Computer Algebra and Particle Physics (CAPP) 2007

29. March 2007

The summation package Sigma evaluates Feynman integrals

Carsten Schneider

RISC-Linz, Austria

Warmup example
(bonus problem 6.69 in "Concrete Mathematics")

FIND a closed form for

$$\sum_{k=1}^{n} k^2 \mathsf{H}_{n+k} = \text{?},$$

where $\mathsf{H}_n := \sum_{k=1}^{n} \frac{1}{k}$.

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Telescoping

GIVEN $f(k)=k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k)=g(k+1)-g(k)}$$

for all $1\leq k\leq n$ and $n\geq 0$.

Sigma computes

$$g(k)=\frac{1}{36}(k(-4k^2+(6n+3)k-12n(n+1)+1)\\ +6(2k^3-3k^2+k+n(n+1)(2n+1))H_{k+n}).$$

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} k^2 \mathrm{H}_{n+k}} = \boxed{g(n+1) - g(1)}$$

$$= -\frac{1}{36} n(n+1)\left(10n + 6(2n+1)H_n - 12(2n+1)H_{2n} - 1\right).$$

Telescoping

GIVEN $f(k) = k^2 H_{n+k}$.

FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

for all $1 \leq k \leq n$ and $n \geq 0$.

Summing this equation over k from 1 to n gives

$$\boxed{\sum_{k=1}^{n} k^2 \mathrm{H}_{n+k}} = \boxed{g(n+1) - g(1)}$$

Try it out!

$$= -\frac{1}{36} n(n+1)\left(10n + 6(2n+1)H_n - 12(2n+1)H_{2n} - 1\right).$$

Telescoping

FIND a closed form for

$$\sum_{k=1}^{n} H_k.$$

A difference field for the summand

Consider the rational function field

$$\mathbb{F} := \mathbb{Q}(k)(h)$$

with the automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ defined by

$$\begin{aligned} \sigma(c) &= c \quad \forall c \in \mathbb{Q}, & & \\ \sigma(k) &= k+1, & \mathcal{S}\,k &= k+1, \\ \sigma(h) &= h + \frac{1}{k+1}, & \mathcal{S}\,H_k &= H_k + \frac{1}{k+1}. \end{aligned}$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

Sigma computes

$$g = (h-1)k \in \mathbb{F}.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

Sigma computes

$$g = (h-1)k \in \mathbb{F}.$$

This gives

$$\boxed{g(k+1) - g(k) = H_k}$$

with

$$g(k) = (H_k - 1)k.$$

Telescoping in the given difference field

FIND $g \in \mathbb{F}$:

$$\boxed{\sigma(g) - g = h.}$$

Sigma computes

$$g = (h-1)k \in \mathbb{F}.$$

This gives

$$\boxed{g(k+1) - g(k) = H_k}$$

with

$$g(k) = (H_k - 1)k.$$

Hence,

$$(H_{n+1} - 1)(n+1) = \sum_{k=1}^{n} H_k.$$

Details General

telescoping

- GIVEN

$$\sum_{k=0}^{n} f(k).$$

- FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

telescoping

- GIVEN

$$\sum_{k=0}^{n} f(k).$$

- FIND $g(k)$:

$$\boxed{f(k) = g(k+1) - g(k)}$$

- THEN (with some mild extra conditions)

$$\sum_{k=0}^{n} f(k) = g(n+1) - g(0)$$

Refined telescoping

- GIVEN

$$\sum_{k=0}^{n} f(k).$$

- FIND $g(k)$ and $f^*(k)$:

$$\boxed{f(k) = g(k+1) - g(k) + f^*(k)}$$

where $f^*(k)$ is simpler than $f(k)$.

- THEN (with some mild extra conditions)

$$\sum_{k=0}^{n} f(k) = g(n+1) - g(0) + \sum_{k=0}^{n} f^*(k).$$

Degree optimal w.r.t the top extension

$$\sum_{k=1}^{n} \frac{1}{k(k+1)^2(k+2)^3} =$$

$$\sum_{k=2}^{n} \frac{2 - kH_k + H_k^4 - kH_k^5}{H_k - kH_k^2} =$$

Sigma

$$\sum_{k=1}^{n} H_k^4 =$$

Degree optimal w.r.t the top extension

$$\sum_{k=1}^{n}\frac{1}{k(k+1)^2(k+2)^3} = -\frac{1}{4}\sum_{k=1}^{n}\frac{9k+2}{k^3} + \frac{n\left(69n^5+585n^4+1967n^3+3283n^2+2728n+904\right)}{16(n+1)^3(n+2)^3}$$

$$\sum_{k=2}^{n}\frac{2-kH_k+H_k^4-kH_k^5}{H_k-kH_k^2} = \frac{1}{2}\sum_{k=2}^{n}\frac{2k^2+H_k}{k^2H_k} + (n+1)H_n^3-(2n+1)\left(\tfrac{3}{2}H_n^2-3H_n+\tfrac{3}{2}\right)+\tfrac{1}{H_n}$$

$$\sum_{k=1}^{n}H_k^4 = -H_n^{(3)}-2H_n^{(2)}+2\sum_{k=1}^{n}\frac{H_k}{k^2} + (n+1)H_n^4-2(2n+1)H_n^3+6(2n+1)H_n^2-12(2n+1)H_n+24n.$$

