

# Symbolic Programming Examples

Thomas Hahn

Max-Planck-Institut für Physik  
München



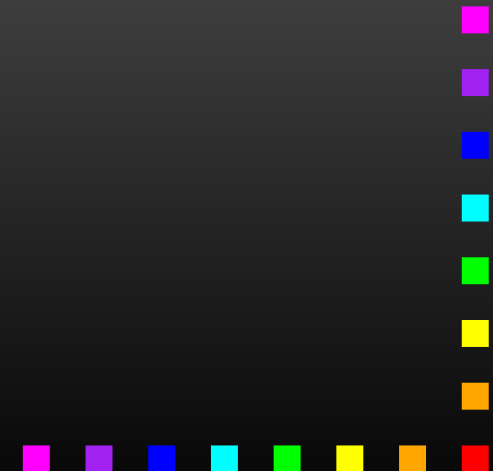
# List of Examples

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- **Calculation of a Fermion Trace**  
Built-in in FORM, complicated in Mathematica.
- **Tensor Reduction**



# Reference Books, Formula Collections

- V.I. Borodulin et al.  
**CORE (Compendium of Relations)**  
hep-ph/9507456.
- Herbert Pietschmann  
**Formulae and Results in Weak Interactions**  
Springer (Austria) 2nd ed., 1983.
- Andrei Grozin  
**Using REDUCE in High-Energy Physics**  
Cambridge University Press, 1997.



# Antisymmetric Tensor

The **Antisymmetric Tensor in  $n$  dimensions** is denoted by  $\varepsilon_{i_1 i_2 \dots i_n}$ . You can think of it as a matrix-like object which has either  $-1$ ,  $0$ , or  $1$  at each position.

For example, the **Determinant** of a matrix, being a **completely antisymmetric** object, can be written with the  $\varepsilon$ -tensor:

$$\det A = \sum_{i_1, \dots, i_n=1}^n \varepsilon_{i_1 i_2 \dots i_n} A_{i_1 1} A_{i_2 2} \cdots A_{i_n n}$$

In practice, the  $\varepsilon$ -tensor is usually contracted, e.g. with vectors. We will adopt the following notation to avoid dummy indices:

$$\varepsilon_{\mu\nu\rho\sigma} p^\mu q^\nu r^\rho s^\sigma = \varepsilon(p, q, r, s).$$



# Momentum Conservation

Problem: **Proliferation of terms** in expressions such as

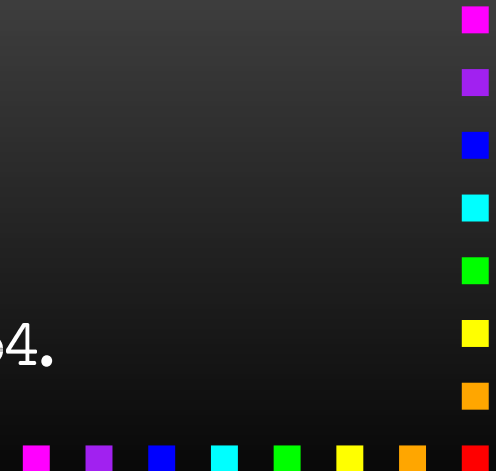
$$d = \frac{1}{(p_1 + p_2 - p_3)^2 + m^2}$$
$$= \frac{1}{p_1^2 + p_2^2 + p_3^2 + 2p_1p_2 - 2p_2p_3 - 2p_1p_3 + m^2},$$

whereas if  $p_1 + p_2 = p_3 + p_4$  we could have instead

$$d = \frac{1}{p_4^2 + m^2}.$$

**In Mathematica: just do** `d /. p1 + p2 - p3 -> p4.`

**Problem: FORM cannot replace sums.**



# Momentum Conservation in FORM

Idea: for each expression  $x$ , add and subtract a zero, i.e. form

$$\{x, y = x + \sigma, z = x - \sigma\}, \quad \text{where e.g.} \quad \sigma = p_1 + p_2 - p_3 - p_4,$$

then select the shortest expression. But: how to select the shortest expression (in FORM)?

Solution: add the number of terms of each argument, i.e.

$$\{x, y, z\} \rightarrow \{x, y, z, n_x, n_y, n_z\}.$$

1 2 3 4 5 6

Then sort  $n_x, n_y, n_z$ , but when exchanging  $n_a$  and  $n_b$ ,  
**exchange also**  $a$  and  $b$ :

```
symm 'foo' (4,1) (5,2) (6,3);
```

This unconventional sort statement is rather typical for FORM.



# Factorization

Because of its **restriction to local operations**, FORM cannot truly factorize expressions.

