DESY theory workshop.

AdS/CFT

$\mathcal{N} = 4$
Super Yang-Mills.
SCFT in 4d

Type *IIB* superstring
in $AdS_5 \times S^5$

$$\lambda = \frac{g_{YM}^2 N_c}{4\pi}$$

AdS/CFT integrability

$\mathcal{N} = 4$ Super Yang-Mills. SCFT in 4d	INTEGRABILITY $(N_c \to \infty)$	Type *IIB* superstring in $AdS_5 \times S^5$

$$\longrightarrow$$

$$\lambda = \frac{g_{YM}^2 N_c}{4\pi}$$

AdS/CFT integrability

$\mathcal{N} = 4$
Super Yang-Mills.
SCFT in 4d

INTEGRABILITY
$(N_c \to \infty)$

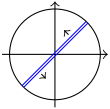
Type *IIB* superstring
in $AdS_5 \times S^5$

$$\lambda = \frac{g_{YM}^2 N_c}{4\pi}$$

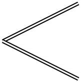
Folded spinning string

[Gubser, Klebanov, Polyakov, 2002; Frolov, Tseytlin, 2002; Belitsky, Gorsky, Korchemsky, 2006; Frolov, Tirziu, Tseytlin, 2007; Kruczenski, Roiban, Tirziu, Tseytlin, 2008]

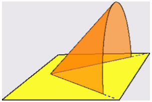
$$\mathrm{Tr}(\phi D_{(\mu_1} \ldots D_{\mu_S)} \phi) \longleftrightarrow$$



$\downarrow S \to \infty$ $\downarrow S \to \infty$



$\longleftrightarrow$

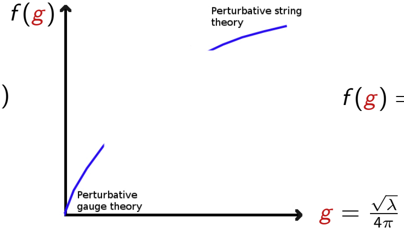


Idea and motivation

Main idea

Discretize the two-dimensional string sigma model on a lattice and study the previous observables at finite coupling (still in the planar limit)

$$f(g) = ag^2 + bg^4 + \mathcal{O}(g^6)$$



$$f(g) = cg + d + \frac{e}{g} + \mathcal{O}(g^{-2})$$

Idea and motivation

Main idea

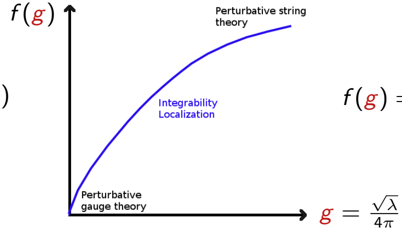
Discretize the two-dimensional string sigma model on a lattice and study the previous observables at finite coupling (still in the planar limit)

Motivation

Many progresses in obtaining exact results within AdS/CFT

- From integrability
- From supersymmetric localization

$$f(g) = ag^2 + bg^4 + \mathcal{O}(g^6)$$



$$f(g) = cg + d + \frac{e}{g} + \mathcal{O}(g^{-2})$$

Idea and motivation

Main idea

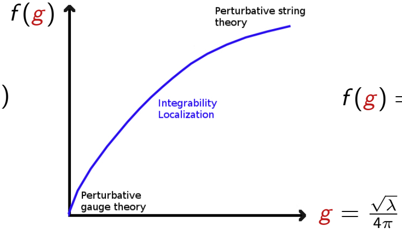
Discretize the two-dimensional string sigma model on a lattice and study the previous observables at finite coupling (still in the planar limit)

Motivation

Many progresses in obtaining exact results within AdS/CFT

- From integrability (assumed at finite coupling)
- From supersymmetric localization (BPS observables)

$$f(g) = ag^2 + bg^4 + \mathcal{O}(g^6)$$



$$f(g) = cg + d + \frac{e}{g} + \mathcal{O}(g^{-2})$$

Idea and motivation

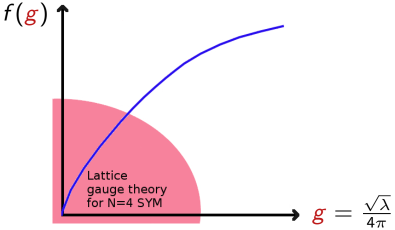
Main idea

Discretize the two-dimensional string sigma model on a lattice and study the previous observables at finite coupling (still in the planar limit)