Simpler w.r.t. the depth

$$\sum_{k=1}^{n} H_k^2 H_k^{(2)} =$$

$$\sum_{k=0}^{a} \left(\sum_{i=0}^{k} \binom{n}{i} \right)^2 =$$

Sigma

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{H_j}{j^2}}{k^3} =$$

Simpler w.r.t. the depth

$$\sum_{k=1}^{n} H_k^2 H_k^{(2)} = \frac{1}{3} H_n^{(3)} - \frac{1}{3} H_n^3 + \left((n+1) H_n^{(2)} + 1\right) H_n^2 + (2n+1)(1 - H_n) H_n^{(2)} - 2H_n$$

$$\sum_{k=0}^{a} \left(\sum_{i=0}^{k} \binom{n}{i}\right)^2 = (n-a)\binom{n}{a} \sum_{i=0}^{a} \binom{n}{i} - \frac{n-2a-2}{2} \left(\sum_{i=0}^{a} \binom{n}{i}\right)^2 - \frac{n}{2} \sum_{i=1}^{a} \binom{n}{i}^2$$

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{H_j}{j^2}}{k^3} = $$

$$= H_n^{(3)} \sum_{j=1}^{n} \frac{H_j}{j^2} - \sum_{j=1}^{n} \frac{H_j H_j^{(3)}}{j^2} + \sum_{j=1}^{n} \frac{H_j}{j^5}$$

Simpler w.r.t. the depth

$$\sum_{k=1}^{n} H_k^2 H_k^{(2)} = \tfrac{1}{3} H_n^{(3)} - \tfrac{1}{3} H_n^3 + \Big((n+1) H_n^{(2)} + 1\Big) H_n^2 + (2n+1)\,(1 - H_n)\, H_n^{(2)} - 2H_n$$

$$\sum_{k=0}^{a} \left(\sum_{i=0}^{k} \binom{n}{i}\right)^2 = (n-a)\binom{n}{a} \sum_{i=0}^{a} \binom{n}{i} - \frac{n-2a-2}{2} \left(\sum_{i=0}^{a} \binom{n}{i}\right)^2 - \frac{n}{2} \sum_{i=1}^{a} \binom{n}{i}^2$$

$$\sum_{k=1}^{n} \frac{\sum_{j=1}^{k} \frac{H_j}{j^2}}{k^3} = S(3,2,1,N) \qquad \text{(Harmonic sum)}$$

$$= H_n^{(3)} \sum_{j=1}^{n} \frac{H_j}{j^2} - \sum_{j=1}^{n} \frac{H_j H_j^{(3)}}{j^2} + \sum_{j=1}^{n} \frac{H_j}{j^5} \qquad \text{(Euler sums)}$$

Further examples

$$\sum_{k=1}^{n} H_k^3 = \frac{1}{2}\Big(2(n+1)H_n^3 - 3(2n+1)H_n^2 + 6(2n+1)H_n - 12n - 1 + H_n^{(2)}\Big)$$

$$\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j} \sum_{i=1}^{j} \frac{1}{i} = \frac{1}{6}\left[H_n^3 + 3H_n\, H_n^{(2)} + 2\, H_n^{(3)}\right]$$

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{k} \frac{H_j^{(2)}}{j^3}\right)^2 = -\left({H_n^{(2)}}^2 + H_n^{(4)}\right)\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^3} + (n+1)\left(\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^3}\right)^2 + \sum_{j=1}^{n}\frac{{H_j^{(2)}}^3}{j^3} - \sum_{j=1}^{n}\frac{{H_j^{(2)}}^2}{j^5} + \sum_{j=1}^{n}\frac{H_j^{(2)} H_j^{(4)}}{j^3}.$$

A problem from quadratic Padé approximation of $log(x)$

For all $n \geq 0$,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\left[3(H_{n-k}-H_k)^2+H_{n-k}^{(2)}+H_k^{(2)}\right]=0$$

where

$$H_k=\sum_{i=1}^{k}\frac{1}{i},\quad H_k^{(2)}=\sum_{i=1}^{k}\frac{1}{i^2}.$$

(K. Driver, H. Prodinger, CS, J.A.C. Weideman; 2006)

A problem from quadratic Padé approximation of $log(x)$

Theorem (Sigma; 2002)

For all $n \geq 0$,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\Big[3(H_{n-k}-H_k)^2+H^{(2)}_{n-k}+H^{(2)}_k\Big]=0$$

where

$$H_k=\sum_{i=1}^{k}\frac{1}{i},\quad H^{(2)}_k=\sum_{i=1}^{k}\frac{1}{i^2}.$$

Proof.

