Algebraic Methods for Multi-Loop Integrals

Robert Harlander

Bergische Universität Wuppertal

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Integration by Parts

- Motivation/Physics $\rightarrow$ later!
- preliminaries: typical Feynman integral:

$$I(n_1,\ldots n_N) = \int \mathrm{d}^D k_1 \cdots \mathrm{d}^D k_l \frac{1}{(p_1^2 - m_1^2)^{n_1} \cdots (p_N^2 - m_N^2)^{n_N}}$$

N	—	number of propagators
n_i	—	indices
l	—	number of loops
p_i	—	linear combinations of
		loop momenta k_i and
		external momenta q_i

- $D = 4 - 2\epsilon$

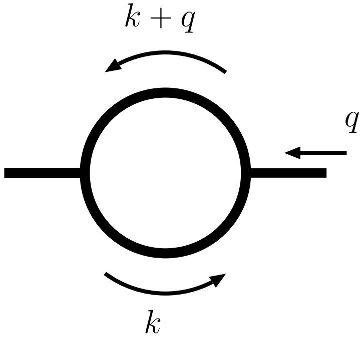
...a word about normalization

- usually: $\frac{i}{(2\pi)^D}$ for each loop
  - angular integration $\rightarrow$ $\pi^{D/2}$

    $\rightarrow$ $\left[\frac{i}{(4\pi)^{D/2}}\right]^l$ in the final result
  - will be dropped!
- typical $\gamma_{\rm E} = 0.577\ldots$ is cancelled by multiplication with $e^{\epsilon\gamma_{\rm E}}$ for each loop
- effectively: replace $\frac{1}{\epsilon} - \gamma_{\rm E} + \ln 4\pi$ by $\frac{1}{\epsilon}$

  and drop $\frac{1}{16\pi^2}$

Example: 1-loop self energy



$$= I(n_1,n_2) = \int \mathrm{d}^D k \frac{1}{(p_1^2 - m_1^2)^{n_1}(p_2^2 - m_2^2)^{n_2}}$$

$$p_1 = k + q\,, \qquad p_2 = -k$$

special cases:

- $m_1 = m_2 = 0$: $\quad [n_1 = a, n_2 = b]$

$$I_q(a,b) = (-1)^{a+b}\,(-q^2)^{D/2-a-b} \cdot \hat{I}(a,b)$$

$$\hat{I}(a,b) = \frac{\Gamma(a+b-D/2)\,\Gamma(D/2-a)\,\Gamma(D/2-b)}{\Gamma(a)\,\Gamma(b)\,\Gamma(D-a-b)}$$

Example: 1-loop self energy

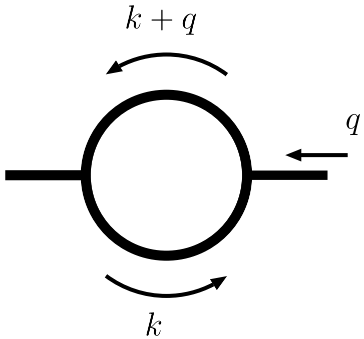
$$\hat{I}(a,b) = \frac{\Gamma(a+b-D/2)\,\Gamma(D/2-a)\,\Gamma(D/2-b)}{\Gamma(a)\,\Gamma(b)\,\Gamma(D-a-b)}$$

- remarks:
  - valid for (almost) all a, b, D (except $-a, -b \in \mathbb{N}$)
  - expansion in ϵ through $a\Gamma(a) = \Gamma(1+a)$ and

$$\Gamma(1+\epsilon) = 1 - \epsilon\gamma_{\rm E} + \frac{\epsilon^2}{2}\left(\gamma_{\rm E}^2 + \frac{\pi^2}{6}\right) + \ldots$$

($\gamma_{\rm E}$ is cancelled by our normalization)

Example: 1-loop self energy

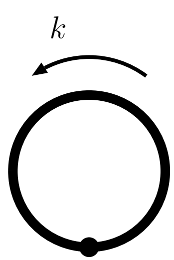


$$= I(n_1,n_2) = \int \mathrm{d}^D k \frac{1}{(p_1^2 - m_1^2)^{n_1} (p_2^2 - m_2^2)^{n_2}}$$

$$p_1 = k + q\,, \qquad p_2 = -k$$

special cases:

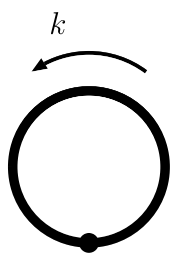
- $n_2 = 0$ [equivalently: $q = 0$, $m_1 = m_2$]



$$= A(a) = \int \mathrm{d}^D k \, \frac{1}{(k^2 - m^2)^a}$$

$$= (-1)^a (m^2)^{D/2-a} \frac{\Gamma(a - D/2)}{\Gamma(a)}$$

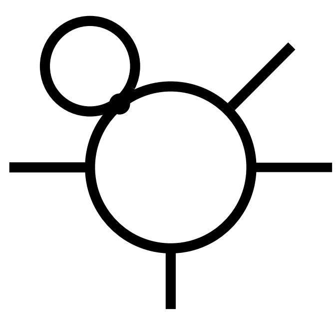
Example: 1-loop self energy



$$= A(a) = \int \mathrm{d}^D k \, \frac{1}{(k^2 - m^2)^a}$$

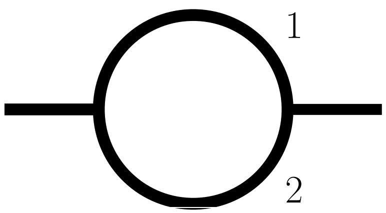
$$= (-1)^a (m^2)^{D/2-a} \frac{\Gamma(a - D/2)}{\Gamma(a)}$$

- note: massless tadpoles $= 0$ (scaleless integrals)

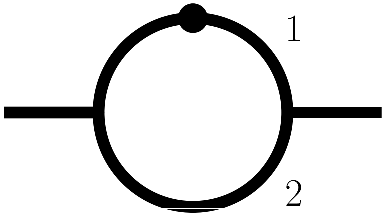
e.g.  $= 0$ in massless case

More notation

- "dots on lines": e.g.



$$= I(n_1, n_2) = \int \mathrm{d}^D k \frac{1}{(p_1^2 - m_1^2)^{n_1}(p_2^2 - m_2^2)^{n_2}}$$



$$= I(n_1 + 1, n_2)$$

useful notation:

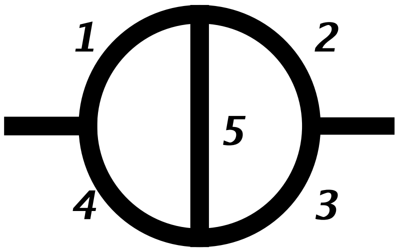
$$\mathbf{1}^{\pm} I(n_1, n_2) = I(n_1 + 1, n_2)$$

$$\mathbf{2}^{\pm} I(n_1, n_2) = I(n_1, n_2 + 1)$$

More notation

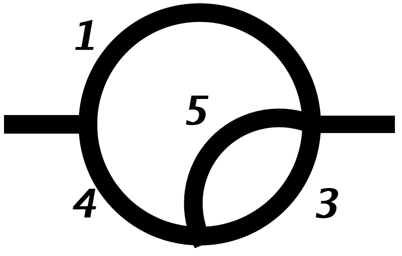
- "shrinking a line to a point (pinching)":

e.g.

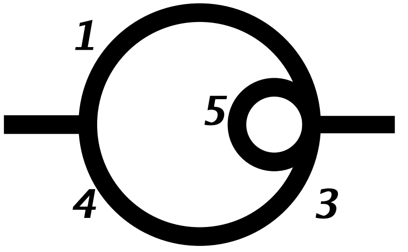


"line 2 shrinks to a point" $\Leftrightarrow$ $n_2 = 0$

graphically:

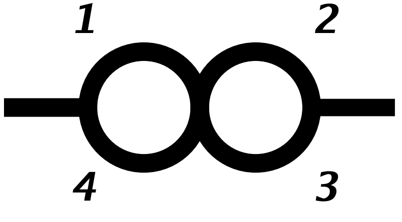


line 3 shrinks $\longrightarrow$



"line 5 shrinks" $\Leftrightarrow$ $n_5 = 0$

graphically:

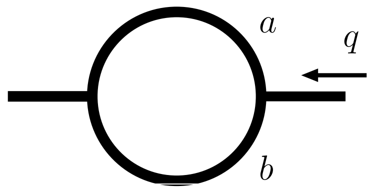


How do indices change?

e.g. derivatives

exercise:

$$q_\mu \frac{\partial}{\partial q_\mu}$$



$$= -a\left[1 - \mathbf{1}^{+}\mathbf{2}^{-} + (m_1^2 - m_2^2 + q^2)\mathbf{1}^{+}\right] I(a,b)$$

$$= -a\Big[I(a,b) - I(a+1, b-1) + (m_1^2 - m_2^2 + q^2) I(a+1, b) \Big]$$

Solution

$$p_1 = k + q\,, \qquad p_2 = -k$$

$$q_\mu \frac{\partial}{\partial q_\mu}\left(\frac{1}{(p_1^2 - m_1^2)^a (p_2^2 - m_2^2)^b}\right) =$$

