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**Beyond complex Langevin equations:  
from simple examples to positive representation  
of Feynman path integrals directly in Minkowski time**

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In general quantum averages result from complex "densities"

- A harmonic oscillator

$$\langle x^2(T) \rangle = \frac{\int dx x^2 K(x, x; T)}{\int dx K(x, x; T)}.$$

- QED/QCD with external charges

$$Z = \int DA \exp \left\{ \frac{i}{\hbar} S[A] \right\} \exp \left\{ \frac{i}{\hbar} \int d^4x A_\mu J_{ext}^\mu \right\} \leftrightarrow \int DU \exp \{ S_{latt}[U] \} W[U] \quad (1)$$

- QCD with finite chemical potential

$$Z = \int DU \exp \{ S_{latt}[U] \} Det[A_0 \rightarrow A_0 + i\mu] \quad (2)$$

**P. Hasenfratz, F. Karsch, 1983**

## I. The Langevin method - real and complex cases

### Real case

$$S(x) \xrightarrow{\dot{x}(\tau) = -\partial_x S + \eta(\tau)} x(\tau) \longrightarrow P(x, \tau) \xrightarrow{\tau \rightarrow \infty} P(x) = e^{-S(x)}$$

$$\int f(x) e^{-S(x)} dx / \int e^{-S(x)} dx = \int f(x) P(x) dx / \int P(x) dx .$$

Complex case, e.g.  $\rho(x) \equiv e^{-S(x)} = e^{-\sigma x^2/2}, \sigma \in \mathcal{C}$ ,

G. Parisi, 1983

J. Klauder, 1983

$$S(x) \xrightarrow{\dot{z}(\tau) = -\partial_z S + \eta(\tau)} z(\tau) \longrightarrow P(x, y, \tau)$$

But  $P(x, y, \tau)$  does not  $\xrightarrow{\tau \rightarrow \infty} e^{-S}$

Nevertheless in some cases  $P(x, y, \infty)$  exists, and indeed

$$\frac{\int f(x) e^{-S(x)} dx}{\int e^{-S(x)} dx} = \frac{\iint f(x + iy) P(x, y, \infty) dx dy}{\iint P(x, y, \infty) dx dy} . \quad (3)$$

Example (Ambjorn and Yang, 1985)

$$\rho(x) = \exp(-S_g(x)) = \exp\left(-\frac{\sigma}{2}x^2\right), \quad \sigma = \sigma_R + i\sigma_I, \quad \sigma_R > 0$$

$$P(x, y) = \exp\left(-\sigma_R(x^2 + 2rxy + (1 + 2r^2)y^2)\right), \quad r = \frac{\sigma_R}{\sigma_I},$$

indeed

$$\langle x^n \rangle_{\rho(x)} = \langle (x + iy)^n \rangle_{P(x,y)}$$

or for the generating function

$$G(t) = \frac{\int_{-\infty}^{\infty} e^{tx} e^{-S_g(x)} dx}{\int_{-\infty}^{\infty} e^{-S_g(x)} dx} = \exp\left(\frac{t^2}{2\sigma}\right) = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy e^{t(x+iy)} P(x, y)}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy P(x, y)}.$$

In general, however

- $P(x, y, \tau)$  not known
- not connected with  $S$

Moreover

- $z(\tau) \rightarrow \infty$ , i.e. diverges
- $z(\tau) \rightarrow$  wrong answer

E. Seiler et al. ( 2 - 6 ), 2005 - present

## II. Avoiding the trouble

- Construct  $\rho, P$  without any reference to the stochastic process.

The only requirement

$$\int_{\mathbb{R}} f(x)\rho(x)dx / \int_{\mathbb{R}} \rho(x)dx = \int \int f(x + iy)P(x, y)dx dy / \int \int P(x, y)dx dy .$$