Sigma

□

(K. Driver, H. Prodinger, CS, J.A.C. Weideman; 2006)

Creative telescoping

Proving $\xrightarrow{\text{Sigma}}$ Finding

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k H_{2n-k} = \frac{(3n)!(-1)^n}{n!n!n!}\frac{1}{12}\big(3H_n^2 - 6H_nH_{3n} + 3H_{3n}^2 + H_n^{(2)} + 12H_{2n}(H_{2n}+H_n-H_{3n}) + 4H_{2n}^{(2)} - 3H_{3n}^{(2)}\big)$$

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k^2 = \frac{(3n)!(-1)^n}{n!n!n!}\frac{1}{12}\big(3H_n^2 - 6H_nH_{3n} + 3H_{3n}^2 - H_n^{(2)} + 12H_{2n}(H_{2n}+H_n-H_{3n}) + 2H_{2n}^{(2)} - 3H_{3n}^{(2)}\big)$$

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k^{(2)} = \frac{1}{2}\frac{(3n)!(-1)^n}{n!n!n!}\big(H_n^{(2)} + H_{2n}^{(2)}\big)$$

Proving $\xrightarrow{\text{Sigma}}$ Finding

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k H_{2n-k} = \frac{(3n)!(-1)^n}{n!n!n!}\frac{1}{12}\big(3H_n^2 - 6H_nH_{3n} + 3H_{3n}^2 + H_n^{(2)} + 12H_{2n}(H_{2n}+H_n-H_{3n}) + 4H_{2n}^{(2)} - 3H_{3n}^{(2)}\big)$$

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k^2 = \frac{(3n)!(-1)^n}{n!n!n!}\frac{1}{12}\big(3H_n^2 - 6H_nH_{3n} + 3H_{3n}^2 - H_n^{(2)} + 12H_{2n}(H_{2n}+H_n-H_{3n}) + 2H_{2n}^{(2)} - 3H_{3n}^{(2)}\big)$$

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k^{(2)} = \frac{1}{2}\frac{(3n)!(-1)^n}{n!n!n!}\big(H_n^{(2)} + H_{2n}^{(2)}\big)$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3\left[3(H_{2n-k}-H_k)^2 + H_{2n-k}^{(2)} + H_k^{(2)}\right] = 0$$

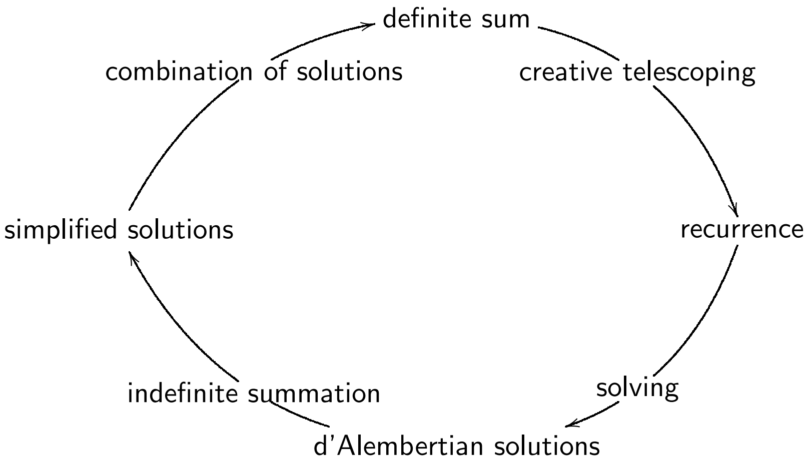
Proving $\xrightarrow{\text{Sigma}}$ Finding

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k H_{2n-k} = \frac{(3n)!(-1)^n}{n!n!n!}\frac{1}{12}\big(3H_n^2 - 6H_nH_{3n} + 3H_{3n}^2 + H_n^{(2)} + 12H_{2n}(H_{2n}+H_n-H_{3n}) + 4H_{2n}^{(2)} - 3H_{3n}^{(2)}\big)$$

$$\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k^2 = \frac{(3n)!(-1)^n}{n!n!n!}\frac{1}{12}\big(3H_n^2 - 6H_nH_{3n} + 3H_{3n}^2 - H_n^{(2)} + 12H_{2n}(H_{2n}+H_n-H_{3n}) + 2H_{2n}^{(2)} - 3H_{3n}^{(2)}\big)$$

$$\boxed{\sum_{k=0}^{2n}\binom{2n}{k}^3(-1)^k H_k^{(2)}} = \text{FIND}$$

The Sigma-summation spiral:



Examples

Evaluation of a quadruple sum

A challenging email

```
From: Doron Zeilberger
To: Robin Pemantle, Herbert Wilf
CC:Carsten Schneider


Robin and Herb,

I am willing to bet that Carsten Schneider's SIGMA package for
handling sums with harmonic numbers (among others) can do it in a
jiffy. I am Cc-ing this to Carsten.

Carsten: please do it, and Cc- the  answer to me.
             -Doron
```

The problem

```
From: Robin Pemantle [University of Pennsylvania]
To: herb wilf; doron zeilberger

Herb, Doron,

I have a sum that, when I evaluate numerically, looks suspiciously
like it comes out to exactly 1. Is there a way I can automatically
decide this? The sum may be written in many ways, but one is:
```

$$\sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}; \qquad H_j := \sum_{i=1}^{j} \frac{1}{i}.$$

```
Of course you can expand out the H's and get a quadruple sum.
There are zillions of ways to play with it, summing by parts,
but I have never managed to get rid of all the summations.

Robin
```