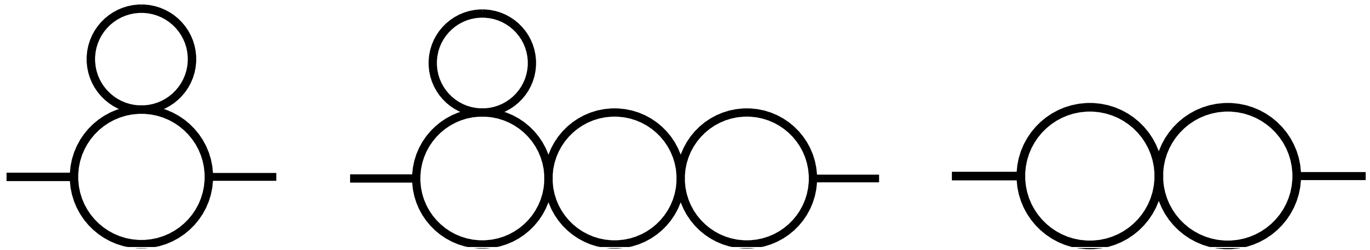
$$= -a\,(2q \cdot p_1)\frac{1}{(p_1^2 - m_1^2)^{a+1}(p_2^2 - m_2^2)^b} =$$

$$= a\,(p_2^2 - q^2 - p_1^2)\frac{1}{(p_1^2 - m_1^2)^{a+1}(p_2^2 - m_2^2)^b} =$$

$$= a\,\frac{(p_2^2 - m_2^2) - (p_1^2 - m_1^2) - q^2 - m_1^2 + m_2^2}{(p_1^2 - m_1^2)^{a+1}(p_2^2 - m_2^2)^b} =$$

$$= -a\,\left[1 - \mathbf{1}^+\mathbf{2}^- + (m_1^2 - m_2^2 + q^2)\mathbf{1}^+\right]\, I(a,b)$$

Products and convolutions of 1-loop

- 
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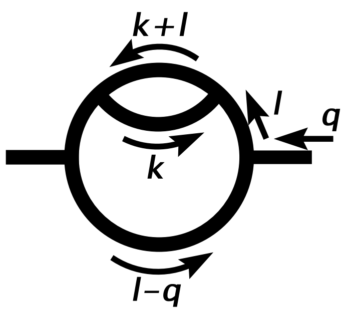
$\longrightarrow$ separate integrations!



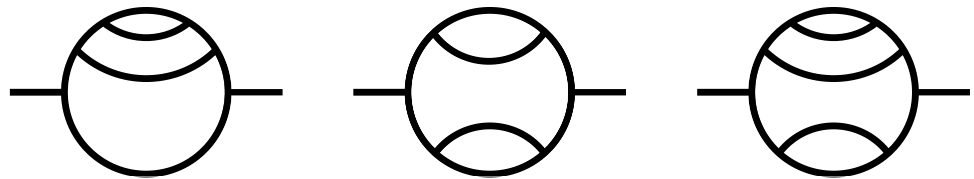
$$= \int \mathrm{d}^D l \frac{1}{(l^2)^{n_1+n_3}[(l-q)^2]^{n_4}} \underbrace{\int \mathrm{d}^D k \frac{1}{(k^2)^{n_5}[(k+l)^2]^{n_2}}}_{\sim (l^2)^{D/2-n_5-n_2}\hat{I}(n_2,n_5)}$$

$$\Big|_{a=n_1+n_3+n_2+n_5-D/2}\Big| \quad \sim \int \mathrm{d}^D l \frac{1}{(l^2)^a[(l-q)^2]^{n_4}} \hat{I}(n_2,n_5)$$

$$\sim (q^2)^{D-\sum n_i} \cdot \hat{I}(n_1+n_3+n_2+n_5-D/2, n_4) \cdot \hat{I}(n_2,n_5)$$

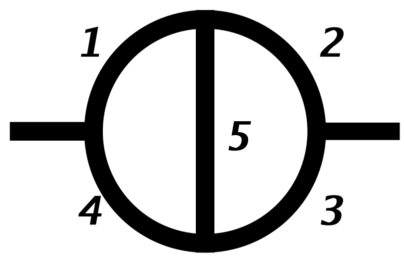
Products and convolutions of 1-loop

- analogously:
  (for $m = 0$)



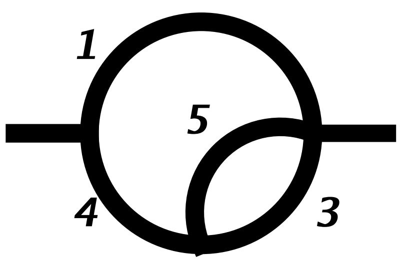
  convolutions of 1-loop self-energy diagrams

- however:
  genuine 2-loop:



  "T_1"–topology

remark: shrinking any line ⇒ convolution of 1-loop!



