Computer algebra in particle physics

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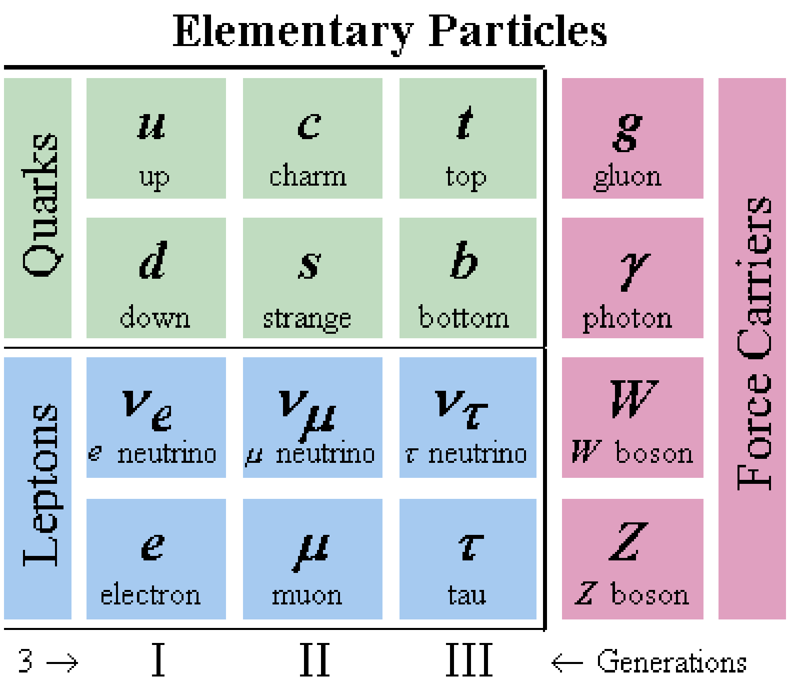
Things we do know

- Elementary particles
  - fermions (leptons, quarks)
    (constituents of matter)
  - bosons
    (carrier particles of forces)



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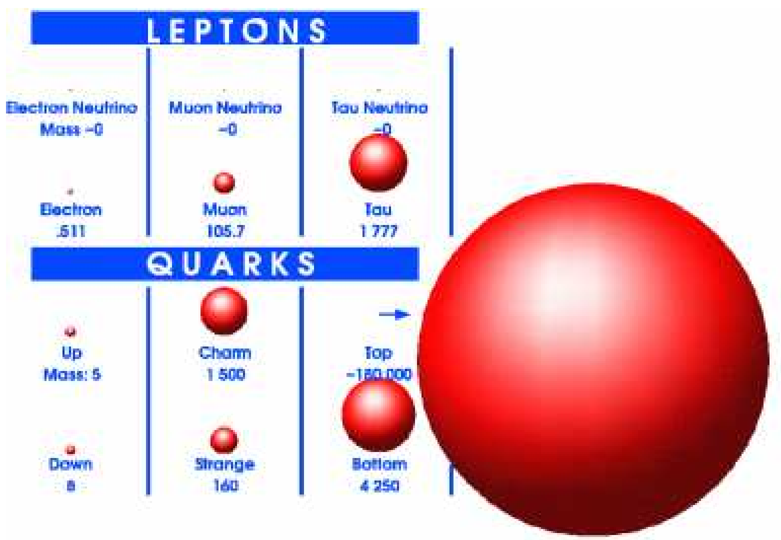
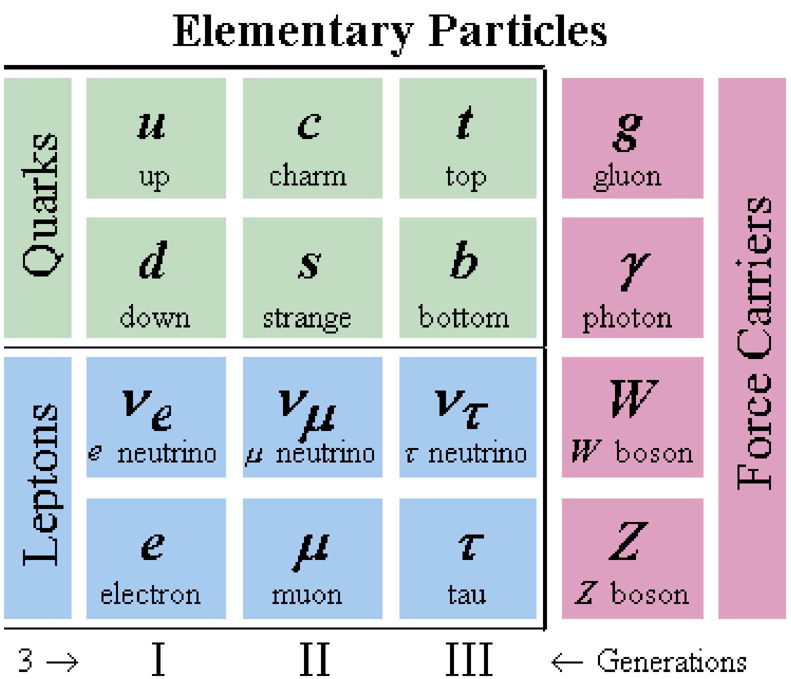
- Masses of fermions





massive neutrinos first glimpse beyond SM

The Nobel Prize in Physics 2008

"for the discovery of the mechanism of spontaneous broken symmetry in subatomic physics"

"for the discovery of the origin of the broken symmetry which predicts the existence of at least three families of quarks in nature"

Photo: SCANPIX

Photo: Kyodo/Reuters

Photo: Kyoto University

Yoichiro Nambu	Makoto Kobayashi	Toshihide Maskawa
1/2 of the prize	1/4 of the prize	1/4 of the prize
USA	Japan	Japan
Enrico Fermi Institute, University of Chicago Chicago, IL, USA	High Energy Accel... Researc...	
b. 1921		b. 1940

Nobel prize 2008 for spontaneous symmetry breaking and CP violation

Titles, data and places gi... ...ove refer to the time of the award.

Things we want to know

The Big Questions

- What is the origin of mass?
- What is 96% of
  the universe made of?
- Why is there no more
  antimatter?
- Do extra dimensions
  of space really exist?
- . . .

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Large Hadron Collider

Highest energies at colliders until 201x

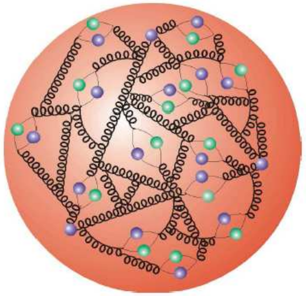
Energy frontier

- Search for Higgs boson, new massive particles at highest energies

$$E = m\,c^2$$

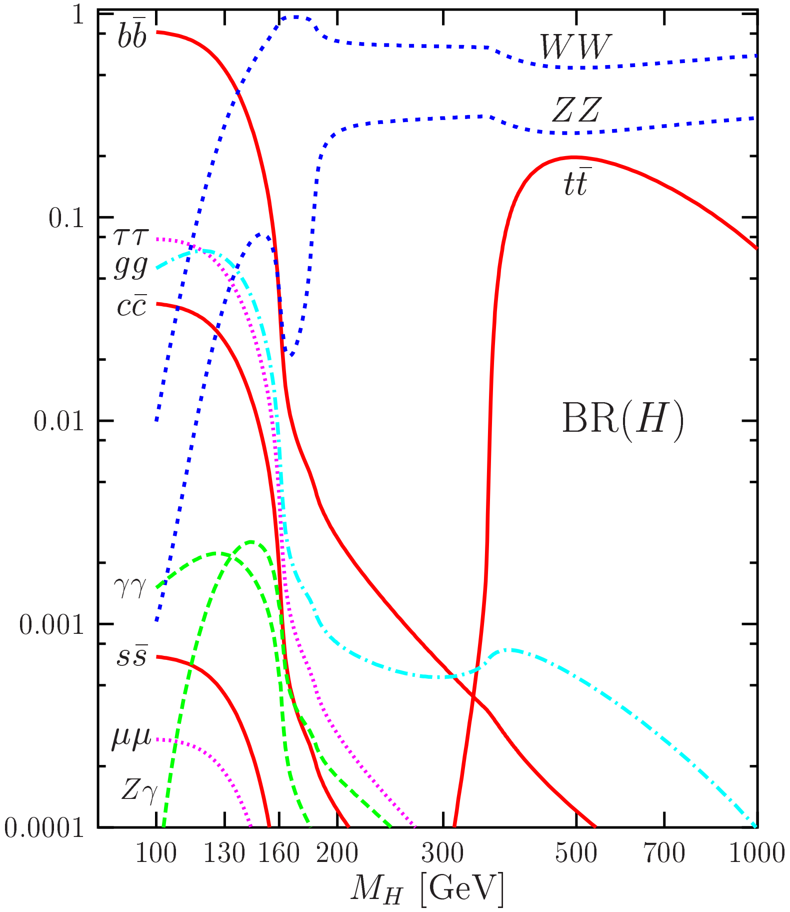
Hadron colliders

- Proton–(anti)-proton collisions reach TeV-scale
  - Tevatron $\sqrt{S} = 1.96$TeV, LHC $\sqrt{S} = 14$TeV