Take the truncated version:

$$S(a,b)=\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{a}\frac{H_j}{j(j+k)},$$

i.e.,

$$\lim_{a,b\to\infty}S(a,b)=S.$$

Take the truncated version:

$$S(a,b)=\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{a}\frac{H_j}{j(j+k)},$$

i.e.,

$$\lim_{a,b\to\infty}S(a,b)=S.$$

$$\sum_{j=1}^{a}\frac{H_j}{j(j+k)}=$$ FIND

Take the truncated version:

$$S(a,b)=\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{a}\frac{H_j}{j(j+k)},$$

i.e.,

$$\lim_{a,b\to\infty}S(a,b)=S.$$

$$\sum_{j=1}^{a}\frac{H_j}{j(j+k)}=\frac{kH_k^2-2H_k+kH_k^{(2)}+2kH_a^{(2)}}{2k^2}$$

$$\underbrace{-\frac{(kH_a-1)}{k^2}\sum_{i=1}^{k}\frac{1}{a+i}-\frac{1}{k}\sum_{i=1}^{k}\frac{1}{i}\sum_{j=1}^{i}\frac{1}{a+j}}_{\text{Limits}}$$

Take the truncated version:

$$S(a,b)=\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\sum_{j=1}^{a}\frac{H_j}{j(j+k)},$$

i.e.,

$$\lim_{a,b\to\infty}S(a,b)=S.$$

Hence, for

$$\boxed{S'(a,b):=\sum_{k=1}^{b}\frac{H_{k+1}-1}{k(k+1)}\frac{kH_k^2-2H_k+kH_k^{(2)}+2kH_a^{(2)}}{2k^2},}$$

we have

$$\lim_{a,b\to\infty}S'(a,b)=S.$$

Further simplification

$$S'(a, b) = \text{Simplify}$$

Further simplification

$$S'(a,b) = A(a,b) + B(a,b) + C(a,b)$$

where

$$A(a,b) := \frac{1}{2(b+1)^2}\Big(6H_b + 4bH_b + 4H_b^2 + 3bH_b^2 + H_b^3 + bH_b^3 - 6bH_a^{(2)} + 2H_bH_a^{(2)} + 2bH_bH_a^{(2)} - 2H_b^{(2)} - 7bH_b^{(2)} + H_bH_b^{(2)} + bH_bH_b^{(2)}\Big),$$

$$B(a,b) := -\frac{2b^2}{(b+1)^2}\Big(H_a^{(2)} + H_b^{(2)}\Big),$$

$$C(a,b) := (H_a^{(2)} - 1)\sum_{i=1}^{b}\frac{H_i}{i^2} - \sum_{i=1}^{b}\frac{H_i^2}{i^3} + \frac{1}{2}\sum_{i=1}^{b}\frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{b}\frac{H_iH_i^{(2)}}{i^2}.$$

Further simplification

$$S'(a,b) = A(a,b) + B(a,b) + C(a,b)$$

where

$$A(a,b) := \frac{1}{2(b+1)^2}\Big(6H_b + 4bH_b + 4H_b^2 + 3bH_b^2 + H_b^3 + bH_b^3 - 6bH_a^{(2)} + 2H_bH_a^{(2)} + 2bH_bH_a^{(2)} - 2H_b^{(2)} - 7bH_b^{(2)} + H_bH_b^{(2)} + bH_bH_b^{(2)}\Big),$$

$$B(a,b) := -\frac{2b^2}{(b+1)^2}\Big(H_a^{(2)} + H_b^{(2)}\Big),$$

$$C(a,b) := (H_a^{(2)} - 1)\sum_{i=1}^{b}\frac{H_i}{i^2} - \sum_{i=1}^{b}\frac{H_i^2}{i^3} + \frac{1}{2}\sum_{i=1}^{b}\frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{b}\frac{H_iH_i^{(2)}}{i^2}.$$

By

$$\lim_{a,b\to\infty} A(a,b) = 0 \qquad \text{and} \qquad \lim_{a,b\to\infty} B(a,b) = -4\zeta(2)$$

we get

$$S = \lim_{a,b\to\infty} S'(a,b) = -4\zeta(2) + \lim_{a,b\to\infty} C(a,b).$$

ζ-relations

This gives

$$
\begin{aligned}
S &= \sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}\\
&= -4\zeta(2) + (\zeta(2)-1)\sum_{i=1}^{\infty}\frac{H_i}{i^2} - \sum_{i=1}^{\infty}\frac{H_i^2}{i^3} + \frac{1}{2}\sum_{i=1}^{\infty}\frac{H_i^3}{i^2} + \frac{1}{2}\sum_{i=1}^{\infty}\frac{H_iH_i^{(2)}}{i^2}.
\end{aligned}
$$

ζ-relations

This gives

$$S=\sum_{j,k=1}^{\infty}\frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}$$

$$=-4\zeta(2)+(\zeta(2)-1)\sum_{i=1}^{\infty}\frac{H_i}{i^2}-\sum_{i=1}^{\infty}\frac{H_i^2}{i^3}+\frac{1}{2}\sum_{i=1}^{\infty}\frac{H_i^3}{i^2}+\frac{1}{2}\sum_{i=1}^{\infty}\frac{H_iH_i^{(2)}}{i^2}.$$

E.g., in J.M. Borwein, Girgensohn. Evaluation of triple Euler sums, 1996.