Integration by Parts identities

[Chetyrkin, Tkachov '81]

observation: (in dimensional regularization)

- integrals are finite
- integration region is infinite

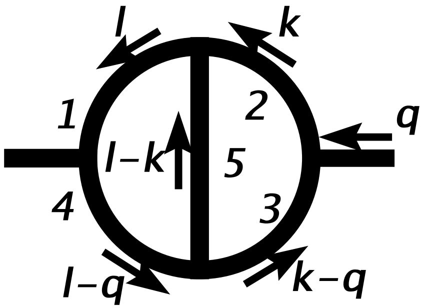
$$\Rightarrow \quad \int \mathrm{d}^D k \, \frac{\partial}{\partial k_\mu} \cdot p_\mu \, f(k,p,\ldots) = 0$$

because surface terms at infinity $= 0$

- if $p = k + q_1 + \cdots$ (q_i external) $\Rightarrow \quad \frac{\partial}{\partial k_\mu} q_\mu = D + q_\mu \frac{\partial}{\partial k_\mu}$

$$\Rightarrow \quad D \int \mathrm{d}^D k \, f(k,p,\ldots) = -\int \mathrm{d}^D k \, p_\mu \frac{\partial}{\partial k_\mu} \, f(k,p,\ldots)$$

Topology T_1 (massless)



$$= \int d^D l \int d^D k \frac{1}{[l^2]^{n_1}[k^2]^{n_2}[(k-q)^2]^{n_3}} \times \frac{1}{[(l-q)^2]^{n_4}[(l-k)^2]^{n_5}}$$

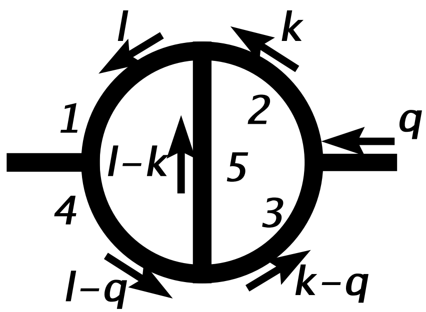
act on integrand with

$$\frac{\partial}{\partial l_\mu} \cdot l_\mu\,, \quad \frac{\partial}{\partial l_\mu} \cdot k_\mu\,, \quad \frac{\partial}{\partial l_\mu} \cdot q_\mu\,, \quad \frac{\partial}{\partial k_\mu} \cdot k_\mu\,, \quad \frac{\partial}{\partial k_\mu} \cdot l_\mu\,, \quad \frac{\partial}{\partial k_\mu} \cdot q_\mu\,,$$

example: $\frac{\partial}{\partial l_\mu} \cdot l_\mu = D + l_\mu \cdot \frac{\partial}{\partial l_\mu}$

$$\frac{\partial}{\partial l_\mu} \cdot l_\mu \frac{1}{[l^2]^{n_1}} = -2n_1 \frac{1}{[l^2]^{n_1}}$$

Topology T_1 (massless)

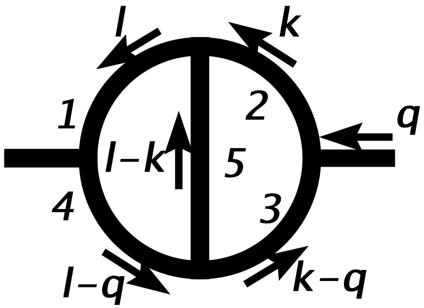


$$== \int \mathrm{d}^D l \int \mathrm{d}^D k \, \frac{1}{[l^2]^{n_1}\,[k^2]^{n_2}\,[(k-q)^2]^{n_3}} \times \frac{1}{[(l-q)^2]^{n_4}\,[(l-k)^2]^{n_5}}$$

$$\frac{\partial}{\partial l_\mu} \cdot l_\mu \frac{1}{[l^2]^{n_1}} = -2n_1 \frac{1}{[l^2]^{n_1}}$$

$$\frac{\partial}{\partial l_\mu} \cdot l_\mu \frac{1}{[(l-q)^2]^{n_4}} = -2n_4 \frac{1}{[(l-q)^2]^{n_4+1}} \left\{ l^2 - l \cdot q \right\}$$

Topology T_1 (massless)