Motivation

Many progresses in obtaining exact results within AdS/CFT

- From integrability (assumed at finite coupling)
- From supersymmetric localization (BPS observables)
- From the lattice on the CFT side [Catterall et al.]



Idea and motivation

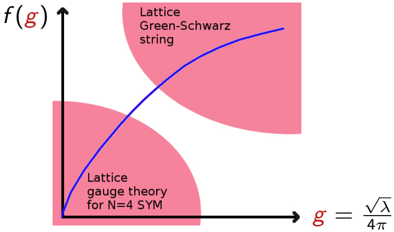
Main idea

Discretize the two-dimensional string sigma model on a lattice and study the previous observables at finite coupling (still in the planar limit)

Motivation

Many progresses in obtaining exact results within AdS/CFT

- From integrability (assumed at finite coupling)
- From supersymmetric localization (BPS observables)
- From the lattice on the CFT side [Catterall et al.] as well as on the string side [McKeown,Roiban], [LB, Bianchi, Forini, Leder, Töpfer, Vescovi]



The Lagrangian

Asymptotic spectrum

- Bosons: 1 mode ϕ $m^2 = 1$; 2 modes x, x^* $m^2 = 1/2$; 5 modes y^a $m^2 = 0$.
- Fermions: 8 modes θ^i, η^i $m^2 = \frac{1}{4}$.

$$S_{\rm cusp} = g \int dtds \, \Big\{ |\partial_t x + \tfrac{1}{2} x|^2 + \tfrac{1}{z^4} |\partial_s x - \tfrac{1}{2} x|^2 + \Big(\partial_t z^M + \tfrac{1}{2} z^M + \tfrac{i}{z^2} z_N \eta_i \left(\rho^{MN}\right)^i{}_j \eta^j \Big)^2$$
$$+ \frac{1}{z^4} \Big(\partial_s z^M - \tfrac{1}{2} z^M \Big)^2 + i \Big(\theta^i \partial_t \theta_i + \eta^i \partial_t \eta_i + \theta_i \partial_t \theta^i + \eta_i \partial_t \eta^i \Big) - \tfrac{1}{z^2} \left(\eta^i \eta_i \right)^2$$
$$+ 2i \Big[\tfrac{1}{z^3} z^M \eta^i (\rho^M)_{ij} \left(\partial_s \theta^j - \tfrac{1}{2} \theta^j - \tfrac{i}{z} \eta^j \left(\partial_s x - \tfrac{1}{2} x \right) \right)$$
$$+ \tfrac{1}{z^3} z^M \eta_i (\rho^\dagger_M)^{ij} \left(\partial_s \theta_j - \tfrac{1}{2} \theta_j + \tfrac{i}{z} \eta_j \left(\partial_s x - \tfrac{1}{2} x \right)^* \right) \Big] \Big\}$$

$$z = e^{\phi} \,, \qquad z^M = e^{\phi} u^M \,, \qquad M = 1, \ldots 6$$

$$u^a = \frac{y^a}{1 + \frac{1}{4} y^2} \,, \qquad u^6 = \frac{1 - \frac{1}{4} y^2}{1 + \frac{1}{4} y^2} \,, \qquad y^2 \equiv \sum_{a=1}^{5} (y^a)^2 \,, \qquad a = 1, \ldots, 5$$

Discretization and numerics

[LB, M. Bianchi, V.Forini, B.Leder, E. Vescovi, 2016]

Features

- Two-dimensional: computationally cheap.
- No worldsheet supersymmetry: GS string has target space SUSY
- No gauge symmetry: only scalar degrees of freedom, some anticommuting