Flajolet, Salvy. Euler sums and contour integral representations, 1998.

we find

$$\sum_{i=1}^{\infty}\frac{H_i}{i^2}=2\zeta(3),\qquad \sum_{i=1}^{\infty}\frac{H_i^2}{i^3}=-\zeta(2)\zeta(3)+\frac{7}{2}\zeta(5),$$

$$\sum_{i=1}^{\infty}\frac{H_i^3}{i^2}=\zeta(2)\zeta(3)+10\zeta(5),\qquad \sum_{i=1}^{\infty}\frac{H_iH_i^{(2)}}{i^2}=\zeta(2)\zeta(3)+\zeta(5).$$

ζ-relations

Theorem.

$$S=\sum_{j,k=1}^{\infty}\frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)}$$
$$=-4\zeta(2)-2\zeta(3)+4\zeta(2)\zeta(3)+2\zeta(5)=0.999222\ldots$$

(R. Pemantle, CS; 2007. When is 0.999... equal to 1?)

Example from particle physics

A. Mitov, S. Moch; 2006.

“QCD corrections to semi-inclusive hadron production in electron-positron annihilation at two loops”.

GIVEN $F(N)$ by

$$F(0) = f_0(0) + f_1(0)\epsilon + f_2(0)\epsilon^2 + f_3(0)\epsilon^3 + f_4(0)\epsilon^4 + O(\epsilon^5)$$

and

$$\text{F(N)} = \frac{2(N-2\epsilon)(-1+2N)}{2(-1+2\epsilon-2N)(-1+3\epsilon-N)}\text{F(N-1)} + \frac{\text{X(2N)}}{2(-1+2\epsilon-2N)(-1+3\epsilon-N)}$$

where

$$X(2N) = x_0(N) + x_1(N)\epsilon + x_2(N)\epsilon^2 + x_3(N)\epsilon^3 + x_4(N)\epsilon^4 + O(\epsilon^5)$$

GIVEN $F(N)$ by

$$F(0) = f_0(0) + f_1(0)\epsilon + f_2(0)\epsilon^2 + f_3(0)\epsilon^3 + f_4(0)\epsilon^4 + O(\epsilon^5)$$

and

Sigma

$$\text{F(N)} = \frac{2(N-2\epsilon)(-1+2N)}{2(-1+2\epsilon-2N)(-1+3\epsilon-N)}\text{F(N-1)} + \frac{\text{X(2N)}}{2(-1+2\epsilon-2N)(-1+3\epsilon-N)}$$

where

$$X(2N) = x_0(N) + x_1(N)\epsilon + x_2(N)\epsilon^2 + x_3(N)\epsilon^3 + x_4(N)\epsilon^4 + O(\epsilon^5)$$

FIND the expansion

$$F(N) = f_0(N) + f_1(N)\epsilon + f_2(N)\epsilon^2 + f_3(N)\epsilon^3 + f_4(N)\epsilon^4 + O(N^5)$$

$$F(N) = F(0)\prod_{i=1}^{N}\frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)} + \frac{1}{2}\prod_{i=1}^{N}\frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}\sum_{i=1}^{N}\frac{\prod_{j=1}^{i}\frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)}X(2i)$$

FIND the expansion

$$F(N) = f_0(N) + f_1(N)\epsilon + f_2(N)\epsilon^2 + f_3(N)\epsilon^3 + f_4(N)\epsilon^4 + O(N^5)$$

$$F(N) = \boxed{F(0)} \cdot \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}}$$

$$+\frac{1}{2}\boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}} \cdot \boxed{\sum_{i=1}^{N} \frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i)}$$

1. FIND expansions for each box, e.g.,

$$\boxed{A(N)} = a_0(N) + a_1(N)\epsilon + a_2(N)\epsilon^2 + a_3(N)\epsilon^3 + a_4(N)\epsilon^4 + O(\epsilon^5)$$

$$\boxed{B(N)} = b_0(N) + b_1(N)\epsilon + b_2(N)\epsilon^2 + b_3(N)\epsilon^3 + b_4(N)\epsilon^4 + O(\epsilon^5)$$

$$F(N) = \boxed{F(0)} \cdot \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}}$$

$$+\frac{1}{2}\boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}} \cdot \boxed{\sum_{i=1}^{N} \frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i)}$$

1. FIND expansions for each box, e.g.,

$$\boxed{A(N)} = a_0(N) + a_1(N)\epsilon + a_2(N)\epsilon^2 + a_3(N)\epsilon^3 + a_4(N)\epsilon^4 + O(\epsilon^5)$$

$$\boxed{B(N)} = b_0(N) + b_1(N)\epsilon + b_2(N)\epsilon^2 + b_3(N)\epsilon^3 + b_4(N)\epsilon^4 + O(\epsilon^5)$$

2. COMBINE:

$$A(N) + B(N) = \cdots + (a_r(N) + b_r(N))\epsilon^r + \dots$$ component wise

$$A(N) \cdot B(N) = \cdots + \Big(\sum_{l=0}^{r} a_l(N) b_{r-l}(N)\Big)\epsilon^r + \dots$$ Cauchy-product

$$\text{GIVEN } F(N) = \boxed{F(0)} \cdot \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}}$$

$$+ \frac{1}{2} \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}} \cdot \boxed{\sum_{i=1}^{N} \frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i)}$$

$$\frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i) =$$

Sigma

$$\text{GIVEN } F(N) = \boxed{F(0)} \cdot \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}}$$

$$+\frac{1}{2}\boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}} \cdot \boxed{\sum_{i=1}^{N} \frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i)}$$

$$\frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i) = \qquad h_0(i) + h_1(i)\epsilon + h_2(i)\epsilon^2 + \dots$$