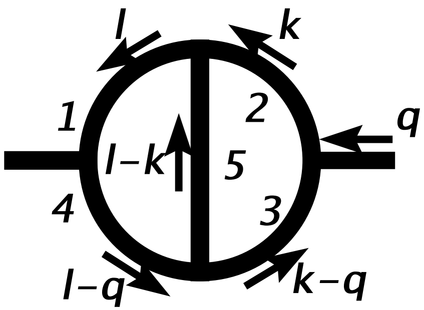


$$== \int d^D l \int d^D k \frac{1}{[l^2]^{n_1}[k^2]^{n_2}[(k-q)^2]^{n_3}} \times \frac{1}{[(l-q)^2]^{n_4}[(l-k)^2]^{n_5}}$$

$$\frac{\partial}{\partial l_\mu} \cdot l_\mu \frac{1}{[l^2]^{n_1}} = -2n_1 \frac{1}{[l^2]^{n_1}}$$

$$\frac{\partial}{\partial l_\mu} \cdot l_\mu \frac{1}{[(l-q)^2]^{n_4}} = -2n_4 \frac{1}{[(l-q)^2]^{n_4+1}} \left\{ l^2 - \frac{1}{2}\left[(l-q)^2 - l^2 - q^2\right] \right\}$$

Topology T_1 (massless)



$$== \int \mathrm{d}^D l \int \mathrm{d}^D k \frac{1}{[l^2]^{n_1}[k^2]^{n_2}[(k-q)^2]^{n_3}} \times \frac{1}{[(l-q)^2]^{n_4}[(l-k)^2]^{n_5}}$$

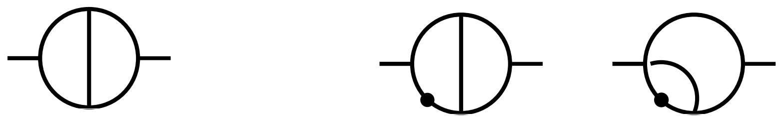
$$\frac{\partial}{\partial l_\mu} \cdot l_\mu \frac{1}{[l^2]^{n_1}} = -2n_1 \frac{1}{[l^2]^{n_1}}$$

$$\frac{\partial}{\partial l_\mu} \cdot l_\mu \frac{1}{[(l-q)^2]^{n_4}} = -2n_4 \underbrace{\frac{1}{[(l-q)^2]^{n_4+1}}}_{\mathbf{4}^+} \left\{ \underbrace{l^2}_{\mathbf{1}^-} - \frac{1}{2}\left[\underbrace{(l-q)^2}_{\mathbf{4}^-} - \underbrace{l^2}_{\mathbf{1}^-} - q^2 \right] \right\}$$

$$\Rightarrow \quad \left[D - 2n_1 - n_4 - n_5 + n_4(q^2 - \mathbf{1}^-)\mathbf{4}^+ + n_5(\mathbf{2}^- - \mathbf{1}^-)\mathbf{5}^+ \right] T_1 = 0$$

IBP identities for T_1

$$\left[\underbrace{D - 2n_1 - n_4 - n_5} + \underbrace{n_4(q^2} - \underbrace{\mathbf{1}^-)\mathbf{4}^+} + n_5(\mathbf{2}^- - \mathbf{1}^-)\mathbf{5}^+ \right] T_1 = 0$$



task: combine all IBP identities such that

$$T_1(n_1, n_2, n_3, n_4, n_5) \to \quad \text{“simpler” integrals, i.e.}$$

- convolutions of 1-loop
- low values of n_i

IBP identities for T_1

$$D - 2\,n_1 + n_4\,(-1 + (q^2 - \mathbf{1}^-)\,\mathbf{4}^+) + n_5\,(-1 + (-\mathbf{1}^- + \mathbf{2}^-)\,\mathbf{5}^+) = 0$$

$$n_1\,(-1 + (-q^2 + \mathbf{4}^-)\,\mathbf{1}^+) + n_4\,(1 + (q^2 - \mathbf{1}^-)\,\mathbf{4}^+) + (-\mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^- + \mathbf{4}^-)\,n_5\,\mathbf{5}^+ = 0$$

$$n_1\,(-1 + (-\mathbf{2}^- + \mathbf{5}^-)\,\mathbf{1}^+) + (q^2 - \mathbf{1}^- - \mathbf{3}^- + \mathbf{5}^-)\,n_4\,\mathbf{4}^+ + n_5\,(1 + (-\mathbf{1}^- + \mathbf{2}^-)\,\mathbf{5}^+) = 0$$