Discretization and numerics

[LB, M. Bianchi, V.Forini, B.Leder, E. Vescovi, 2016]

Features

- Two-dimensional: computationally cheap.
- No worldsheet supersymmetry: GS string has target space SUSY
- No gauge symmetry: only scalar degrees of freedom, some anticommuting

Observables

- Action $\leftrightarrow$ Derivative of cusp anomalous dimension
$$\langle S \rangle = -g\partial_g \log Z \leftrightarrow Z = e^{-f(g)V}$$
- Mass of the x excitations
$$\sum_{s_1,s_2} \langle x(t,s_1)x^*(o,s_2)\rangle \overset{t\to\infty}{\sim} e^{-tm_x}$$

Discretization and numerics

[LB, M. Bianchi, V.Forini, B.Leder, E. Vescovi, 2016]

Features

- Two-dimensional: computationally cheap.
- No worldsheet supersymmetry: GS string has target space SUSY
- No gauge symmetry: only scalar degrees of freedom, some anticommuting

Observables

- Action $\leftrightarrow$ Derivative of cusp anomalous dimension
$$\langle S \rangle = -g\partial_g \log Z \leftrightarrow Z = e^{-f(g)V}$$
- Mass of the x excitations
$$\sum_{s_1,s_2} \langle x(t,s_1)x^*(o,s_2)\rangle \overset{t\to\infty}{\sim} e^{-tm_x}$$

Technical obstacles

- Fermion doubling: introduce a proper Wilson term .
- Quartic fermionic interaction: Hubbard-Stratonovich linearization
- Sign problem

Sign problem

Hubbard-Stratonovich

$$\exp\Big\{-g\int dtds\Big[-\tfrac{1}{z^2}\left(\eta^i\eta_i\right)^2+\Big(\tfrac{i}{z^2}z_N\eta_i\rho^{MN\,i}{}_j\eta^j\Big)^2\Big]\Big\}$$

$$\sim\int D\phi D\phi_M\,\exp\Big\{-g\int dtds\,[\tfrac{1}{2}\phi^2+\tfrac{\sqrt{2}}{z}\phi\,\eta^2+\tfrac{1}{2}(\phi_M)^2-i\,\tfrac{\sqrt{2}}{z^2}\phi_M z_N\left(i\,\eta_i\rho^{MN\,i}{}_j\eta^j\right)]\Big\}$$

$$\mathcal{L}=|\partial_t x+\tfrac{m}{2}x|^2+\tfrac{1}{z^4}|\partial_s x-\tfrac{m}{2}x|^2+(\partial_t z^M+\tfrac{1}{2}z^M)^2+\tfrac{1}{z^4}(\partial_s z^M-\tfrac{m}{2}z^M)^2+\tfrac{1}{2}\phi^2+\tfrac{1}{2}(\phi_M)^2+\psi^T O_F\psi$$

$$\int D\psi\;e^{-\int dtds\,\psi^T O_F\psi}=\mathrm{Pf}\,O_F\equiv(\det O_F\,O_F^\dagger)^{\frac{1}{4}}=\int D\xi D\bar{\xi}\,e^{-\int dtds\,\bar{\xi}(O_F O_F^\dagger)^{-\frac{1}{4}}\xi}$$

There is a sign problem.

Sign problem

Hubbard-Stratonovich

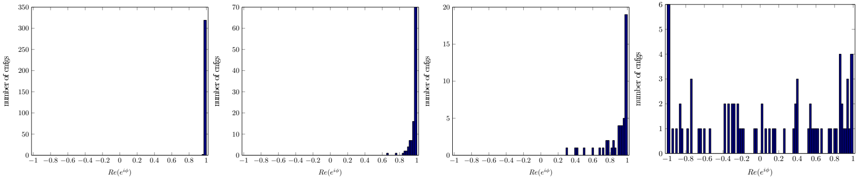
$$\exp\Big\{-g\int dtds\Big[-\frac{1}{z^2}\left(\eta^i\eta_i\right)^2+\Big(\frac{i}{z^2}z_N\eta_i\rho^{MN\,i}{}_j\eta^j\Big)^2\Big]\Big\}$$