GIVEN $F(N) = \boxed{F(0)} \cdot \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}}$

$$+\frac{1}{2}\boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}} \cdot \boxed{\sum_{i=1}^{N} \frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i)}$$

$$\sum_{j=1}^{N} \frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i) = \underbrace{\sum_{i=1}^{N} h_0(i)}_{\text{sum 0}} + \epsilon \underbrace{\sum_{i=1}^{N} h_1(i)}_{\text{sum 1}} + \epsilon^2 \underbrace{\sum_{i=1}^{N} h_2(i)}_{\text{sum 2}} + \dots$$

Sigma

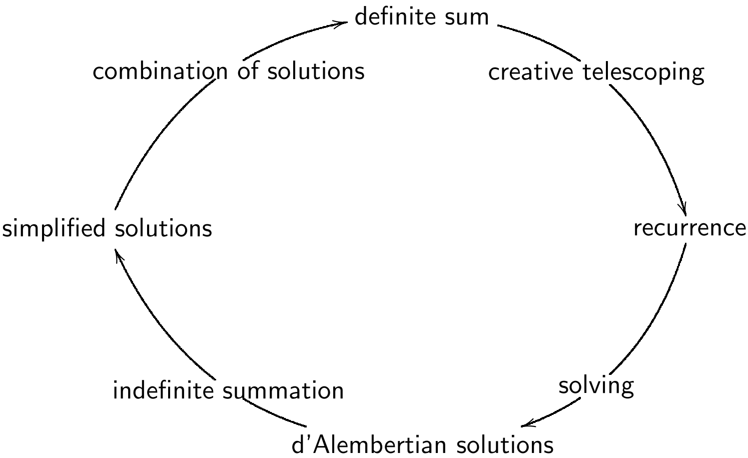
$$\text{GIVEN } F(N) = \boxed{F(0)} \cdot \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}}$$

$$+ \frac{1}{2} \boxed{\prod_{i=1}^{N} \frac{(i-2\epsilon)(2i-1)}{(2\epsilon-2i-1)(3\epsilon-i-1)}} \cdot \boxed{\sum_{i=1}^{N} \frac{\prod_{j=1}^{i} \frac{(2\epsilon-2j-1)(3\epsilon-j-1)}{(j-2\epsilon)(2j-1)}}{(2\epsilon-2i-1)(3\epsilon-i-1)} X(2i)}$$

$$= \boxed{f_0(N) + f_1(N)\epsilon + f_2(N)\epsilon^2 + f_3(N)\epsilon^3 + f_4(N)\epsilon^4 + O(N^5)}$$

COMBINE

The Sigma-summation spiral:



Examples

Appendix

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Back

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]$:

$d = 1$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\,d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

Back

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]$:

$d = 1$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\, d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $b \geq 0$:

$b = 2$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

Back

FIND $g \in \mathbb{Q}(k)(h)$:

$$\sigma(g) - g = h.$$

Denominator bound: COMPUTE a polynomial $d \in \mathbb{Q}(k)[h]$:

$d = 1$

$$\forall g \in \mathbb{Q}(k)(h): \quad \sigma(g) - g = h \quad \Rightarrow \quad g\, d \in \mathbb{Q}(k)[h].$$

FIND $g' \in \mathbb{Q}(k)[h]$ with

$$\sigma(\frac{g'}{d}) - \frac{g'}{d} = h.$$

Degree bound: COMPUTE $b \geq 0$:

$b = 2$

$$\forall g \in \mathbb{Q}(k)[h] \quad \sigma(g) - g = h \quad \Rightarrow \quad \deg(g) \leq b.$$

Polynomial Solution: FIND

g=hk-k

$$g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h].$$

Back

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\sigma(g) - g = h$$

◂ Back

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h + \tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = h$$

Back

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\, h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

Back

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h+\tfrac{1}{k+1}\right)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\,h^2 + g_1 h + g_0\right] = h$$

coeff. comp.

$$\boxed{\sigma(g_2) - g_2 = 0}$$

$g_2 = c \in \mathbb{Q}$

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\big] - \big[g_2\,h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\big[\tfrac{2h(k+1)+1}{(k+1)^2}\big]$$

Back

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\big(h + \tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\right] - \left[g_2\,h^2 + g_1 h + g_0\right] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\left[\tfrac{2h(k+1)+1}{(k+1)^2}\right]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}$$

Back

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h+\tfrac{1}{k+1}\right)^2+\sigma(g_1h+g_0)\right] - \left[g_2\,h^2+g_1h+g_0\right] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\left[\tfrac{2h(k+1)+1}{(k+1)^2}\right]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}$$

$c = 0,\quad g_1 = k + d,\ d \in \mathbb{Q}$

Back

ANSATZ $g = g_2\, h^2 + g_1\, h + g_0 \in \mathbb{Q}(k)[h]$

$$\left[\sigma(g_2)\left(h+\tfrac{1}{k+1}\right)^2+\sigma(g_1 h+g_0)\right] - \left[g_2\, h^2 + g_1 h + g_0\right] = h$$

coeff. comp.