- • Introduce two, independent complex variables

$$z = x + iy, \quad \bar{z} = x - iy$$

then

$$\int_{\Gamma_z} f(z)\rho(z)dz / \int_{\Gamma_z} \rho(z)dz = \int_{\Gamma_z} \int_{\Gamma_{\bar{z}}} f(z)P(z, \bar{z})dz d\bar{z} / \int_{\Gamma_z} \int_{\Gamma_{\bar{z}}} P(z, \bar{z})dz d\bar{z}$$

sufficient condition

$$\rho(z) = \int_{\Gamma_{\bar{z}}} P(z, \bar{z})d\bar{z}$$

additionally require

$$P(z, \bar{z})|_{x+iy, x-iy} \quad \text{positive and normalizable}$$

## The construction

1. Find  $P(z, \bar{z})$  which satisfies

$$\rho(z) = \int_{\Gamma_{\bar{z}}} P(z, \bar{z}) d\bar{z}$$

and

2. Is positive and normalizable at

$$P(z, \bar{z})|_{x+iy, x-iy} = P(x, y)$$

• Then

$$\int f(z) \rho(z) dz / \int \rho(z) dz = \int \int f(x + iy) P(x, y) dx dy / \int \int P(x, y) dx dy$$

## Example 1 - generalized gaussian model

$$\begin{aligned}
 S(z, \bar{z}) &= a^* z^2 + 2bz\bar{z} + a\bar{z}^2 \quad |_{x+iy, x-iy} = 2(b + \alpha)x^2 + 4\beta xy + 2(b - \alpha)y^2, \\
 a &= \alpha + i\beta, \quad b = b^*, & \lambda_{\pm} &= 2(b \pm |a|) \\
 P(x, y) &= e^{-S(x,y)} & & \text{is positive and normalizable for } b > |a|
 \end{aligned}$$

At the same time

$$\rho(z) = \int_{\Gamma_{\bar{z}}} P(z, \bar{z}) d\bar{z} = \frac{1}{2} \sqrt{\frac{\pi}{-a}} \exp(-sz^2), \quad s = \frac{|a|^2 - b^2}{a}.$$

This reduces to  $\exp\left(-\frac{\sigma}{2}z^2\right)$  if

$$b = \frac{\sigma_R}{2}(1 + r^2), \quad \alpha = -\frac{\sigma_R}{2}r^2, \quad \beta = \frac{\sigma_R}{2}r, \quad r = \frac{\sigma_R}{\sigma_I}, \quad \sigma_R > 0$$



- But is more general.
- Provides positive representation for any complex  $a$ .
- For example:

$\alpha = 0, \beta \neq 0 - e^{i|s|x^2} - \text{”Minkowski” integrals}$

$\alpha > 0, \beta = 0 - e^{+|s|x^2} - \text{”a striking example:”}$

complex stochastic process has the correct fixed point also for  $\sigma_R < 0$ ”

$$\int f(x + iy)P(x, y)dxdy = \text{analytic continuation of } \int \rho(x)f(x)dx$$

H. Okamoto et al., 1988  
 L.L. Salcedo, 1997  
 D. Weingarten, 2002

## Example 2 - quartic action

$$S_4(z, \bar{z}) = \frac{d}{b}(a^* z^2 + 2bz\bar{z} + a\bar{z}^2)^2,$$

$$\rho_4(z) = \frac{i}{2} \int_{\Gamma_{\bar{z}}} d\bar{z} e^{-S_4(z, \bar{z})} = \frac{i}{2} \left( \frac{b}{2da^2} \right)^{\frac{1}{4}} \exp(-\sigma z^4) (\sigma z^4)^{\frac{1}{4}} K_{\frac{1}{4}}(\sigma z^4),$$

with an arbitrary complex

$$\sigma = \frac{d(b^2 - |a|^2)^2}{2ba^2}.$$

$$\rho(z) \sim e^{(-2\sigma z^4)}, \quad z \longrightarrow \infty$$

again

$$\langle \mathbf{x}^n \rangle_{\rho(x)} \longleftarrow \langle z^n \rangle_{\rho(z)} = \langle (x + iy)^n \rangle_{P(x,y)}$$

### III. Path integrals in Minkowski time – a free particle

**2N variables**  $z_i, \bar{z}_i$  ( $\equiv z, \bar{z}$ ) **with periodic boundary conditions**

$$S_N(z, \bar{z}) = \sum_{i=1}^N a \bar{z}_i^2 + 2b \bar{z}_i z_i + 2c \bar{z}_i z_{i+1} + 2c^* z_i \bar{z}_{i+1} + a^* z_i^2, \quad a, c \in C, b \in R.$$

$$\rho(z) \sim \int \prod_{i=1}^N d\bar{z}_i \exp(-S_N(z, \bar{z})) \sim \exp\left(\mathcal{A} \sum_{i=1}^N (z_{i+1} - z_i)^2 - r (z_{i+1} - z_{i-1})^2\right)$$

$$2c \equiv 2\gamma = -b + |a|, \quad \mathcal{A} = \frac{b(b - |a|)}{a}, \quad r = \frac{b - |a|}{4b}.$$

to be compared with 
$$S_N^{free} = \frac{im}{2\hbar\epsilon} \sum_{i=1}^N (z_{i+1} - z_i)^2$$