$$\sim\int D\phi D\phi_M\,\exp\Big\{-g\int dtds\,\big[\tfrac{1}{2}\phi^2+\tfrac{\sqrt{2}}{z}\phi\,\eta^2+\tfrac{1}{2}(\phi_M)^2-i\,\tfrac{\sqrt{2}}{z^2}\phi_M z_N\left(i\,\eta_i\rho^{MN\,i}{}_j\eta^j\right)\big]\Big\}$$

$$\mathcal{L}=\left|\partial_t x+\tfrac{m}{2}x\right|^2+\tfrac{1}{z^4}\left|\partial_s x-\tfrac{m}{2}x\right|^2+\left(\partial_t z^M+\tfrac{1}{2}z^M\right)^2+\tfrac{1}{z^4}\left(\partial_s z^M-\tfrac{m}{2}z^M\right)^2+\tfrac{1}{2}\phi^2+\tfrac{1}{2}(\phi_M)^2+\psi^T O_F\psi$$

$$\int D\psi\; e^{-\int dtds\,\psi^T O_F\psi}=\mathrm{Pf}\,O_F\equiv\left(\det O_F\,O_F^\dagger\right)^{\frac{1}{4}}=\int D\xi D\bar{\xi}\;e^{-\int dtds\,\bar{\xi}(O_F O_F^\dagger)^{-\frac{1}{4}}\,\xi}$$

There is a sign problem.



A partial solution

$$\mathcal{L}_4 = \frac{1}{z^2}\left(-(\eta^2)^2 + \left(i\,\eta_i(\rho^{MN})^i{}_j n^N \eta^j\right)^2\right) = \frac{1}{z^2}\left(-4\,(\eta^2)^2 + 2\left|\eta_i(\rho^N)^{ik} n_N \eta_k\right|^2\right)$$

$$\Sigma_i{}^j = \eta_i \eta^j \qquad \tilde{\Sigma}_j{}^i = (\rho^N)^{ik} n_N (\rho^L)_{jl} n_L \eta_k \eta^l$$

$$\Sigma_{\pm}{}_i^j = \Sigma_i^j \pm \tilde{\Sigma}_i^j$$

A partial solution

$$\mathcal{L}_4 = \frac{1}{z^2}\left(-(\eta^2)^2 + \left(i\,\eta_i(\rho^{MN})^i{}_j n^N \eta^j\right)^2\right) = \frac{1}{z^2}\left(-4\,(\eta^2)^2 + 2\left|\eta_i(\rho^N)^{ik} n_N \eta_k\right|^2\right)$$

$$\Sigma_i{}^j = \eta_i \eta^j \qquad \tilde{\Sigma}_j{}^i = (\rho^N)^{ik} n_N (\rho^L)_{jl} n_L \eta_k \eta^l$$

$$\Sigma_\pm{}_i^j = \Sigma_i^j \pm \tilde{\Sigma}_i^j$$

$$\mathcal{L}_4 = \frac{1}{z^2}\left(-4\,(\eta^2)^2 \mp 2(\eta^2)^2 \mp \Sigma_\pm{}_i^j \Sigma_\pm{}_j^i\right)$$

A partial solution

$$\mathcal{L}_4 = \frac{1}{z^2}\left(-(\eta^2)^2 + \left(i\,\eta_i(\rho^{MN})^i{}_j n^N \eta^j\right)^2\right) = \frac{1}{z^2}\left(-4\,(\eta^2)^2 + 2\left|\eta_i(\rho^N)^{ik} n_N \eta_k\right|^2\right)$$

$$\Sigma_i{}^j = \eta_i \eta^j \qquad \tilde{\Sigma}_j{}^i = (\rho^N)^{ik} n_N (\rho^L)_{jl} n_L \eta_k \eta^l$$