$$\sigma(g_2) - g_2 = 0$$

$g_2 = c \in \mathbb{Q}$

$$\sigma(g_1\, h + g_0) - (g_1\, h + g_0) = h - c\left[\tfrac{2h(k+1)+1}{(k+1)^2}\right]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}$$

$c = 0$, $g_1 = k + d$, $d \in \mathbb{Q}$

$$\sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}$$

Back

ANSATZ $g = g_2\,h^2 + g_1\,h + g_0 \in \mathbb{Q}(k)[h]$

$$\Big[\sigma(g_2)\big(h+\tfrac{1}{k+1}\big)^2 + \sigma(g_1 h + g_0)\Big] - \big[g_2\,h^2 + g_1 h + g_0\big] = h$$

coeff. comp.

g=hk-k

$$\sigma(g_2) - g_2 = 0$$

$$g_2 = c \in \mathbb{Q}$$

$$\sigma(g_1\,h + g_0) - (g_1\,h + g_0) = h - c\Big[\tfrac{2h(k+1)+1}{(k+1)^2}\Big]$$

coeff. comp.

$$\sigma(g_1) - g_1 = 1 - c\,\tfrac{2}{k+1}$$

$c = 0,\quad g_1 = k + d,\ d \in \mathbb{Q}$

$$g_0 = -k,\ d = 0 \quad \leftarrow \quad \sigma(g_0) - g_0 = -1 - d\frac{1}{k+1}$$

Back

The general case (Karr's algorithm; 1981)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

Back

The general case (Karr's algorithm; 1981)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

Back

The general case (Karr's algorithm; 1981)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = \alpha_1\, t_1 + \beta_1, \qquad \alpha_1 \in \mathbb{K}^*, \qquad \beta_1 \in \mathbb{K}$$

Back

The general case (Karr's algorithm; 1981)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\sigma(c) = c \quad \forall c \in \mathbb{K}$$

$$\sigma(t_1) = \alpha_1\, t_1 + \beta_1, \qquad \alpha_1 \in \mathbb{K}^*, \qquad \beta_1 \in \mathbb{K}$$

$$\sigma(t_2) = \alpha_2\, t_2 + \beta_2, \qquad \alpha_2 \in \mathbb{K}(t_1)^*, \qquad \beta_2 \in \mathbb{K}(t_1)$$

Back

The general case (Karr's algorithm; 1981)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$
\begin{array}{lll}
\sigma(c) = c \quad \forall c \in \mathbb{K} & & \\
\sigma(t_1) = \alpha_1\, t_1 + \beta_1, & \alpha_1 \in \mathbb{K}^*, & \beta_1 \in \mathbb{K} \\
\sigma(t_2) = \alpha_2\, t_2 + \beta_2, & \alpha_2 \in \mathbb{K}(t_1)^*, & \beta_2 \in \mathbb{K}(t_1) \\
\sigma(t_3) = \alpha_3\, t_3 + \beta_3, & \alpha_3 \in \mathbb{K}(t_1, t_2)^*, & \beta_3 \in \mathbb{K}(t_1, t_2) \\
\vdots & &
\end{array}
$$

Back

The general case (Karr's algorithm; 1981)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= \alpha_1\, t_1 + \beta_1, && \alpha_1 \in \mathbb{K}^*, && \beta_1 \in \mathbb{K} \\
\sigma(t_2) &= \alpha_2\, t_2 + \beta_2, && \alpha_2 \in \mathbb{K}(t_1)^*, && \beta_2 \in \mathbb{K}(t_1) \\
\sigma(t_3) &= \alpha_3\, t_3 + \beta_3, && \alpha_3 \in \mathbb{K}(t_1,t_2)^*, && \beta_3 \in \mathbb{K}(t_1,t_2) \\
&\vdots
\end{aligned}$$

such that $\{c \in \mathbb{K}(t_1)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}$.

Back

The general case (Karr's algorithm; 1981)

GIVEN a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$:
a rational function field $\mathbb{F} := \mathbb{K}(t_1)(t_2)\dots(t_e)$ with an automorphism

$$
\begin{aligned}
\sigma(c) &= c \quad \forall c \in \mathbb{K} \\
\sigma(t_1) &= \alpha_1\, t_1 + \beta_1, && \alpha_1 \in \mathbb{K}^*, && \beta_1 \in \mathbb{K} \\
\sigma(t_2) &= \alpha_2\, t_2 + \beta_2, && \alpha_2 \in \mathbb{K}(t_1)^*, && \beta_2 \in \mathbb{K}(t_1) \\
\sigma(t_3) &= \alpha_3\, t_3 + \beta_3, && \alpha_3 \in \mathbb{K}(t_1, t_2)^*, && \beta_3 \in \mathbb{K}(t_1, t_2) \\
&\vdots
\end{aligned}
$$

such that $\{c \in \mathbb{K}(t_1)\dots(t_e) | \sigma(c) = c\} = \mathbb{K}$.