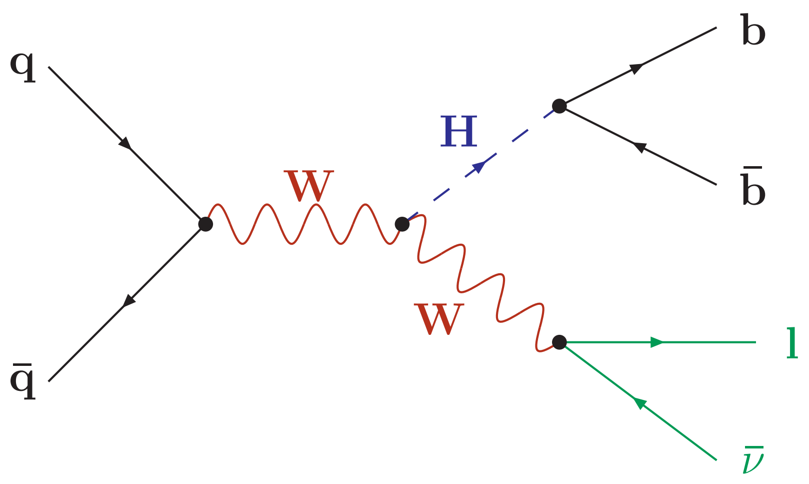
- Proton: composite multi-particle bound state
  - collider: "wide-band beams" of quarks and gluons
  - understand SM background (LHC is a QCD machine)
- Theory has to match or exceed accuracy of LHC data
  - perturbative QCD is essential and established part of toolkit
  - electroweak corrections important for precision predictions

Higgs production at LHC



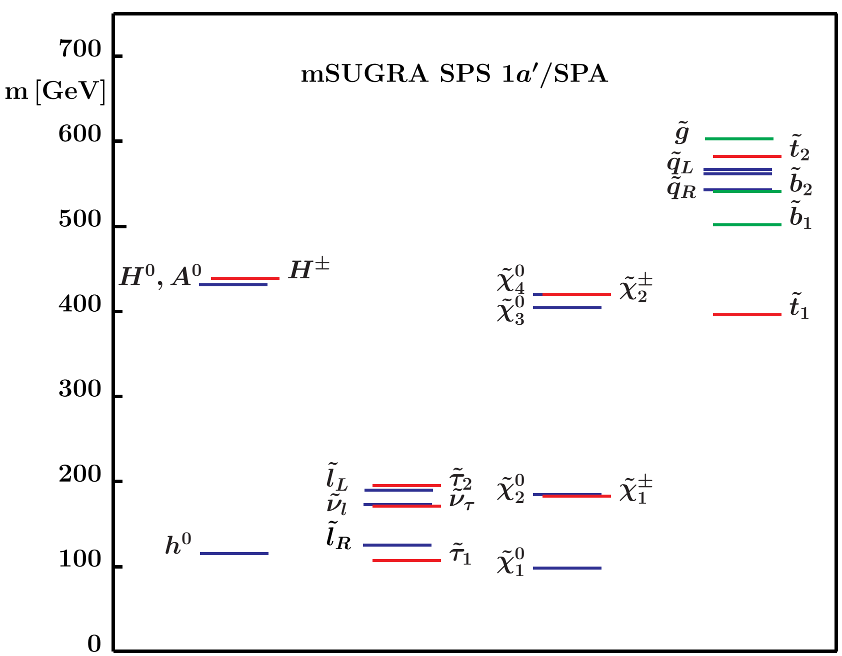
- Branching ratios for decay of Standard Model Higgs

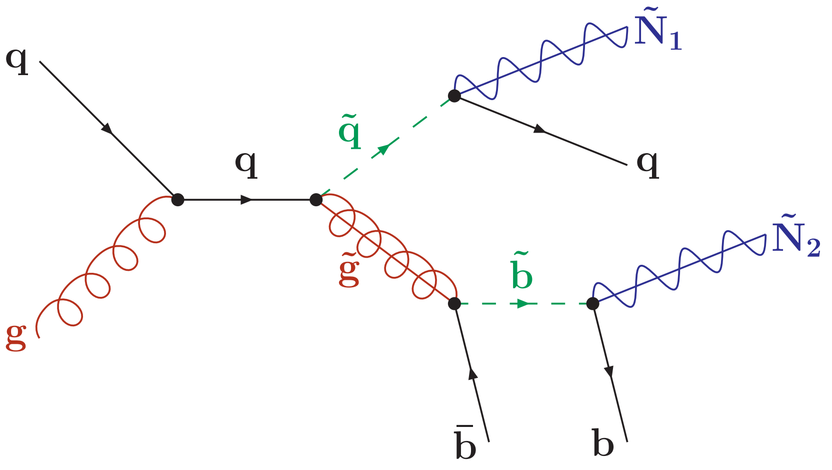
- High-multiplicity final states
  - typical SM process is accompanied by radiation of multiple jets
- Example: Higgs-strahlung
  - channel $q\bar{q} \to W(Z)H$ (third largest rate at LHC)
  - dominant decay $H \to b\bar{b}$



Supersymmetry

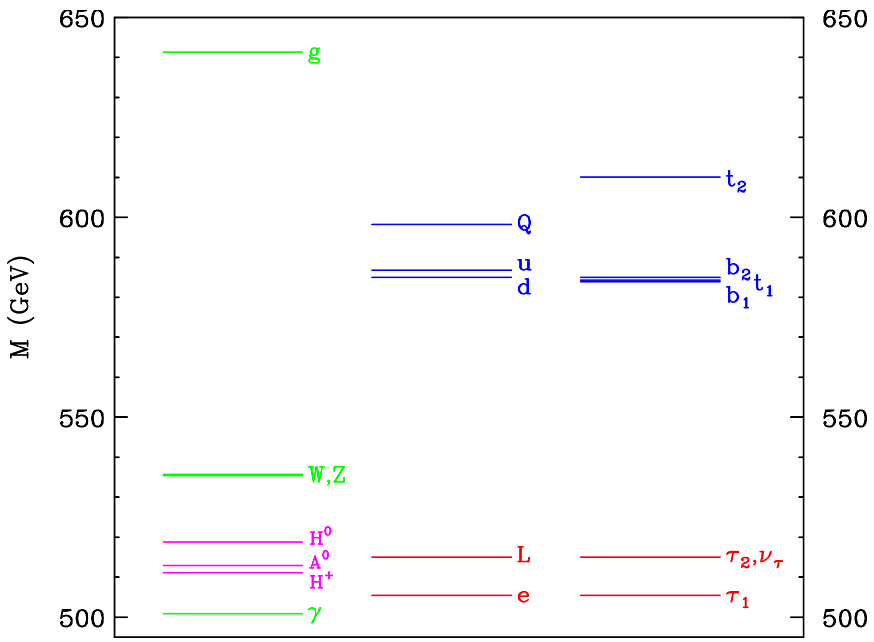
- Pair-production of supersymmetric particles (R-parity)
  - lightest supersymmetric particle (LSP) must be absolutely stable
- MSSM spectrum
  - typical signature: multiple jets, leptons and missing energy
- Example: neutralino production $\tilde{N}^0_{1,2}$
  - electric and color-neutral (dark matter candidate)