$$\Sigma_{\pm}{}_i^j = \Sigma_i^j \pm \tilde{\Sigma}_i^j$$

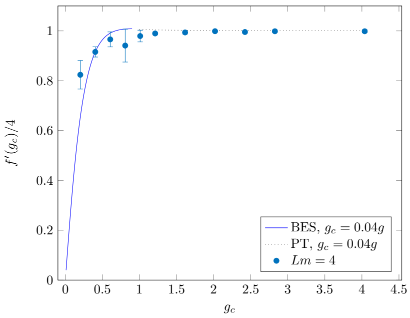
$$\mathcal{L}_4 = \frac{1}{z^2}\left(-4\,(\eta^2)^2 \mp 2(\eta^2)^2 \mp \Sigma_{\pm}{}_i^j \Sigma_{\pm}{}_j^i\right)$$

Different linearization

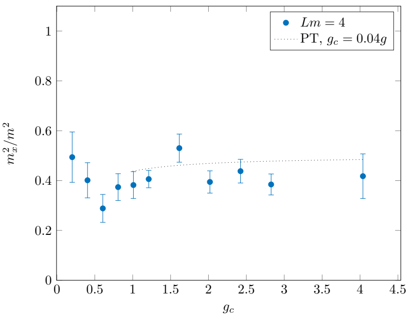
$$\mathcal{L}_4 \to \frac{12}{z}\eta^2\phi + 6\phi^2 + \frac{2}{z}\eta_j\phi_i^j\eta^i + \frac{2}{z}(\rho^N)^{ik} n_N \eta_k \phi_i^j (\rho^L)_{jl} n_L \eta^l + \phi_j^i \phi_i^j$$

Data

Action expectation value

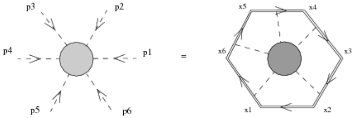


Mass of the x



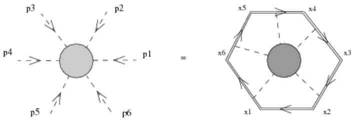
THANK YOU

Scattering amplitudes



GKP string

Scattering amplitudes



GKP string

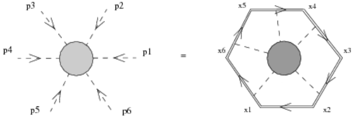
- Free energy: $\log Z = \Gamma_{\text{cusp}}(\lambda) V$

Square and pentagon [Drummond, Henn, Korchemsky, Sokatchev, 2007]

$$\log F_4 = \frac{1}{4}\Gamma_{\text{cusp}}(a)\log^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \text{const}$$

$$\log F_5 = -\frac{1}{8}\Gamma_{\text{cusp}}(a)\sum_{i=1}^{5}\log\left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\right)\log\left(\frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2}\right) + \text{const}$$

Scattering amplitudes



GKP string

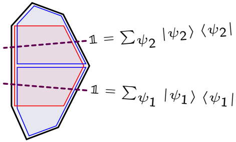
- Free energy: $\log Z = \Gamma_{\text{cusp}}(\lambda) V$
- Dispersion relation: $E_i = E_i(p_i) \Leftrightarrow \{E(\mathbf{u}), p(\mathbf{u})\}$
- S-matrix: $S(\mathbf{u},\mathbf{v}) \Leftrightarrow P(\mathbf{u}|\mathbf{v})$

Square and pentagon [Drummond, Henn, Korchemsky, Sokatchev, 2007]

$$\log F_4 = \frac{1}{4}\Gamma_{\text{cusp}}(a) \log^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \text{const}$$

$$\log F_5 = -\frac{1}{8}\Gamma_{\text{cusp}}(a) \sum_{i=1}^{5} \log\left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\right) \log\left(\frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2}\right) + \text{const}$$

Hexagon and higher [Basso, Sever, Vieira, 2013]



$$\mathcal{W}_{\text{hep}} = \sum_{\psi_1,\psi_2} P(0|\psi_1)P(\psi_1|\psi_2)P(\psi_2|0) \times e^{-E_1\tau_1 + ip_1\sigma_1 + im_1\phi_1 - E_2\tau_2 + ip_2\sigma_2 + im_2\phi_2}$$