$P_N(x_i, y_i)$  is real, positive and normalisable, up to a zero mode

$$\lambda_0 = 2(b - |a| + 2\gamma),$$

## The continuum limit of the propagator

$$K_N(z_N, z_1) = e^{-\mathcal{A}(z_N - z_1)^2} \int dz_2 \dots dz_{N-1} e^{-S_N^\rho(z_1, \dots, z_N)},$$

### Transfer matrix $\rightarrow$ recursion

$$K_N(z_N, z_1) \sim \exp\left(\mathcal{A}\sigma_N(\mathbf{r})(z_N - z_1)^2\right), \quad \sigma_N(\mathbf{r}) = \frac{P_N(\mathbf{r})}{Q_N(\mathbf{r})}$$

$$\sigma_N(\mathbf{r}) = v_0(\mathbf{r}) + \frac{v_1(\mathbf{r})}{N} + \frac{v_2(\mathbf{r})}{N^2} + \dots, N \rightarrow \infty$$

$r = 0 \implies v_1 = 1, v_i = 0, i \neq 1$  – Feynman case

$r \neq 0 \implies v_i \neq 0$  – no continuum limit since  $\mathcal{A} \sim 1/\epsilon$

• • • define the new limit ( $\lim_1$ )

$$|a|, b \rightarrow \infty, b - |a| = \frac{m}{2\hbar\epsilon} = \text{const.}, \quad a = -i|a|.$$

• • • than physics =  $\lim_{N \rightarrow \infty} \lim_1 \langle O \rangle_{P_N(x,y)}$

## VI. Path integrals in Minkowski time – an harmonic oscillator

$$\rho(z) = \left(\frac{i}{2}\right)^N \int \prod_{i=1}^N d\bar{z}_i P(z, \bar{z}) \sim \exp(-S_N^\rho(z))$$

gaussian integrations

$$-S_N^\rho(z) = \sum_{i=1}^N -E \left( (z_{i+1} - z_i)^2 - \left(\frac{D}{E} + 1\right) (z_i^2 + z_{i+1}^2) \right)$$

with

$$D = \frac{b^2 + 4\gamma^2 - |a|^2}{2a}, \quad E = \frac{2b\gamma}{a}$$

compare with an harmonic oscillator

$$\frac{im}{2\hbar\epsilon} \left( (x_1 - x_2)^2 - \frac{\omega^2\epsilon^2}{2} (x_1^2 + x_2^2) \right)$$

## The continuum limit for HO

$$\rho = \frac{\omega^2 T^2}{2(N-1)^2}, \quad \mu = \frac{m(N-1)}{2\hbar T},$$

$$a = -i|a|, \quad b = \frac{\mu}{\nu}, \quad |a| = \frac{\mu}{\nu} \zeta(\nu, \rho), \quad 2\gamma = -\mu \zeta(\nu, \rho)$$

where

$$\zeta(\nu, \rho) = \frac{\sqrt{1 - 2\nu^2\rho + \nu^2\rho^2} - \nu(1 - \rho)}{1 - \nu^2}$$

$$\langle O \rangle = \overbrace{\lim_{N \rightarrow \infty}}^{\text{lim2}} \overbrace{\lim_{\nu \rightarrow 0}}^{\text{lim1}} \langle O \rangle_{P_N(x,y)}$$