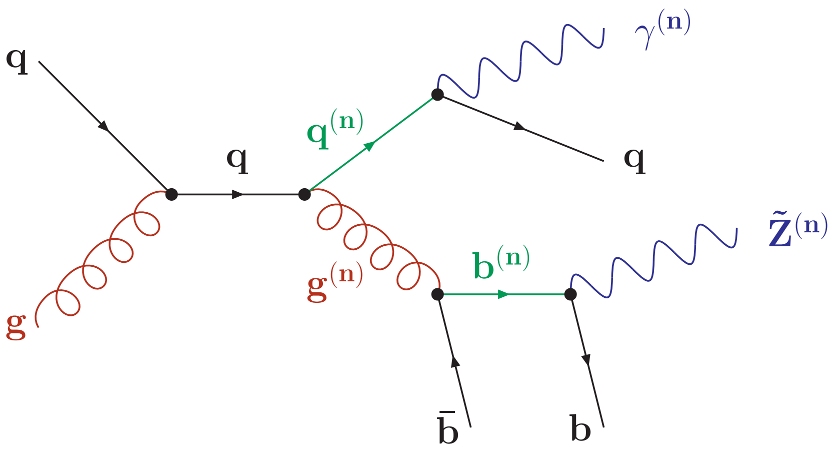




Large extra dimensions

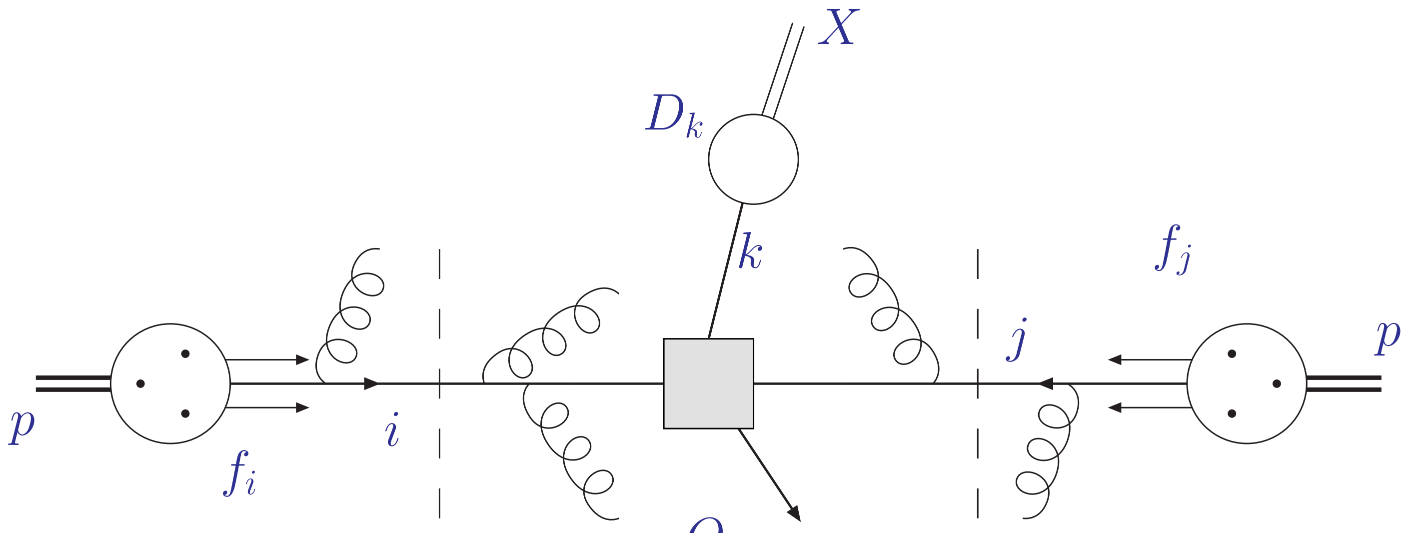
- Spectrum of first Kaluza-Klein excitations
  - effective mass $\simeq$ (compactification radius)$^{-1}$, $m^{(n)} \simeq 1/R$
- Pair-production of excited KK-modes in interactions
  - phenomenology: missing energy in subsequent chain decays





Perturbative QCD at colliders

- Hard hadron-hadron scattering
  - constituent partons from each incoming hadron interact at short



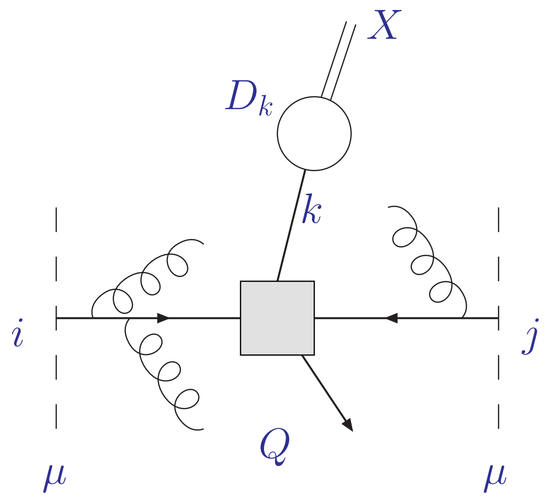
- QCD factorization
  - separate sensitivity to dynamics from different scales

$$\sigma_{pp\to X} = \sum_{ijk} f_i(\mu^2) \otimes f_j(\mu^2) \otimes \hat{\sigma}_{ij\to k}\left(\alpha_s(\mu^2), Q^2, \mu^2\right) \otimes D_{k\to X}(\mu^2)$$

  - factorization scale μ, subprocess cross section $\hat{\sigma}_{ij\to k}$ for parton types i, j and hadronic final state X

Hard scattering cross section

- Parton cross section $\hat{\sigma}_{ij\to k}$ calculable pertubatively in powers of α_s
  - constituent partons interact at short distances $\mathcal{O}(1/Q)$



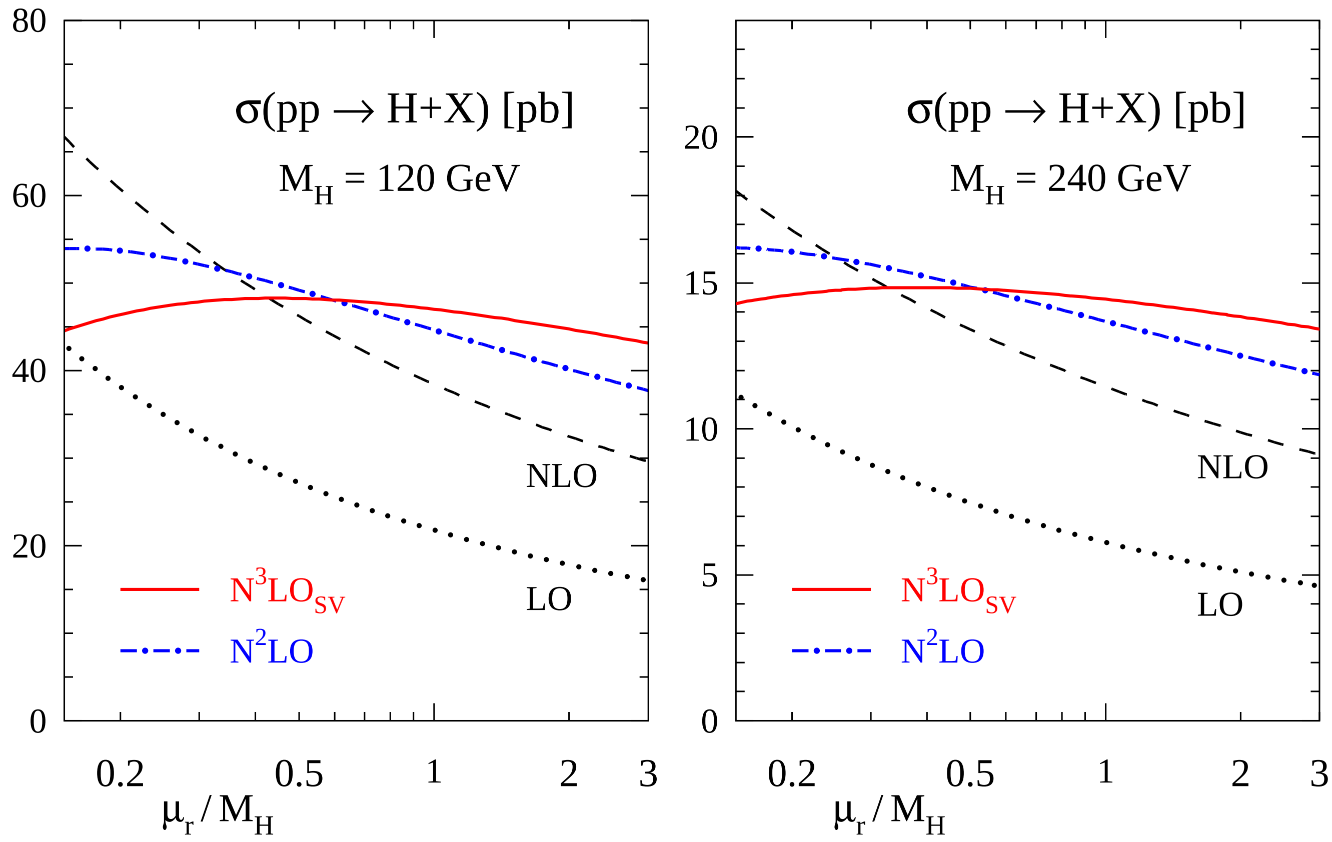
- Accuracy of perturbative predictions
  - LO (leading order)
  - NLO (next-to-leading order)
  - NNLO (next-to-next-to-leading order)
  - N^3LO (next-to-next-to-next-to-leading order)
  - ...

Recent result on third-order QCD corrections

- Charged-current structure function F_3 at N^{3}LO in QCD

  S.M., Vermaseren, Vogt `arXiv:0812.4168 [hep-ph]`

Cross section for Higgs production



- Variation of cross section at LHC with renormalizaion scale for different Higgs masses: $M_H = 120$GeV (left) and $M_H = 240$GeV (right)
  - NNLO corrections
    Harlander, Kilgore '02; Anastasiou, Melnikov '02; Ravindran, Smith, van Neerven '03
  - complete soft N^3LO corrections (dominant part at three loops) S.M., Vogt '05

Needs (I)

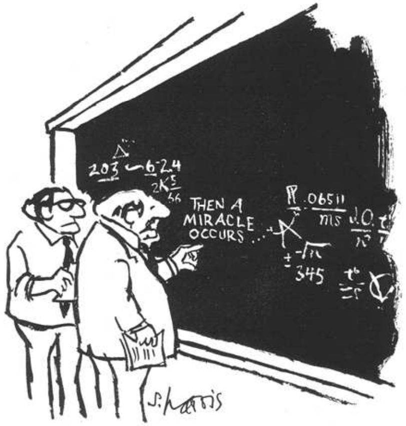
Particle physics

- Perturbative approach straightforward in principle
  - draw all Feynman diagrams and evaluate them,
  - use standard reduction techniques for tree/loop amplitudes
- (Extremely) hard in practice
  - intermediate expressions more complicated than final results
- Known bottlenecks
  - **many diagrams** — many diagrams are related by gauge invariance
  - **many terms in each diagram** — nonabelian gauge boson self-interactions are complicated
  - **many kinematic variables** — allowing the construction of very complicated expressions
- Computer algebra programs are a standard tool

Needs (II)

Computer algebra

- Naive wish-list related to computer algebra systems
  - completeness: every branch of modern mathematics is covered
  - intelligence: output of all solutions we ever wanted to have
  - efficiency in performance: compilation
  - short development cycles: interactive use and high-quality output
  - standardized programming language: development tools exist
  - source code freely available: portability and bug hunting
- Unfortunately, life is not easy



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Needs (III)

- Choice for computer algebra system depends on specific needs

Local problems

- Problem expands into sum of different terms; each term can be solved independently of others
  - complications: number of terms can become large
  - requirements: bookkeeping, handling large amounts of data

Non-local problems

- Standardized non-local operations (e.g. factorization)
  - requirements: implementation of algorithms

Non-standard problems

- Dedicated algorithms, developed by user to solve a specific problem
  - requirements: modelling of abstract mathematical concepts in programming language of computer algebra system

Needs (IV)

Requirements in particle physics

- Symbolic calculations characterized by need for basic operations
  - sorting, gcd, factorization, multiplication
  - symbolic integration/summation
  - solution of systems of equations
  - ...
- Specialized code usually written by the user
  - largely dependent on the physics problem
  - add-on libraries
- No need for a system which knows more than the user

Computer algebra systems

- History
  - early realizations: Schoonschip ‘67, Reduce ‘68, . . .
- Today
  - Commercial systems: Mathematica, Maple, . . .
  - Free software: FORM, GiNaC, . . .