- “Small Scale” – factarg

$f(s^2 + s*t + s*u)$

factarg f;       $f(s, s + t + u)$

Only trivial factorization possible this way.

- “Large Scale” – Bracketing & Collecting

$a*c + a*d + b*c + b*d$

bracket a, b;

.sort

  $a*(c + d) + b*(c + d)$

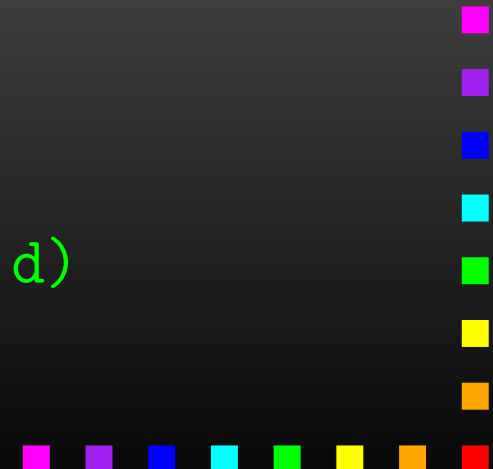
collect f;

  $a*f(c + d) + b*f(c + d)$

bracket f;

.sort

  $(a + b)*f(c + d)$



# Abbreviating

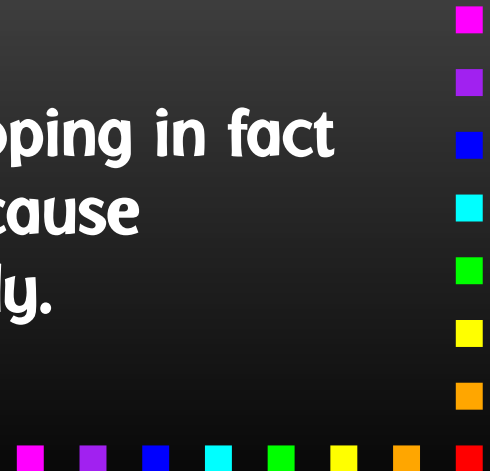
One of the most powerful tricks to both **reduce the size** of an expression and **reveal its structure** is to substitute subexpressions by new variables.

The essential function here is `Unique` with which new symbols are introduced. For example,

```
Unique["test"]
```

generates e.g. the symbol `test1`, which is **guaranteed not to be in use so far**.

The `Module` function which implements lexical scoping in fact uses `Unique` to rename the symbols internally because **Mathematica can really do dynamical scoping only**.

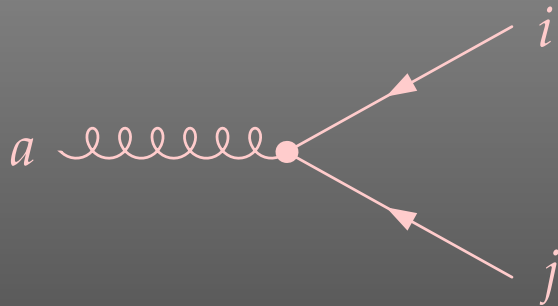




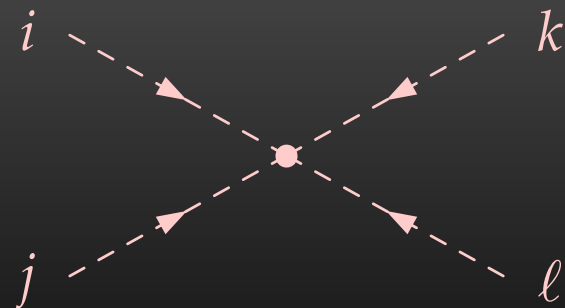
# Colour Structures

In Feynman diagrams four type of **Colour structures** appear:

Natural Representation

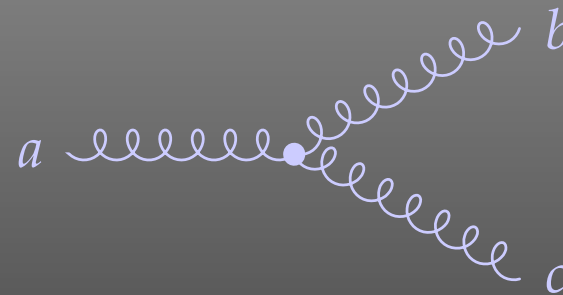


$$\sim T_{ij}^a = \text{SUNT}[a, i, j]$$

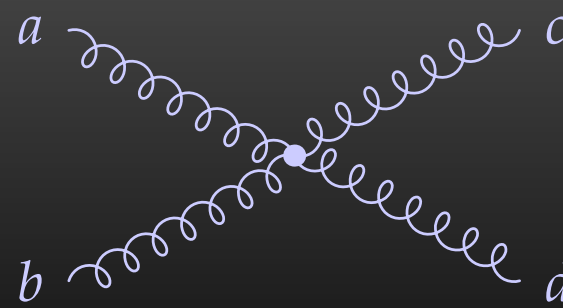


$$\sim T_{ij}^a T_{kl}^a = \text{SUNTSum}[i, j, k, l]$$

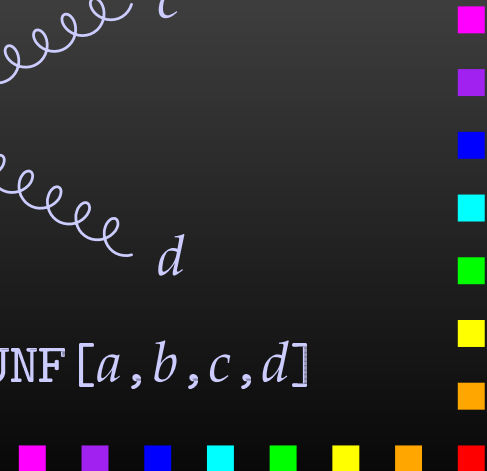
Adjoint Representation



$$\sim f^{abc} = \text{SUNF}[a, b, c]$$



$$\sim f^{abx} f^{xcd} = \text{SUNF}[a, b, c, d]$$



# Unified Notation

The SUNF's can be converted to SUNT's via

$$f^{abc} = 2i [\text{Tr}(T^c T^b T^a) - \text{Tr}(T^a T^b T^c)] .$$

We can now represent all colour objects by just SUNT:

- $\text{SUNT}[i, j] = \delta_{ij}$
- $\text{SUNT}[a, b, \dots, i, j] = (T^a T^b \dots)_{ij}$
- $\text{SUNT}[a, b, \dots, 0, 0] = \text{Tr}(T^a T^b \dots)$

This notation again avoids **unnecessary dummy indices**.  
(Mainly namespace problem.)

For purposes such as the “large- $N_c$  limit” people like to use **SU(N)** rather than an explicit **SU(3)**.



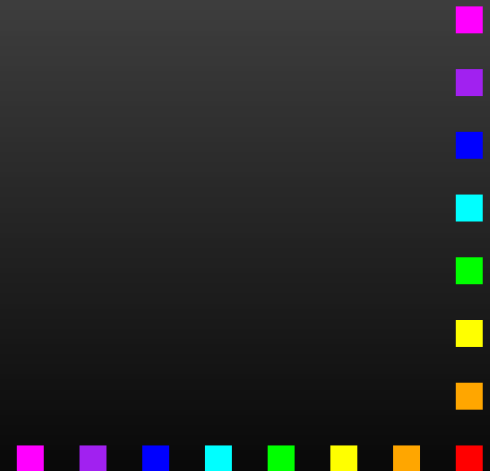
# Fierz Identities

The **Fierz Identities** relate expressions with **different orderings of external particles**. The Fierz identities essentially express completeness of the underlying matrix space.

They were originally found by Markus Fierz in the context of Dirac spinors, but can be generalized to any finite-dimensional matrix space [hep-ph/0412245].

For  $SU(N)$  (colour) reordering, we need

$$T_{ij}^a T_{kl}^a = \frac{1}{2} \left( \delta_{il} \delta_{kj} - \frac{1}{N} \delta_{ij} \delta_{kl} \right).$$



# Cvitanovich Algorithm

For an **Amplitude**:

- convert all colour structures to (generalized) SUNT objects,
- simplify as much as possible, i.e. use the Fierz identity on all internal gluon lines.

For a **Squared Amplitude**:

- use the Fierz identity for  $SU(N)$  to get rid of all SUNT objects.

For “hand” calculations, a pictorial version of this algorithm exists in the literature.



## Translation to Colour-Chain Notation

In colour-chain notation we can distinguish two cases:

a) Contraction of **different chains**:

$$\langle A | T^a | B \rangle \langle C | T^a | D \rangle = \frac{1}{2} \left( \langle A | D \rangle \langle C | B \rangle - \frac{1}{N} \langle A | B \rangle \langle C | D \rangle \right),$$

b) Contraction on the **same chain**:

$$\langle A | T^a | B | T^a | C \rangle = \frac{1}{2} \left( \langle A | C \rangle \text{Tr } B - \frac{1}{N} \langle A | B | C \rangle \right).$$