## IV. Applications – a free particle

The propagator from the new  $(x, y)$  representation

$$K_N(z_N, z_1) = e^{(-\mathcal{A}(z_1 - z_N)^2)} \int d\bar{z}_1 \prod_{j=2}^{N-1} dx_j dy_j d\bar{z}_N \exp(-X^T M X)$$

$$X^T = (z_1, \bar{z}_1, x_2, y_2, x_3, \dots, y_{N-1}, z_N, \bar{z}_N)$$

$$K_N(z_N, z_1) \sim \exp(\sigma_N(a, b)(z_N - z_1)^2),$$

$N$	$\sigma_N(a, b)$	$lim\ 1$
5	$\frac{id(16b^2 + 28bd - 19d^2)}{8(8b - 3d)(b - d)}$	$i\frac{d}{4}$
8	$\frac{id(16b^4 + 40b^3d - 70b^2d^2 + 23bd^3 - d^4)}{(b - d)(112b^3 - 120b^2d + 30bd^2 - d^3)}$	$i\frac{d}{7}$
11	$\frac{id(1024b^5 + 3328b^4d - 9472b^3d^2 + 6832b^2d^3 - 1700bd^4 + 109d^5)}{(b - d)(1280b^4 - 2304b^3d + 1344b^2d^2 - 280bd^3 + 15d^4)}$	$i\frac{d}{10}$

$$lim1 : \quad |a|, b \rightarrow \infty, b - |a| = \frac{m}{2\hbar\epsilon} = const. \equiv d, \quad a = -i|a|.$$

$$\lim_{N \rightarrow \infty} lim1 K_N \sim \exp\left(\frac{im}{2\hbar} \frac{(z_N - z_1)^2}{T}\right) \longrightarrow \exp\left(\frac{im}{2\hbar} \frac{(x_N - x_1)^2}{T}\right) \equiv K$$

## An average

### Classics

$$\langle x^2(t) \rangle = \int K(0, x; T-t) x^2 K(x, 0; t) dx / K(0, 0; T) = \frac{i\hbar t(T-t)}{m T}$$

### New representation

$$\begin{aligned} \langle z_k^2 \rangle \Big|_{z_1=z_N=0} &= \int d\bar{z}_1 dx_2 dy_2 \dots dy_{N-1} d\bar{z}_N (x_k + iy_k)^2 \exp(-X^T M X) / Z \\ &= \frac{1}{2} \left( M_{2k-2, 2k-2}^{-1} + i(M_{2k-2, 2k-1}^{-1} + M_{2k-1, 2k-2}^{-1}) - M_{2k-1, 2k-1}^{-1} \right), \end{aligned}$$

$$\langle z_k^2 \rangle \xrightarrow{\text{lim}_1} \frac{i}{2d} \frac{(k-1)(N-k)}{N-1} \xrightarrow{\text{lim}_2} \frac{i\hbar t(T-t)}{m T}$$

zero mode  $\implies$  fixed end points  $\implies$  two integrals (  $d\bar{z}_1, d\bar{z}_N$  ) not positive



## V. Applications – an harmonic oscillator

$$\langle x^2(T) \rangle = \langle x^2(0) \rangle = \frac{\int dx x^2 K(x, x; T)}{\int dx K(x, x; T)}.$$

### Classics

$$K(x_b, x_a; T) \sim \exp \left\{ \frac{i}{\hbar} \frac{m\omega}{2 \sin \omega T} \left( (x_a^2 + x_b^2) \cos \omega T - 2x_a x_b \right) \right\},$$

gives

$$\langle x^2(T) \rangle = -\frac{i\hbar T \cot \frac{\omega T}{2}}{4m \frac{\omega T}{2}}.$$

### The new way

$$\langle z_1^2 \rangle = \frac{1}{Z} \int \prod_{j=1}^N dx_j dy_j (x_1 + iy_1)^2 \exp \{ -X^T M X \}.$$

$$\lim_{m \rightarrow \infty} \langle z_1^2 \rangle = \lim_{\nu \rightarrow 0} \langle z_1^2 \rangle = -\frac{i\hbar T P_N(\omega T/2)}{m Q_N(\omega T/2)}.$$

TABLE II

$N$	$P_N(x)/Q_N(x)$
5	$\frac{(x^2-2x-4)(x^2+2x-4)}{x^2(x^4-20x+80)}$
8	$\frac{7(128x^8-12544x^6+384160x^4-3764768x^2+5764801)}{32x^2(x^2-49)(2x^2-49)(8x^4-392x^2+2401)}$
11	$\frac{5(x^5-5x^4-100x^3+375x^2+1875x-3125)(x^5+5x^4-100x^3-375x^2+1875x+3125)}{2x^2(x^{10}-275x^8+27500x^6-1203125x^4+21484375x^2-107421875)}$