Examples in particle physics

- Feynman integrals
- QCD corrections at NLO to scattering processes
- Deep-inelastic scattering and Mellin moments

Loops

- Feynman integral for a given process
- Tensor integrals

$$I^{\mu_1,\mu_2,\dots}(D;\nu_1,\dots,\nu_n) = \int d^D p_1 \,\dots\, d^D p_l \, \frac{p_1^{\mu_1} p_2^{\mu_2} \dots}{(p_1^2)^{\nu_1} \,\dots\, (p_n^2)^{\nu_n}}$$

  - Lorentz indices $\mu_1, \mu_2, \dots$
  - l-loops, n-propagators
  - D (complex) space-time dimensions
    (dimensional regularization)
    Bollini, Giambiagi ‘72; Ashmore ‘72; Cicuta, Montaldi ‘72; ‘t Hooft, Veltman ‘72

Loops

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- Tensor integrals can be mapped to scalar integrals
  - expand $I^{\mu_1,\mu_2,\dots}(D;\dots)$ in tensor structures
  - coefficients of tensor structures are scalar integrals (projectors)
  - scalar products in numerator cancelled via denominators (sometimes, there are irreducible scalar products)

Upshot

- Scalar integral (l-loops, n-propagators, D dimensions)

$$I(D;\nu_1,\ldots,\nu_n) = \int d^D p_1 \,\ldots\, d^D p_l \,\frac{1}{(p_1^2)^{\nu_1}\,\ldots\,(p_n^2)^{\nu_n}}$$

  - $p_i = f(p_1, ..., p_l)$ (energy-momentum conservation)

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  - $p_i = f(p_1, ..., p_l)$ (energy-momentum conservation)
- Classification of scalar integrals
  - topology, n-point function (number of loops, legs)
  - scales (number of non-vanishing scalar products of momenta and masses)

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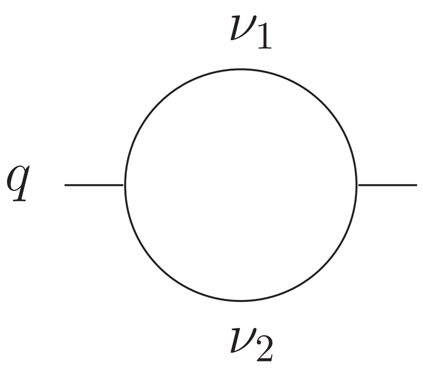
Task

- Calculation of scalar (loop) integrals
- Analytic expressions for $I(D;\nu_1,\ldots,\nu_n)$
  - expansion around integer-valued D space-time dimension (typically four space-time dimensions $D = 4 - 2\epsilon$)

Two-point integrals

- Easy example:
  massless one-loop two-point function L1

$$\mathrm{L1} = \int d^D p_1 \, \frac{1}{(p_1^2)^{\nu_1} \, ((p_1 - q)^2)^{\nu_2}}$$



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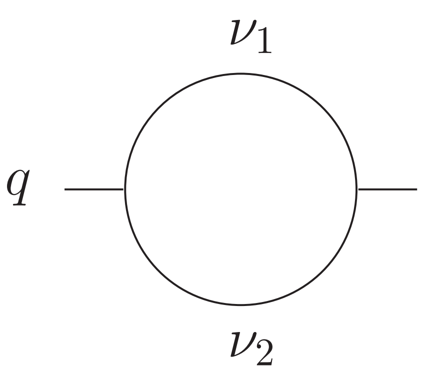

- Results for L1

$$\mathrm{L1} \quad = \quad \mathrm{i}\,(-1)^{\nu_1+\nu_2}\,\pi^{-D/2}\,(-p^2)^{D/2-\nu_1-\nu_2} \; \times$$
$$\times \, \frac{\Gamma(\nu_1+\nu_2-D/2)\Gamma(D/2-\nu_1)\Gamma(D/2-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(D-\nu_1-\nu_2)}$$

Two-point integrals

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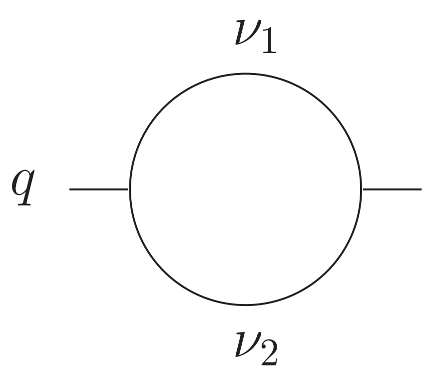


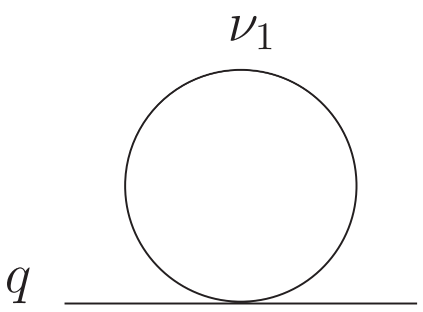
- Results for $\mathrm{L1}$

$$\begin{aligned} \mathrm{L1} \quad &= \quad \mathrm{i}\,(-1)^{\nu_1+\nu_2}\,\pi^{-D/2}\,(-p^2)^{D/2-\nu_1-\nu_2} \times \\ & \quad \times \frac{\Gamma(\nu_1+\nu_2-D/2)\Gamma(D/2-\nu_1)\Gamma(D/2-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(D-\nu_1-\nu_2)} \end{aligned}$$

- Expansion of Gamma-function in $\epsilon = 2 - \frac{D}{2}$ around positive integers values ($\nu_i \geq 0$)
  - Riemann zeta values $\Gamma(1+\epsilon) = 1 - \epsilon\gamma_E + \frac{\epsilon^2}{2}(\zeta_2 + \gamma_E^2) + \ldots$
  - $\overline{\mathrm{MS}}$-scheme puts $\gamma_E = 0$

- Easiest example: massless tadpole (scaleless integral)
  - vanishes in dimensional regularization

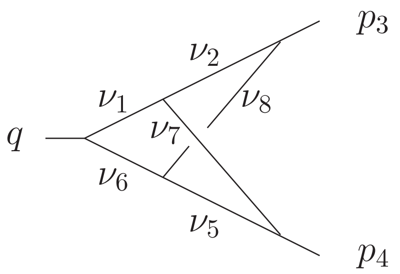
$$0 = \int d^D p_1 \, \frac{1}{(p_1^2)^{\nu_1}}$$



Massless two-loop non-planar vertex

- Another one-scale example

$$V_{\rm NO} = \int d^D p_1 \, d^D p_2 \, \frac{1}{(p_1^2)^{\nu_1} \, (p_2^2)^{\nu_2} \, (p_5^2)^{\nu_5} \, (p_6^2)^{\nu_6} \, (p_7^2)^{\nu_7} \, (p_8^2)^{\nu_8}}$$



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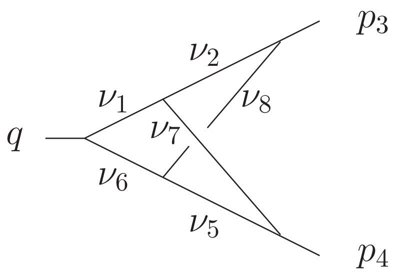


- Result in terms of multiple zeta values
- Laurent series (propagator powers $\nu_i = 1$) to order ϵ^5 S.M. '05

$$V_{\rm NO} =$$

$$S_\Gamma^2 \left(-q^2\right)^{-2-2\epsilon} \Bigg[- \frac{1}{\epsilon^4} + \frac{5\zeta_2}{\epsilon^2} + \frac{27\zeta_3}{\epsilon} + 23\zeta_2^2 - \epsilon(48\zeta_2\zeta_3 - 117\zeta_5)$$