## Fermion Trace

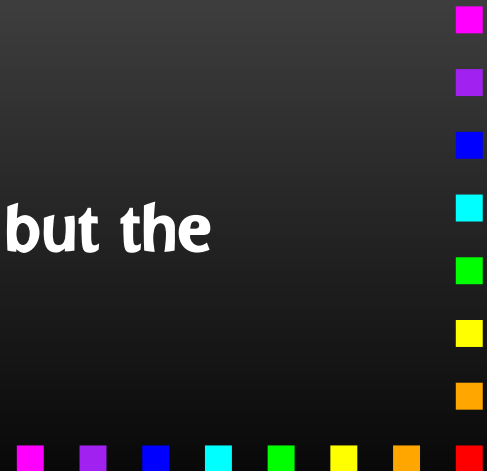
Leaving apart problems due to  $\gamma_5$  in  $d$  dimensions, we have as the main algorithm for the 4d case:

$$\begin{aligned}\text{Tr } \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \cdots &= + g_{\mu\nu} \text{Tr } \gamma_\rho \gamma_\sigma \cdots \\ &\quad - g_{\mu\rho} \text{Tr } \gamma_\nu \gamma_\sigma \cdots \\ &\quad + g_{\mu\sigma} \text{Tr } \gamma_\nu \gamma_\rho \cdots\end{aligned}$$

This algorithm is recursive in nature, and we are ultimately left with

$$\text{Tr } \mathbb{1} = 4.$$

(Note that this 4 is not the space-time dimension, but the dimension of spinor space.)



# Tensor Reduction

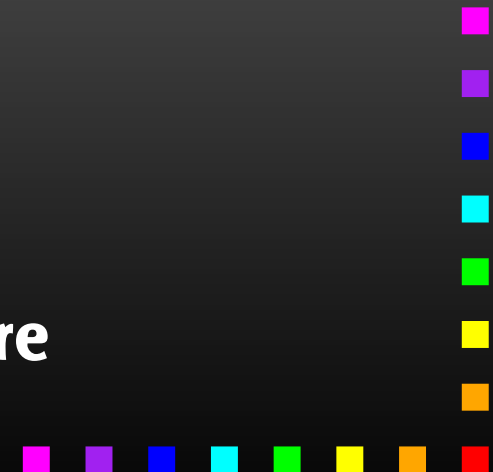
The loop integrals corresponding to closed loops in a Feynman integral in general have a **tensor structure** due to **integration momenta in the numerator**. For example,

$$B_{\mu\nu}(p) = \int d^d q \frac{q_\mu q_\nu}{(q^2 - m_1^2) ((q - p)^2 - m_2^2)}.$$

Such tensorial integrals are rather unwieldy in practice, therefore they are reduced to linear combinations of Lorentz-covariant tensors, e.g.

$$B_{\mu\nu}(p) = B_{00}(p) g_{\mu\nu} + B_{11}(p) p_\mu p_\nu.$$

It is the **coefficient functions**  $B_{00}$  and  $B_{11}$  which are implemented in a library like LoopTools.



# Tensor Reduction Algorithm

The first step is to **convert the integration momenta** in the numerator to an actual tensor, e.g.  $q_\mu q_\nu \rightarrow N_{\mu\nu}$ . FORM has the special command `totensor` for this:

```
totensor q1, NUM;
```

The next step is to **take out  $g_{\mu\nu}$ 's in all possible ways**. We do this in form of a sum:

$$N_{\mu_1 \dots \mu_n} = \sum_{i=0,2,4,\dots}^n \pi(0)^i \sum_{\substack{\text{all } \{\nu_1, \dots, \nu_i\} \\ \in \{\mu_1, \dots, \mu_n\}}} g_{\nu_1 \nu_2} \cdots g_{\nu_{i-1} \nu_i} N_{\mu_1 \dots \mu_n \setminus \nu_1 \dots \nu_i}$$

The  $\pi(0)^i$  **keeps track of the indices** of the tensor coefficients, i.e. it later provides the two zeros for every  $g_{\mu\nu}$  in the index, as in  $D_{0012}$ .





# Tensor Reduction Algorithm

To fill in the remaining  $\pi(i)$ 's, we start off by **tagging the arguments** of the loop function, which are just the momenta. For example:

$$C(p_1, p_2, \dots) \rightarrow \tau(\pi(1)p_1 + \pi(2)p_2) C(p_1, p_2, \dots)$$

The temporary function  $\tau$  keeps its argument, the 'tagged' momentum  $p$ , separate from the rest of the amplitude.

Now **add the indices** of  $N_{\mu_1 \dots \mu_n}$  to the momentum in  $\tau$ :

$$\tau(p) N_{\mu_i \dots \mu_n} = p_{\mu_i} \cdots p_{\mu_n}.$$

Finally, collect all  $\pi$ 's into the tensor-coefficient index.