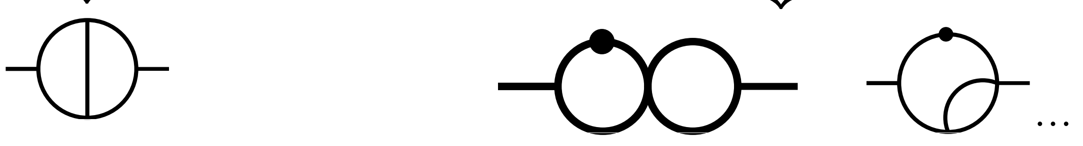
$$n_2\,(-1 + (-\mathbf{1}^- + \mathbf{5}^-)\,\mathbf{2}^+) + (q^2 - \mathbf{2}^- - \mathbf{4}^- + \mathbf{5}^-)\,n_3\,\mathbf{3}^+ + n_5\,(1 + (\mathbf{1}^- - \mathbf{2}^-)\,\mathbf{5}^+) = 0$$

$$n_2\,(-1 + (-q^2 + \mathbf{3}^-)\,\mathbf{2}^+) + n_3\,(1 + (q^2 - \mathbf{2}^-)\,\mathbf{3}^+) + (\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^- - \mathbf{4}^-)\,n_5\,\mathbf{5}^+ = 0$$

$$D - 2\,n_2 + n_3\,(-1 + (q^2 - \mathbf{2}^-)\,\mathbf{3}^+) + n_5\,(-1 + (\mathbf{1}^- - \mathbf{2}^-)\,\mathbf{5}^+) = 0$$

(1) – (3):

$$\Big[\underbrace{D - 2n_5 - n_1 - n_4} - \underbrace{n_1(\mathbf{5}^- - \mathbf{2}^-)\mathbf{1}^+ - n_4(\mathbf{5}^- - \mathbf{3}^-)\mathbf{4}^+}\Big]T_1 = 0$$



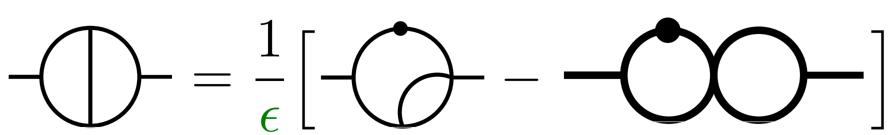
Triangle rule

$$T(n_1,n_2,n_3,n_4,n_5) = \frac{1}{D-2n_5-n_1-n_4}$$
$$\times \Big[n_1(\mathbf{5}^- - \mathbf{2}^-)\mathbf{1}^+ + n_4(\mathbf{5}^- - \mathbf{3}^-)\mathbf{4}^+\Big] T_1(n_1,n_2,n_3,n_4,n_5)$$

$\rightarrow$ recurrence relation

example: $n_1 = n_2 = n_3 = n_4 = n_5 = 1$

$$T_1(1,1,1,1,1) = \frac{1}{D-4}\Big[T_1(2,1,1,1,0) - T_1(2,0,1,1,1)$$
$$+ T_1(1,1,1,2,0) - T_1(1,1,0,2,1)\Big]$$



Triangle rule

indices > 1 $\quad\longrightarrow\quad$ apply rec.-rel. repeatedly

$$
\begin{array}{lllllll}
T_1(1,3,1,1,1) & \to & T_1(2,3,1,1,0) & & & & \\
 & & T_1(2,2,1,1,1) & \to & T_1(3,2,1,1,0) & & \\
 & & T_1(1,3,1,2,0) & & T_1(3,1,1,1,1) & \to & T_1(4,1,1,1,0) \\
 & & T_1(1,3,0,2,1) & & T_1(2,2,1,2,0) & & T_1(4,0,1,1,1) \\
 & & & & T_1(2,2,0,2,1) & & T_1(3,1,1,2,0) \\
 & & & & & & T_1(3,1,0,2,1)
\end{array}
$$

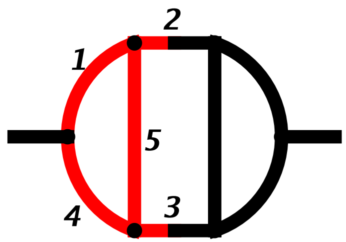
$\longrightarrow$ generates many terms

$\qquad\longrightarrow$ needs computer algebra!