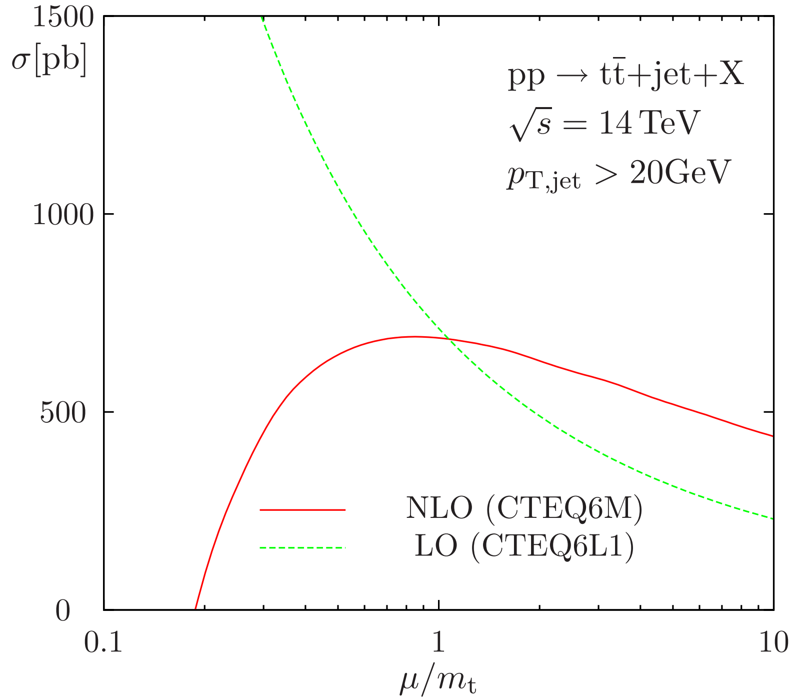
$$+\epsilon^2 \Big(\frac{456}{35}\zeta_2^3 - 267\zeta_3^2\Big) - \epsilon^3 \Big(\frac{1962}{5}\zeta_2^2\zeta_3 + 240\zeta_2\zeta_5 + 6\zeta_7\Big)$$

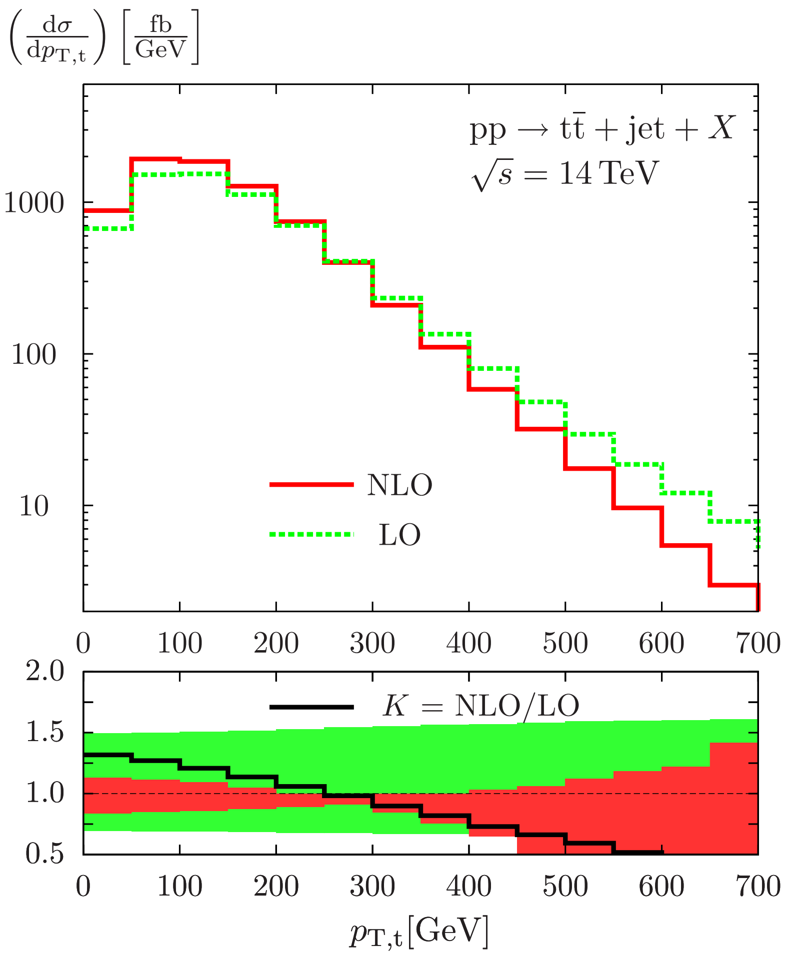
$$-\epsilon^4 \Big(\frac{3219}{7}\zeta_2^4 - 264\zeta_2\zeta_3^2 + 2466\zeta_3\zeta_5 - 264\zeta_{5,3}\Big)$$

$$-\epsilon^5 \Big(\frac{2832}{5}\zeta_2^3\zeta_3 + 2718\zeta_2^2\zeta_5 + 1218\zeta_2\zeta_7 + 1626\zeta_3^3 + \frac{20777}{3}\zeta_9\Big) \Bigg]$$

QCD corrections at NLO

$t\bar{t}+$ jet production

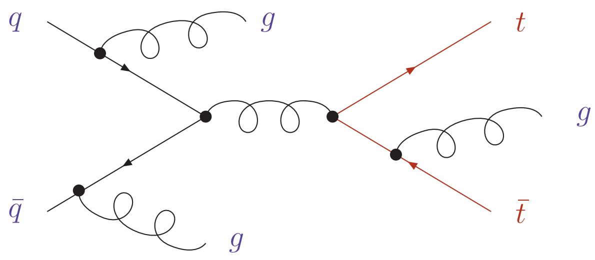


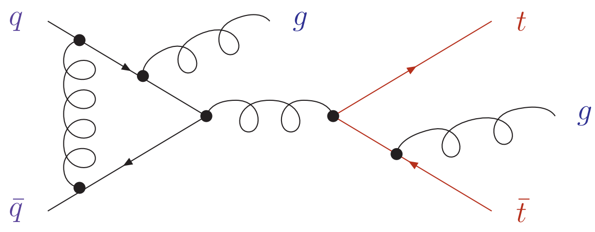


- Impressive state-of-the-art NLO QCD result Dittmaier, Uwer, Weinzierl '07-'08
  - much improved scale dependence for total rate and distributions

Why are one-loop corrections difficult ?

- Outline of a generic NLO calculation





Real corrections
- subtractions (IR-divergent)

Virtual corrections
+ subtractions (IR-divergent)

Cancellation of singularities
Finite partonic cross sections
Phase space integration
Convolution with PDFs
Monte Carlo

- All conceptual issues solved, but no general libraries available
- Important in practice: **speed, stability** and **automatization**

NLO virtual amplitudes

- Straightforward in principle – hard in practice with known bottlenecks
  - one-loop tensor integrals (standard Passarino-Veltman reduction)

$$I^{\mu_1,\mu_2,\dots}(k_1,\dots) = \int d^D p_1 \, \frac{p_1^{\mu_1} p_2^{\mu_2} \dots}{(p_1^2 - m_1^2)((p_1 - k_1)^2 - m_2^2) \, \dots}$$

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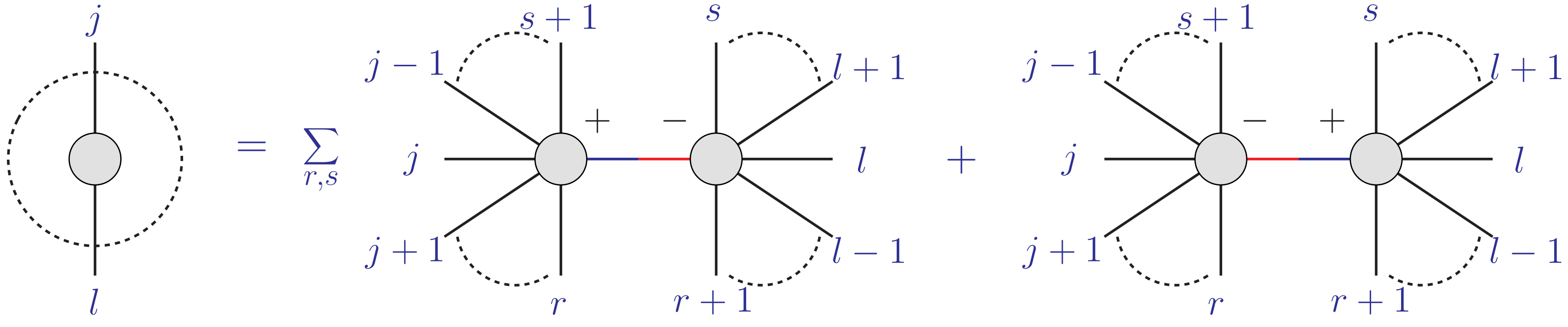
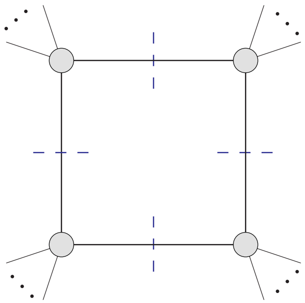
On-shell recursions

- **New** recursive on-shell approach Bern, Dixon, Dunbar, Kosower '94; Britto, Cachazo, Feng '04; Ossola, Pittau, Papadopolous '06; + [many others]
  - use complex momenta (Cauchy's theorem)
- Tree level: physical amplitude from sum over all cyclic orderings of amplitudes $n-2$ legs Britto, Cachazo, Feng, Witten '05



Generalized unitarity

- One-loop amplitudes from generalized unitarity
  - cut conditions for propagators freeze integrand of box integrals



$$\int d^4l \quad \longrightarrow \quad \delta(p_1^2)\,\delta(p_2^2)\;\delta(p_3^2)\,\delta(p_4^2)$$