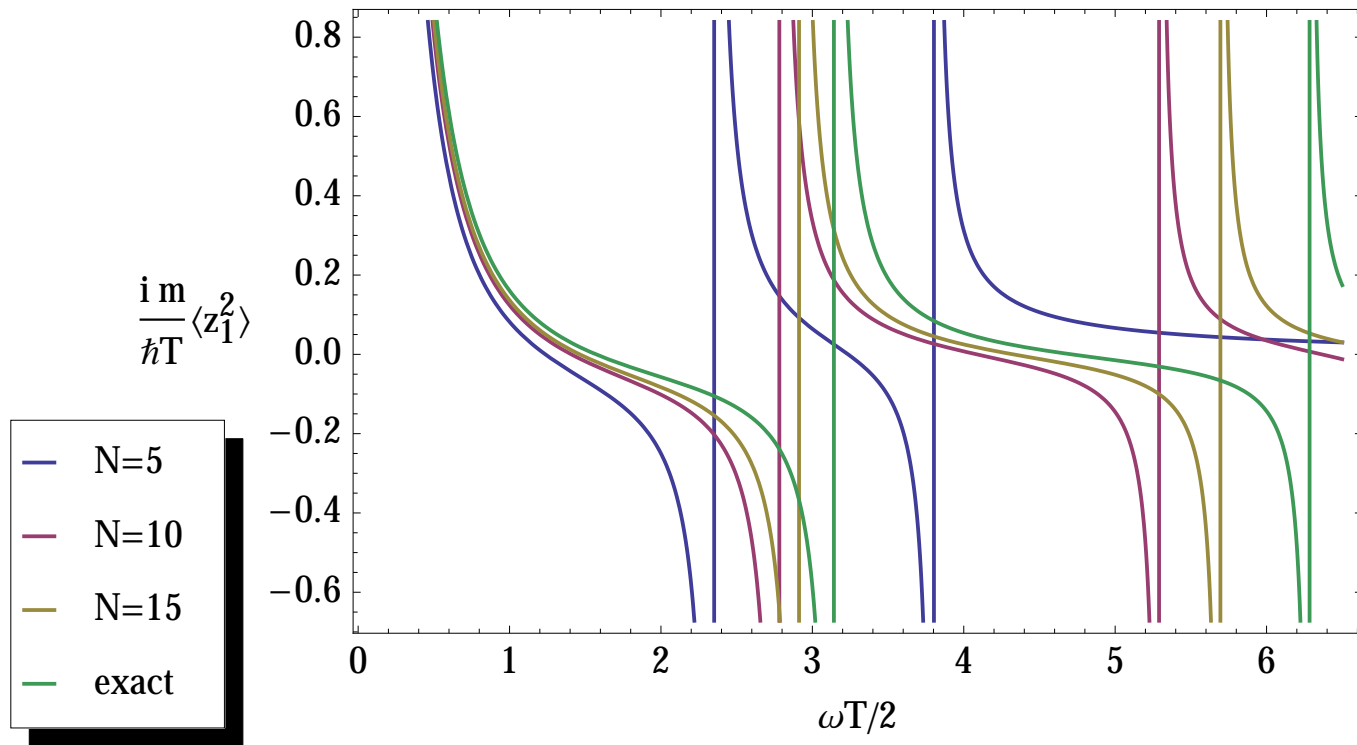


Figure 1:

- the average is over positive probability
- $\lim$  of our average reproduces the standard discretization
- one negative mode

$$S_N(x, y) = 2 \sum_{i=1}^N (b + \alpha)x_i^2 + 2\beta x_i y_i + (b - \alpha)y_i^2 + 2\gamma(x_i x_{i+1} + y_i y_{i+1}).$$

The lowest eigenvalue

$$\lambda_0 = 2(b - |a| + 2\gamma) = \begin{cases} 0 & \text{a free particle} \\ \xrightarrow{\lim} -\frac{m\omega^2 T}{4\hbar(N-1)}, & \text{an harmonic oscillator} \end{cases} \quad (4)$$

One can:

fix it

regularize it –  $M^{-1}$  exists

→ use our trick again

relax periodic boundary conditions

$\exp\{-S_N\}$  provides a positive representation for the path integral of HO

## VI. Applications – a charged particle in a constant magnetic field

### A cute observable

One can see classical orbits looking at the time dependence of quantum averages with fixed-end-points propagators as weights.