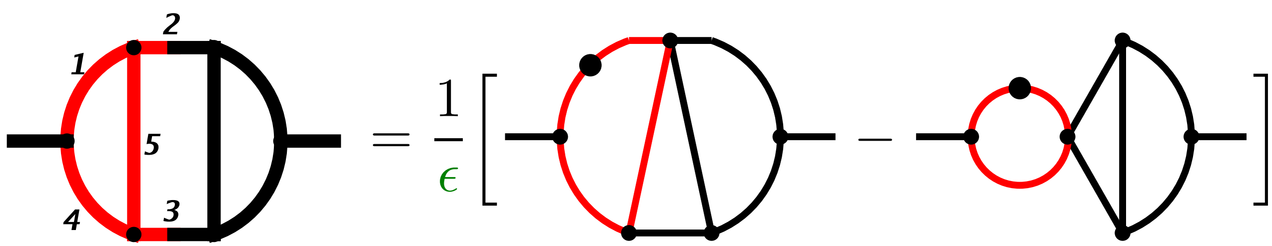
Triangle rule

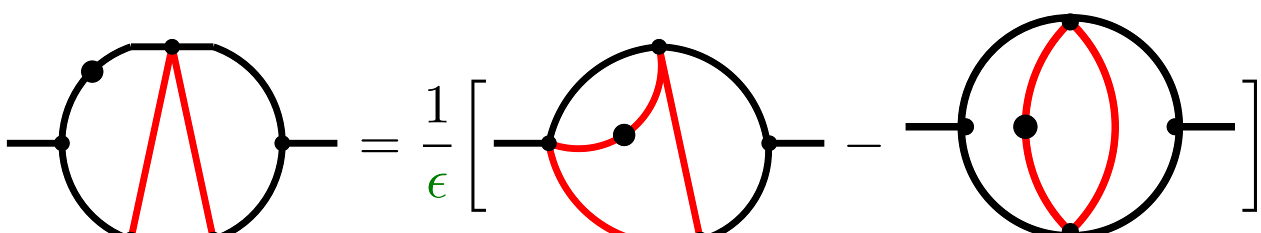
observation: this rec.-relation applies to any triangular sub-loop!

e.g. 3-loop:



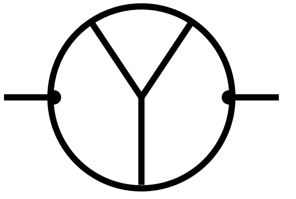
$$D - 2n_5 - n_1 - n_4 \quad = \quad n_1(\mathbf{5}^- - \mathbf{2}^-)\mathbf{1}^+ + n_4(\mathbf{5}^- - \mathbf{3}^-)\mathbf{4}^+$$





Triangle rule – exercise

Which convolutions of 1- and 2-loop integrals does the following integral reduce to?



Triangle rule

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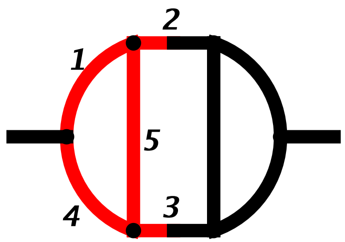
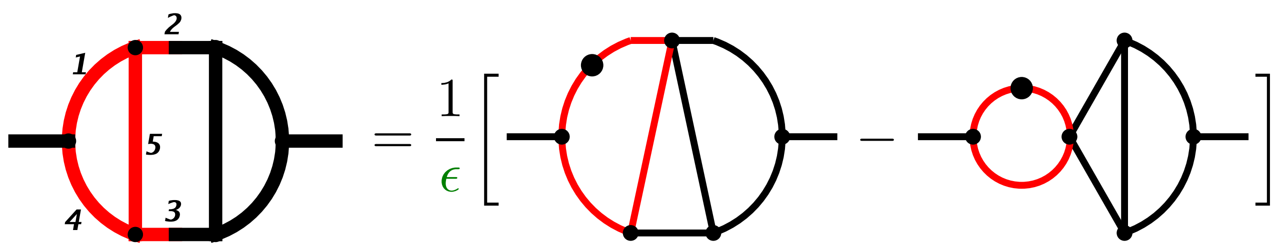
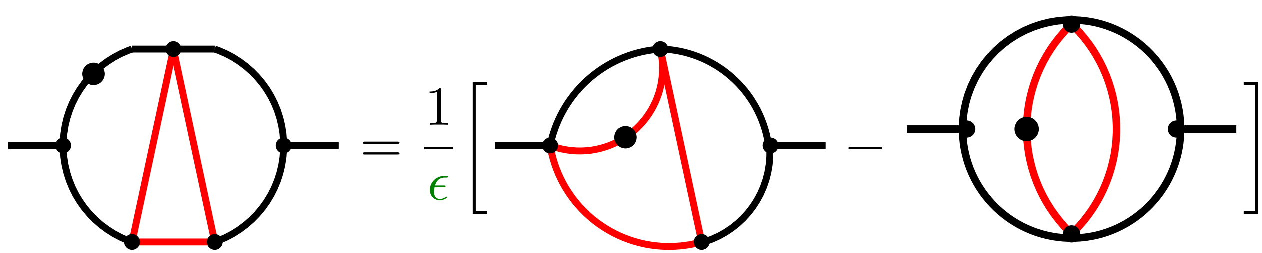
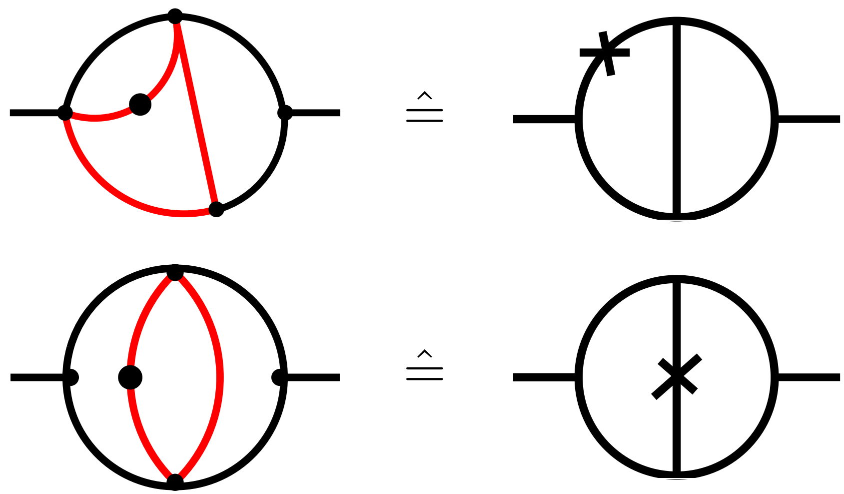
e.g. 3-loop:



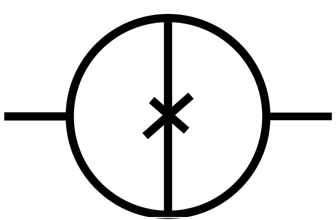
$$D - 2n_5 - n_1 - n_4 \quad = \quad n_1(\mathbf{5}^- - \mathbf{2}^-)\mathbf{1}^+ + n_4(\mathbf{5}^- - \mathbf{3}^-)\mathbf{4}^+$$



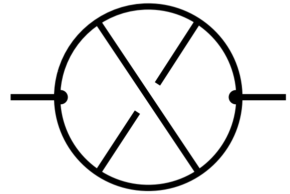




 non-integer power

 can not be reduced by triangle rule!

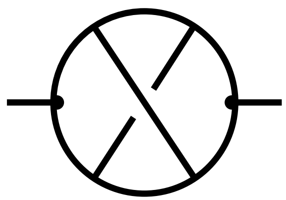
Non-planar diagram: Topology NO



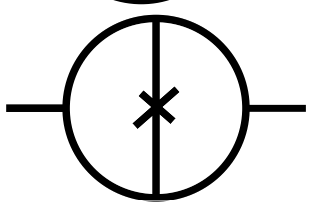
— no triangular sub-loop!

$$\mathrm{NO}(1,1,1,1,1,1,1,1) = 20\,\zeta_5 + \mathcal{O}(\epsilon)$$

cannot be reduced to simpler topology



$\mathrm{NO}(1,1,1,1,1,1,1,1)$



$\mathrm{T}_1(1,1,1,1,\epsilon)$

} master integrals

The crucial idea

Any integral

$$I(n_1,\dots n_N) = \int \mathrm{d}^D k_1 \cdots \mathrm{d}^D k_l \frac{1}{(p_1^2 - m_1^2)^{n_1} \cdots (p_N^2 - m_N^2)^{n_N}}$$

can be written as

$$I(n_1,\dots n_N) = C_1(D) I_1 + \cdots + C_M(D) I_M$$

$I_1,\dots,I_M$: finite set of master integrals

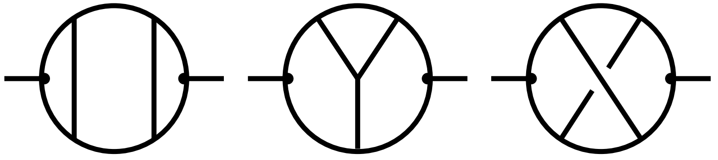
$C(D)$: rational functions of $D = 4 - 2\epsilon$

- universal problem: identify and evaluate $I_1,\dots,I_M$
- specific problem: evaluate $C_i(D)$

Approaches

- MINCER approach
  - write down IBP identities
  - combine them to achieve reduction

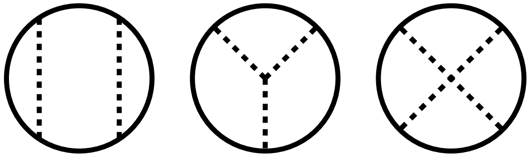
$\longrightarrow$ successful for massless 3-loop self-energies



[Chetyrkin, Tkachov]
[Larin, Tkachov, Vermaseren]

$\longrightarrow$ MINCER

massive 3-loop tadpoles



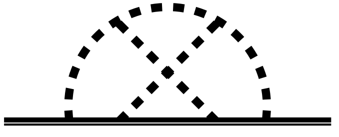
[Broadhurst]
[Chetyrkin, Kühn, Steinhauser]

$\longrightarrow$ MATAD

Approaches

- MINCER approach
  - write down IBP identities
  - combine them to achieve reduction

3-loop on-shell



[Melnikov, v.Ritbergen]

3-loop HQET [Grozin]