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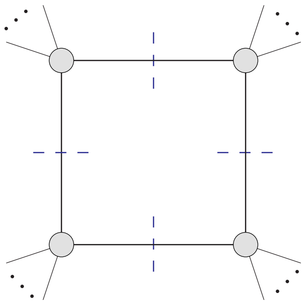
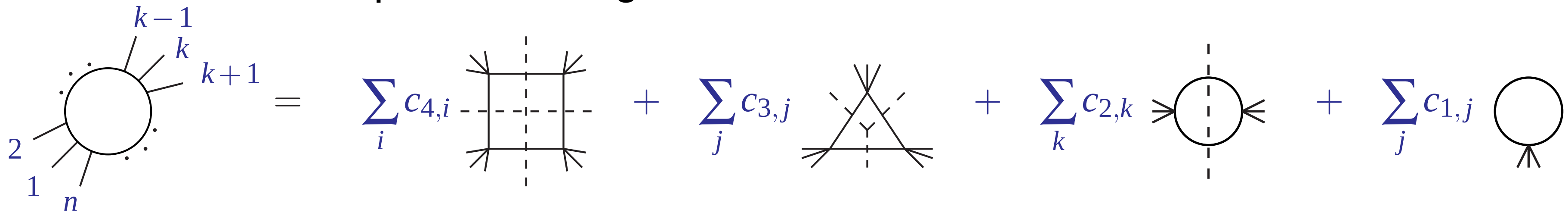


$$\int d^4 l \quad \longrightarrow \quad \delta(p_1^2)\,\delta(p_2^2)\;\delta(p_3^2)\,\delta(p_4^2)$$

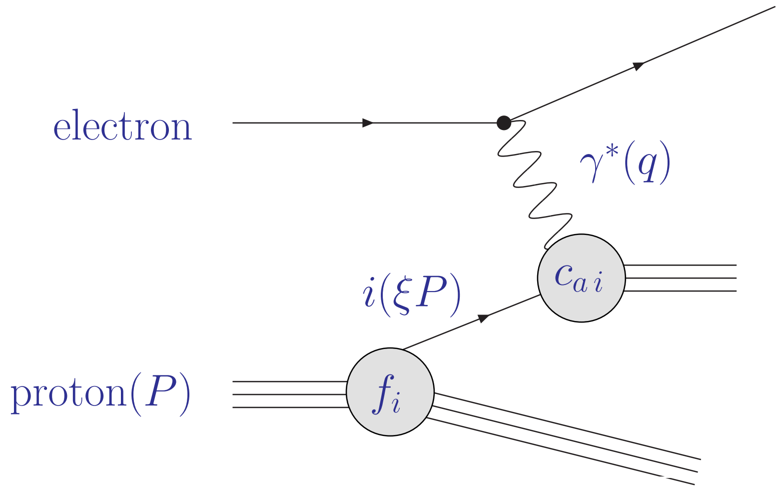
- Constructive approach to one-loop amplitudes
  - sew tree level amplitudes together (on-shell with complex momenta)
  - algebraic extraction of coefficients of 4-, 3- , 2- and 1-point one-loop scalar integrals



Deep-inelastic scattering

QCD corrections to structure functions

- Cross section $\sigma \simeq L^{\mu\nu} W_{\mu\nu}$ with leptonic tensor $L_{\mu\nu}$
  - hadronic tensor $L_{\mu\nu}$ parametrized through structure functions



- Structure function F_2 (up to terms $\mathcal{O}(1/Q^2)$)
  - coefficient functions c_a (hard scattering)

$$F_2(x, Q^2) \;=\; x \sum_i \int_x^1 \frac{d\xi}{\xi}\, c_{2,i}\left(\frac{x}{\xi}, \alpha_s(\mu^2), \frac{\mu^2}{Q^2}\right) f_i(\xi, \mu^2)$$

Mellin moments

- Cross section $\sigma(N)$ in Mellin N-space

$$
\begin{aligned}
\sigma(N) &= \int_0^1 dx\, x^N \, \frac{d\sigma(x)}{dx} \\
&= \int_0^1 dx\, x^N \int dLIPS^{(m)} \, |\mathcal{A}(\{\mathrm{in}\} \to \{\mathrm{out}\})|^2 \, \delta(x - f(p)) \\
&= \int dLIPS^{(m)} \, |\mathcal{A}(\{\mathrm{in}\} \to \{\mathrm{out}\})|^2 \, \frac{1}{(f)^{-N}}
\end{aligned}
$$

  - $f(p)$ an invariant depended on internal and external momenta

- Upshot
  - mapping to discrete set of variables (positive integer Mellin N)

Reductions in Mellin space

- Reductions with integration-by-parts in Mellin space
- N case leads to monomials $\prod_i \frac{1}{P_i(\epsilon)^{\nu_i - N}}$
  - P_i relevant propagators in the problem
  - ν_i fixed integers

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Integration-by-parts

- Integration-by-parts exploits translational invariance
  - in p-space

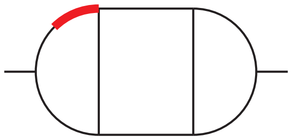
$$0 \;=\; \int d^D p \, \frac{\partial}{\partial p^\mu} f(p, \dots)$$

  - in x-space

$$0 \;=\; (x_1 - x_2)_\mu + (x_2 - x_3)_\mu + (x_3 - x_1)_\mu$$

Example from DIS

- Scalar diagram with external momenta P and Q (four-point function with underlying ladder topology)

   $= \int \prod_{n}^{3} d^D p_n \, \frac{1}{(P-p_1)^2} \, \frac{1}{p_1^2 \ldots p_8^2}$

  coefficient $\frac{(2P \cdot p_1)^N}{(p_1)^N}$

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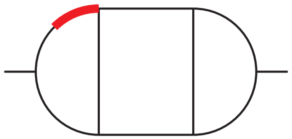


$$= \int \prod_{n}^{3} d^D p_n \, \frac{1}{(P-p_1)^2} \, \frac{1}{p_1^2 \, \ldots p_8^2}$$

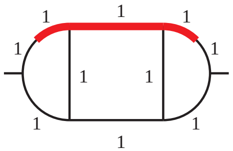
coefficient $\frac{(2P \cdot p_1)^N}{(p_1)^N}$

- N-th moment: coefficient of $(2P \cdot Q)^N$

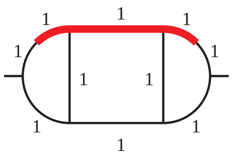
$$= \frac{(2\, P \cdot Q)^N}{(Q^2)^{N+\alpha}} \, C_N$$

Difference equations

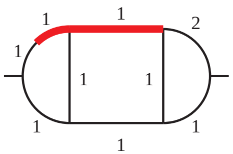
- Single-step difference equation in N
  - extremely simple example



$$= -\frac{N+3+3\varepsilon}{N+2}\,\frac{2\,p\cdot q}{q^2}$$



$$+\frac{2}{N+2}$$



Difference equations

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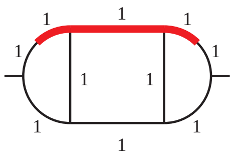
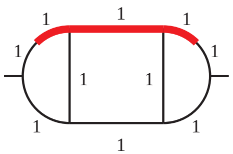
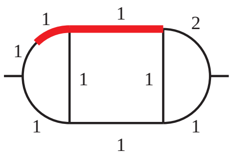


$= -\frac{N+3+3\varepsilon}{N+2}\,\frac{2\,p\cdot q}{q^2}$



$+\frac{2}{N+2}$

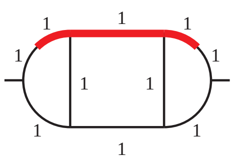


- Formal equation

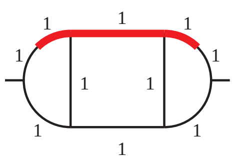
$$\mathbf{I}(\mathbf{N}) \quad = \quad -\frac{N+3+3\epsilon}{N+2}\,\mathbf{I}(\mathbf{N}-\mathbf{1}) + \frac{2}{N+2}\,\mathbf{G}(\mathbf{N})$$

Difference equations

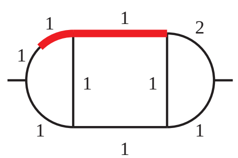
- Single-step difference equation in N
  - extremely simple example



$$= -\frac{N+3+3\varepsilon}{N+2}\,\frac{2\,p\cdot q}{q^2}$$



$$+\frac{2}{N+2}$$

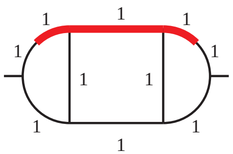
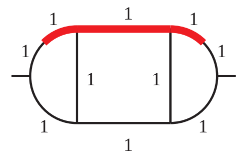
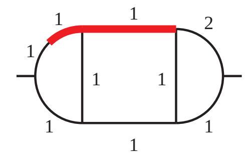


- Formal equation, formal solution

$$\mathbf{I(N)} \quad = \quad (-1)^N \frac{\prod_{j=1}^{N}(j+3+3\epsilon)}{\prod_{j=1}^{N}(j+2)}\mathbf{I(0)} + (-1)^N \sum_{\mathbf{i=1}}^{\mathbf{N}} (-1)^j \frac{\prod_{j=i+1}^{N}(j+3+3\epsilon)}{\prod_{j=i}^{N}(j+2)}\mathbf{G(i)}$$