### Define

$$\langle \vec{x} \rangle = \int d^2x K(\vec{x}_b, \vec{x}; T-t) \vec{x} K(\vec{x}, \vec{x}_a; t) / K(\vec{x}_b, \vec{x}_a; T) = x_{x_a, x_b, T}^{cl}(t)$$

- is gauge invariant
- can be reduced to HO averages

$$\begin{aligned} \langle x(t) \rangle_B &= \langle x(t) \rangle_{O=O_x} \\ \langle y(t) \rangle_B &= \langle y(t) \rangle_{O=O_y} \end{aligned}$$

Two gauges

$$\vec{A}^{FG} = \frac{B}{2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \quad \vec{A}^{LG} = B \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix} \quad (5)$$

Landau: particle in a magnetic field  $\leftrightarrow$  1 d harmonic oscillator

$$H = \frac{1}{2m} \hat{P}_x^2 + \frac{1}{2m} \left( \hat{P}_y - \frac{eB}{c} x \right)^2 = \frac{1}{2m} \hat{P}_x^2 + \frac{m\omega^2}{2} (x - O_x)^2$$

$$\Psi(x, y) = \exp\left(\frac{i}{\hbar} p_y y\right) \psi(x), \quad \omega = \frac{eB}{mc}, \quad O_x = \frac{c}{Be} p_y,$$

Feynman kernel in the Landau gauge

$$K_{LG} = \exp \left\{ \frac{im}{2\hbar} \left( \frac{\omega}{2} \cot \frac{\omega T}{2} (\Delta x^2 + \Delta y^2) + \omega(x_a + x_b)\Delta y \right) \right\}$$

→ split the LG propagator as follows

$$K_{LG} = \underbrace{\exp \left\{ \frac{i m}{\hbar} \frac{1}{2} (\omega(x_a + x_b)\Delta y + \omega ct\Delta y^2) \right\}}_{\exp \left( \frac{i}{\hbar} p_y (y_b - y_a) \right) = \exp \left( \frac{i}{\hbar} m\omega O_x (y_b - y_a) \right)} \underbrace{\exp \left\{ \frac{i m}{\hbar} \frac{1}{2} \left( \frac{\omega}{2} \cot \frac{\omega T}{2} (\Delta x^2 - \Delta y^2) \right) \right\}}_{K_{O_x}^{HO}} \quad (6)$$

with the classical expression for the center of oscillations

$$O_x = \frac{1}{2}(x_a + x_b) + \frac{1}{2} \cot \frac{\omega T}{2} (y_b - y_a)$$

more precisely

$$K^B(\vec{x}_b, \vec{x}_a; T) = \int dO \exp \left\{ \frac{i}{\hbar} m\omega O (y_b - y_a) \right\} K_O^{HO}(x_b, x_a; T)|_{saddlepoint} = (6)$$

→ Need positive distribution representing shifted harmonic oscillator.

Simply add linear terms

$$S_N(z, \bar{z}) \rightarrow S_N(z, \bar{z}) + \sum_i e^* z_i + e \bar{z}_i \leftrightarrow S_N(z - z_c, \bar{z} - z_c^*),$$