GIVEN $f \in \mathbb{F}$;

FIND $g \in \mathbb{F}$ with

$$\sigma(g) - g = f.$$

Back

Quadratic Padé approximation to $\log(x)$ at $x = 1$

Back

FIND $r_n(x) = \sum_{k=0}^{n} a_k x^k, \quad s_n(x) = \sum_{k=0}^{n} b_k x^k, \quad t_n(x) = \sum_{k=0}^{n} c_k x^k$:

$$r_n(x)\,(\log x)^2 + s_n(x)\,\log(x) + t_n(x) = O((x-1)^{3n+2}).$$

Quadratic Padé approximation to $\log(x)$ at $x = 1$

Back

FIND $r_n(x) = \sum_{k=0}^{n} a_k x^k, \quad s_n(x) = \sum_{k=0}^{n} b_k x^k, \quad t_n(x) = \sum_{k=0}^{n} c_k x^k$:

$$r_n(x)\,(\log x)^2 + s_n(x)\,\log(x) + t_n(x) = O((x-1)^{3n+2}).$$

A. Weideman finds

$$\begin{aligned} r_n(x) &= c_3 A(n,x) \\ s_n(x) &= c_2 A(n,x) + 2c_3 B(n,x) \\ t_n(x) &= c_1 A(n,x) + c_2 B(n,x) + c_3 C(n,x) \end{aligned}$$

where

$$A(n,x) = \sum_{k=0}^{n} \binom{n}{k}^3 (-x)^k \qquad B(n,x) = \sum_{k=0}^{n} \left[\frac{d}{dk}\binom{n}{k}^3\right](-x)^k$$

$$C(n,x) = \sum_{k=0}^{n} \left[\frac{d^2}{dk^2}\binom{n}{k}^3\right](-x)^k.$$

Quadratic Padé approximation to $\log(x)$ at $x = 1$

Back

FIND $r_n(x) = \sum_{k=0}^{n} a_k x^k, \quad s_n(x) = \sum_{k=0}^{n} b_k x^k, \quad t_n(x) = \sum_{k=0}^{n} c_k x^k$:

$$r_n(x)\,(\log x)^2 + s_n(x)\,\log(x) + t_n(x) = O((x-1)^{3n+2}).$$

A. Weideman finds

$$\begin{aligned} r_n(x) &= c_3 A(n,x) \\ s_n(x) &= c_2 A(n,x) + 2c_3 B(n,x) \\ t_n(x) &= c_1 A(n,x) + c_2 B(n,x) + c_3 C(n,x) \end{aligned}$$

where

$$A(n,x) = \sum_{k=0}^{n} \binom{n}{k}^3 (-x)^k \qquad B(n,x) = \sum_{k=0}^{n} \left[\frac{d}{dk}\binom{n}{k}^3\right](-x)^k$$

$$C(n,x) = \sum_{k=0}^{n} \left[\frac{d^2}{dk^2}\binom{n}{k}^3\right](-x)^k.$$

Tests at $x = 1$:
$c_1 = \pi^2, \quad c_2 = 0, \quad c_3 = 1$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{(-1)^k \binom{n}{k}^3 \left[3(H_{n-k} - H_k)^2 + H_{n-k}^{(2)} + H_k^{(2)}\right]}_{=:\, f(n,k)}.$$

FIND $c_0(n)$, $c_1(n)$, $c_2(n)$, and $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $0 \leq k \leq n$ and all $n \geq 0$.

Back

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{(-1)^k \binom{n}{k}^3 \left[3(H_{n-k}-H_k)^2 + H_{n-k}^{(2)} + H_k^{(2)}\right]}_{=:\, f(n,k)}.$$

FIND $c_0(n)$, $c_1(n)$, $c_2(n)$, and $g(n,k)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)\,f(n,k)+c_1(n)\,f(n+1,k)+c_2(n)\,f(n+2,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Sigma: $c_0(n) := 3(3n+2)(3n+4)(3n+8), \quad c_1(n) := 0,$
$c_2(n) := (n+2)^2(3n+8),$

$$g(n,k) := (-1)^k \binom{n}{k}^3 \frac{p_1(k,n,H_k,H_k^{(2)},H_{n-k},H_{n-k}^{(2)})}{(n-k+1)^5(n-k+2)^5},$$

$$g(n,k+1) := (-1)^k \binom{n}{k}^3 \frac{p_2(k,n,H_k,H_k^{(2)},H_{n-k},H_{n-k}^{(2)})}{(n-k+1)^5}.$$

Back

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{SUM}(n) := \sum_{k=0}^{n} \underbrace{(-1)^k \binom{n}{k}^3 \left[3(H_{n-k} - H_k)^2 + H_{n-k}^{(2)} + H_k^{(2)}\right]}_{=:\, f(n,k)}.$$

GIVEN $c_0(n)$, $c_1(n)$, $c_2(n)$, and $g(n,k)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k) + c_2(n)\, f(n+2,k)}$$

for all $0 \le k \le n$ and all $n \ge 0$.

Summing this equation over k from 0 to n gives:

$$\boxed{g(n,n+1) - g(n,0)}$$

$$= \boxed{\begin{array}{l} c_0(n)\,\mathsf{SUM}(n) + \\ c_1(n)\left[\mathsf{SUM}(n+1) - f(n+1,n+1)\right] \\ c_2(n)\left[\mathsf{SUM}(n+2) - f(n+2,n+1) - f(n+2,n+2)\right]. \end{array}}$$

Back