Difference equations

- Single-step difference equation in N
  - extremely simple example

 $= -\frac{N+3+3\varepsilon}{N+2}\,\frac{2\,p\cdot q}{q^2}$  $+\ \frac{2}{N+2}$ 

- Formal equation, formal solution, input to solution

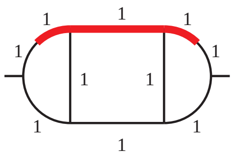
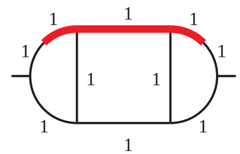
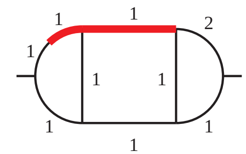
$$\mathbf{I(N)} = (-1)^N \frac{\prod_{j=1}^{N}(j+3+3\epsilon)}{\prod_{j=1}^{N}(j+2)}\mathbf{I(0)} + (-1)^N \sum_{\mathbf{i=1}}^{\mathbf{N}} (-1)^j \frac{\prod_{j=i+1}^{N}(j+3+3\epsilon)}{\prod_{j=i}^{N}(j+2)}\mathbf{G(i)}$$

$$\mathbf{I(0)} = -\frac{2}{3}\frac{1}{\epsilon^2} + \frac{23}{3}\frac{1}{\epsilon} - 42$$

$$\mathbf{G(i)} = \frac{(-1)^i}{\epsilon^2}\frac{2}{3}\left(\frac{S_1(i+2)}{i+2} - \frac{S_{1,2}(i)}{2} - \frac{S_2(i+1)}{2(i+1)} - S_2(i) - \frac{1}{(i+1)^2} - \frac{1}{(i+2)^2}\right) + \ldots$$

Difference equations

- Single-step difference equation in N
  - extremely simple example

 $= -\dfrac{N+3+3\varepsilon}{N+2}\,\dfrac{2\,p\cdot q}{q^2}$  $+\,\dfrac{2}{N+2}$ 

- Formal equation, formal solution, input to solution

$$\mathbf{I(N)} = (-1)^N \frac{\prod_{j=1}^{N}(j+3+3\epsilon)}{\prod_{j=1}^{N}(j+2)}\mathbf{I(0)} + (-1)^N \sum_{\mathbf{i=1}}^{\mathbf{N}} (-1)^j \frac{\prod_{j=i+1}^{N}(j+3+3\epsilon)}{\prod_{j=i}^{N}(j+2)}\mathbf{G(i)}$$

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$$\mathbf{G(i)} = \frac{(-1)^i}{\epsilon^2}\frac{2}{3}\left(\frac{S_1(i+2)}{i+2} - \frac{S_{1,2}(i)}{2} - \frac{S_2(i+1)}{2(i+1)} - S_2(i) - \frac{1}{(i+1)^2} - \frac{1}{(i+2)^2}\right) + \ldots$$

- **Upshot**
  - automatic build-up of **nested sums**
  - efficient implementation in FORM

I(N) =

```
sign(N)*ep^-2 * ( 4/3*S(R(1),1 + N)*den(1 + N) + 8/3*S(R(1),1 + N)*den(
  1 + N)^2 + 4/3*S(R(1),2 + N)*den(2 + N) + 4/3*S(R(1),2 + N)*den(2 + N)^2 + 4/3*
  S(R(1),N) + 2/3*S(R(1,2),N) + 2/3*S(R(2),1 + N)*den(1 + N) + 2/3*S(R(2),2 + N)*
  den(2 + N) - 2*S(R(2),N) - 4/3*S(R(2),N)*N + 4*S(R(2,1),N) + 4/3*S(R(2,1),N)*N
   - 6*S(R(3),N) - 2*S(R(3),N)*N - 8/3*den(1 + N)^2 - 4*den(1 + N)^3 - 4/3*den(2
   + N)^2 - 2*den(2 + N)^3 )

+ sign(N)*ep^-1 * (  - 16*S(R(1),1 + N)*den(1 + N) - 88/3*S(R(1),1 + N)*
  den(1 + N)^2 - 20/3*S(R(1),1 + N)*den(1 + N)^3 - 16*S(R(1),2 + N)*den(2 + N) -
  44/3*S(R(1),2 + N)*den(2 + N)^2 - 10/3*S(R(1),2 + N)*den(2 + N)^3 - 20*S(R(1),N)
   + 8/3*S(R(1,1),1 + N)*den(1 + N) + 8/3*S(R(1,1),1 + N)*den(1 + N)^2 + 8/3*S(R(1
  ,1),2 + N)*den(2 + N) + 8/3*S(R(1,1),N) + 10/3*S(R(1,1,2),N) + 10/3*S(R(1,2),1
   + N)*den(1 + N) + 10/3*S(R(1,2),2 + N)*den(2 + N) - 16*S(R(1,2),N) - 4*S(R(1,2)
  ,N)*N + 14*S(R(1,2,1),N) + 4*S(R(1,2,1),N)*N - 24*S(R(1,3),N) - 6*S(R(1,3),N)*N
   - 58/3*S(R(2),1 + N)*den(1 + N) - 40/3*S(R(2),1 + N)*den(1 + N)^2 - 46/3*S(R(2)
  ,2 + N)*den(2 + N) - 6*S(R(2),2 + N)*den(2 + N)^2 + 56/3*S(R(2),N) + 20*S(R(2),N
  )*N + 10*S(R(2,1),1 + N)*den(1 + N) + 6*S(R(2,1),2 + N)*den(2 + N) - 134/3*S(R(2
  ,1),N) - 56/3*S(R(2,1),N)*N + 16/3*S(R(2,1,1),N) + 8/3*S(R(2,1,1),N)*N - 62/3*S(
  R(2,2),N) - 22/3*S(R(2,2),N)*N - 18*S(R(3),1 + N)*den(1 + N) - 12*S(R(3),2 + N)*
  den(2 + N) + 76*S(R(3),N) + 100/3*S(R(3),N)*N - 10*S(R(3,1),N) - 10/3*S(R(3,1),N
  )*N + 36*S(R(4),N) + 12*S(R(4),N)*N + 32*den(1 + N)^2 + 164/3*den(1 + N)^3 + 24*
  den(1 + N)^4 + 16*den(2 + N)^2 + 82/3*den(2 + N)^3 + 12*den(2 + N)^4 )

+ sign(N) * ( 100*S(R(1),1 + N)*den(1 + N) + 168*S(R(1),1 + N)*den(1 + N)
  ^2 + 268/3*S(R(1),1 + N)*den(1 + N)^3 - 16/3*S(R(1),1 + N)*den(1 + N)^4 + 100*S(
  R(1),2 + N)*den(2 + N) + 84*S(R(1),2 + N)*den(2 + N)^2 + 134/3*S(R(1),2 + N)*
  den(2 + N)^3 - 8/3*S(R(1),2 + N)*den(2 + N)^4 + 160*S(R(1),N) - 32*S(R(1,1),1 +
  N)*den(1 + N) - 80/3*S(R(1,1),1 + N)*den(1 + N)^2 - 20/3*S(R(1,1),1 + N)*den(1
   + N)^3 - 32*S(R(1,1),2 + N)*den(2 + N) - 4/3*S(R(1,1),2 + N)*den(2 + N)^2 - 10/
  3*S(R(1,1),2 + N)*den(2 + N)^3 - 40*S(R(1,1),N) + 4/3*S(R(1,1,1),1 + N)*den(1 +
  N) - 40/3*S(R(1,1,1),1 + N)*den(1 + N)^2 + 4/3*S(R(1,1,1),2 + N)*den(2 + N) - 44/
  3*S(R(1,1,1),2 + N)*den(2 + N)^2 + 4/3*S(R(1,1,1),N) + 38/3*S(R(1,1,1,2),N) + 38/
  3*S(R(1,1,2),1 + N)*den(1 + N) + 38/3*S(R(1,1,2),2 + N)*den(2 + N) - 68*S(R(1,1,
  2),N) - 12*S(R(1,1,2),N)*N + 42*S(R(1,1,2,1),N) + 12*S(R(1,1,2,1),N)*N - 76*S(R(
  1,1,3),N) - 18*S(R(1,1,3),N)*N - 170/3*S(R(1,2),1 + N)*den(1 + N) + 40/3*S(R(1,2
  ),1 + N)*den(1 + N)^2 - 134/3*S(R(1,2),2 + N)*den(2 + N) + 14*S(R(1,2),2 + N)*
  den(2 + N)^2 + 430/3*S(R(1,2),N) + 60*S(R(1,2),N)*N + 30*S(R(1,2,1),1 + N)*den(1
   + N) + 18*S(R(1,2,1),2 + N)*den(2 + N) - 452/3*S(R(1,2,1),N) - 56*S(R(1,2,1),N)
  *N + 74/3*S(R(1,2,1,1),N) + 8*S(R(1,2,1,1),N)*N - 248/3*S(R(1,2,2),N) - 22*S(R(1
  ,2,2),N)*N - 58*S(R(1,3),1 + N)*den(1 + N) - 40*S(R(1,3),2 + N)*den(2 + N) + 886/
  3*S(R(1,3),N) + 100*S(R(1,3),N)*N - 116/3*S(R(1,3,1),N) - 10*S(R(1,3,1),N)*N +
  410/3*S(R(1,4),N) + 36*S(R(1,4),N)*N + 186*S(R(2),1 + N)*den(1 + N) + 448/3*S(R(
  2),1 + N)*den(1 + N)^2 + 160/3*S(R(2),1 + N)*den(1 + N)^3 + 138*S(R(2),2 + N)*
  den(2 + N) + 206/3*S(R(2),2 + N)*den(2 + N)^2 + 80/3*S(R(2),2 + N)*den(2 + N)^3
   - 70*S(R(2),N) - 160*S(R(2),N)*N - 338/3*S(R(2,1),1 + N)*den(1 + N) - 64/3*S(R(
  2,1),1 + N)*den(1 + N)^2 - 206/3*S(R(2,1),2 + N)*den(2 + N) - 10/3*S(R(2,1),2 +
  N)*den(2 + N)^2 + 760/3*S(R(2,1),N) + 140*S(R(2,1),N)*N + 50/3*S(R(2,1,1),1 + N)
  *den(1 + N) + 26/3*S(R(2,1,1),2 + N)*den(2 + N) - 170/3*S(R(2,1,1),N) - 100/3*S(
  R(2,1,1),N)*N - 12*S(R(2,1,1,1),N) + 4/3*S(R(2,1,1,1),N)*N + 38/3*S(R(2,1,2),N)
   - 2/3*S(R(2,1,2),N)*N - 182/3*S(R(2,2),1 + N)*den(1 + N) - 116/3*S(R(2,2),2 + N
  )*den(2 + N) + 676/3*S(R(2,2),N) + 308/3*S(R(2,2),N)*N - 118/3*S(R(2,2,1),N) -
  18*S(R(2,2,1),N)*N + 296/3*S(R(2,3),N) + 36*S(R(2,3),N)*N + 694/3*S(R(3),1 + N)*
  den(1 + N) + 188/3*S(R(3),1 + N)*den(1 + N)^2 + 448/3*S(R(3),2 + N)*den(2 + N)
   + 80/3*S(R(3),2 + N)*den(2 + N)^2 - 1454/3*S(R(3),N) - 290*S(R(3),N)*N - 86/3*
  S(R(3,1),1 + N)*den(1 + N) - 56/3*S(R(3,1),2 + N)*den(2 + N) + 440/3*S(R(3,1),N)
   + 164/3*S(R(3,1),N)*N - 10*S(R(3,1,1),N) - 10/3*S(R(3,1,1),N)*N + 80*S(R(3,2),N
  ) + 80/3*S(R(3,2),N)*N + 302/3*S(R(4),1 + N)*den(1 + N) + 194/3*S(R(4),2 + N)*
  den(2 + N) - 434*S(R(4),N) - 556/3*S(R(4),N)*N - 8*S(R(4,1),N) - 8/3*S(R(4,1),N)
  *N - 150*S(R(5),N) - 50*S(R(5),N)*N - 200*den(1 + N)^2 - 380*den(1 + N)^3 - 896/
  3*den(1 + N)^4 - 100*den(1 + N)^5 - 100*den(2 + N)^2 - 190*den(2 + N)^3 - 448/3*
  den(2 + N)^4 - 50*den(2 + N)^5 );
```