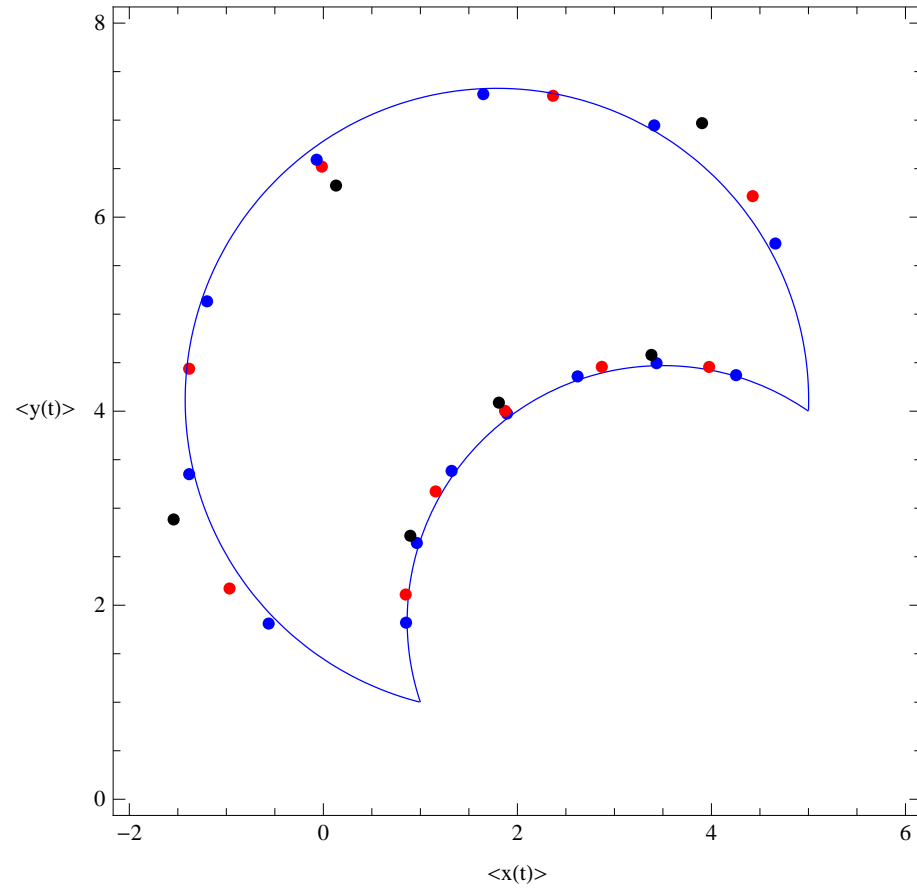
then the average

$$\langle z_k \rangle_{P(x,y)} = \langle x_k + iy_k \rangle_{P(x,y)}$$

reproduces, after taking *lim1*, the magnetic field average.



$N = 5, 7, 9;$      $\omega T / 2\pi = 0.4, 0.7$



## VII. Summary

Problems of complex Langevin approach were avoided.

Instead of simulating badly converging complex random walks, pairs of corresponding weights were directly constructed.

- Two complex variables  $z$  and  $\bar{z}$ .
- The sum rule  $\rho(z) = \int d\bar{z} P(z, \bar{z})$  was derived and used to explicitly obtain  $P$  and  $\rho$  in few cases of physical interest.

One degree of freedom:

gaussian model was generalized to arbitrary complex slope  
a particular quartic problem was also solved  
a class of nonlinear models

Quantum mechanical path integrals directly in Minkowski time:

a free particle  
an harmonic oscillator  
a particle in a constant magnetic field – standard Wick rotation  
does not give positive representation in this case

## VIII. To be done – plenty

- Explore a freedom in  $P(z, \bar{z})$  to derive it from given  $\rho(z)$
- Mathematics: how to construct positive and normalizable  $P$  from  $\rho$  ?  
– deformations (?)
- Generalizations – 1 DOF: nonabelian models, compact variables, fermions
- Generalizations – field theory ...

## IX. An amusing interpretation

New variables ( $\bar{z}'s$ )  $\longrightarrow$

- I. Quantum amplitudes became positive
- II.  $\longrightarrow$  no interference.
- III. Path integrals have to be evaluated in the limit of  $|a|, b \rightarrow \infty$
- IV.  $\longrightarrow$  a single trajectory might dominate the path integral (saddle point).
- V. The variables are integrated over – they are hidden !

Are these the hidden variables of quantum mechanics?

$\Rightarrow$  There are some similarities but there is **no revolution** .

## X. Demystification

- I. Quantum amplitudes became positive
  - only integrands of  $\int dx dy$  did –
  - full quantum amplitudes are integrals over new variables with fixed end-points  $\rightarrow$  they are complex.
- II.  $\rightarrow$  No interference
  - but true quantum amplitudes add only after integrating new ones at fixed points  $\rightarrow$  there is an interference.
- III. Path integrals have to be evaluated in the limit of  $|a|, b \rightarrow \infty$ 
  - but the difference  $b - |a|$  remains fixed and it is  $O(1/\hbar)$ .
- IV.  $\rightarrow$  A single trajectory might dominate path integral
  - $\rightarrow$  finite difference  $b - |a|$  drives quantum fluctuations even at infinite  $b, |a|$ .
- V. Nevertheless the variables are integrated over – they are hidden !

- ● Therefore quantum mechanics remains intact, Bell inequalities are violated (as they should), etc., etc.
- ● Nevertheless some new structure may have shown up.
- ● It remains to be seen if these variables tell us more than just help with some numerical calculations.