I(N) =

```
sign(N)*ep^-2 * ( 4/3*S(R(1),1 + N)*den(1 + N) + 8/3*S(R(1),1 + N)*den(
  1 + N)^2 + 4/3*S(R(1),2 + N)*den(2 + N) + 4/3*S(R(1),2 + N)*den(2 + N)^2 + 4/3*
  S(R(1),N) + 2/3*S(R(1,2),N) + 2/3*S(R(2),1 + N)*den(1 + N) + 2/3*S(R(2),2 + N)*
  den(2 + N) - 2*S(R(2),N) - 4/3*S(R(2),N)*N + 4*S(R(2,1),N) + 4/3*S(R(2,1),N)*N
   - 6*S(R(3),N) - 2*S(R(3),N)*N - 8/3*den(1 + N)^2 - 4*den(1 + N)^3 - 4/3*den(2
   + N)^2 - 2*den(2 + N)^3 )

+ sign(N)*ep^-1 * (  - 16*S(R(1),1 + N)*den(1 + N) - 88/3*S(R(1),1 + N)*
  den(1 + N)^2 - 20/3*S(R(1),1 + N)*den(1 + N)^3 - 16*S(R(1),2 + N)*den(2 + N) -
  44/3*S(R(1),2 + N)*den(2 + N)^2 - 10/3*S(R(1),2 + N)*den(2 + N)^3 - 20*S(R(1),N)
   + 8/3*S(R(1,1),1 + N)*den(1 + N) + 8/3*S(R(1,1),1 + N)*den(1 + N)^2 + 8/3*S(R(1
  ,1),2 + N)*den(2 + N) + 8/3*S(R(1,1),N) + 10/3*S(R(1,1,2),N) + 10/3*S(R(1,2),1
   + N)*den(1 + N) + 10/3*S(R(1,2),2 + N)*den(2 + N) - 16*S(R(1,2),N) - 4*S(R(1,2)
  ,N)*N + 14*S(R(1,2,1),N) + 4*S(R(1,2,1),N)*N - 24*S(R(1,3),N) - 6*S(R(1,3),N)*N
   - 58/3*S(R(2),1 + N)*den(1 + N) - 40/3*S(R(2),1 + N)*den(1 + N)^2 - 46/3*S(R(2)
  ,2 + N)*den(2 + N) - 6*S(R(2),2 + N)*den(2 + N)^2 + 56/3*S(R(2),N) + 20*S(R(2),N
  )*N + 10*S(R(2,1),1 + N)*den(1 + N) + 6*S(R(2,1),2 + N)*den(2 + N) - 134/3*S(R(2
  ,1),N) - 56/3*S(R(2,1),N)*N + 16/3*S(R(2,1,1),N) + 8/3*S(R(2,1,1),N)*N - 62/3*S(
  R(2,2),N) - 22/3*S(R(2,2),N)*N - 18*S(R(3),1 + N)*den(1 + N) - 12*S(R(3),2 + N)*
  den(2 + N) + 76*S(R(3),N) + 100/3*S(R(3),N)*N - 10*S(R(3,1),N) - 10/3*S(R(3,1),N
  )*N + 36*S(R(4),N) + 12*S(R(4),N)*N + 32*den(1 + N)^2 + 164/3*den(1 + N)^3 + 24*
  den(1 + N)^4 + 16*den(2 + N)^2 + 82/3*den(2 + N)^3 + 12*den(2 + N)^4 )

+ sign(N) * ( 100*S(R(1),1 + N)*den(1 + N) + 168*S(R(1),1 + N)*den(1 + N)
  ^2 + 268/3*S(R(1),1 + N)*den(1 + N)^3 - 16/3*S(R(1),1 + N)*den(1 + N)^4 + 100*S(
  R(1),2 + N)*den(2 + N) + 84*S(R(1),2 + N)*den(2 + N)^2 + 134/3*S(R(1),2 + N)*
  den(2 + N)^3 - 8/3*S(R(1),2 + N)*den(2 + N)^4 + 160*S(R(1),N) - 32*S(R(1,1),1 +
  N)*den(1 + N) - 80/3*S(R(1,1),1 + N)*den(1 + N)^2 - 20/3*S(R(1,1),1 + N)*den(1
   + N)^3 - 32*S(R(1,1),2 + N)*den(2 + N) - 4/3*S(R(1,1),2 + N)*den(2 + N)^2 - 10/
  3*S(R(1,1),2 + N)*den(2 + N)^3 - 40*S(R(1,1),N) + 4/3*S(R(1,1,1),1 + N)*den(1 +
  N) - 40/3*S(R(1,1,1),1 + N)*den(1 + N)^2 + 4/3*S(R(1,1,1),2 + N)*den(2 + N) - 44/
  3*S(R(1,1,1),2 + N)*den(2 + N)^2 + 4/3*S(R(1,1,1),N) + 38/3*S(R(1,1,1,2),N) + 38/
  3*S(R(1,1,2),1 + N)*den(1 + N) + 38/3*S(R(1,1,2),2 + N)*den(2 + N) - 68*S(R(1,1,
  2),N) - 12*S(R(1,1,2),N)*N + 42*S(R(1,1,2,1),N) + 12*S(R(1,1,2,1),N)*N - 76*S(R(
  1,1,3),N) - 18*S(R(1,1,3),N)*N - 170/3*S(R(1,2),1 + N)*den(1 + N) + 40/3*S(R(1,2
  ),1 + N)*den(1 + N)^2 - 134/3*S(R(1,2),2 + N)*den(2 + N) + 14*S(R(1,2),2 + N)*
  den(2 + N)^2 + 430/3*S(R(1,2),N) + 60*S(R(1,2),N)*N + 30*S(R(1,2,1),1 + N)*den(1
   + N) + 18*S(R(1,2,1),2 + N)*den(2 + N) - 452/3*S(R(1,2,1),N) - 56*S(R(1,2,1),N)
  *N + 74/3*S(R(1,2,1,1),N) + 8*S(R(1,2,1,1),N)*N - 248/3*S(R(1,2,2),N) - 22*S(R(1
  ,2,2),N)*N - 58*S(R(1,3),1 + N)*den(1 + N) - 40*S(R(1,3),2 + N)*den(2 + N) + 886/
3*S(R(1,3),N) + 100*S(R(1,3),N)*N - 116/3*S(R(1,3,1),N) - 10*S(R(1,3,1),N)*N +
410/3*S(R(1,4),N) + 36*S(R(1,4),N)*N + 186*S(R(2),1 + N)*den(1 + N) + 448/3*S(R(
2),1 + N)*den(1 + N)^2 + 160/3*S(R(2),1 + N)*den(1 + N)^3 + 138*S(R(2),2 + N)*
den(2 + N) + 206/3*S(R(2),2 + N)*den(2 + N)^2 + 80/3*S(R(2),2 + N)*den(2 + N)^3
 - 70*S(R(2),N) - 160*S(R(2),N)*N - 338/3*S(R(2,1),1 + N)*den(1 + N) - 64/3*S(R(
2,1),1 + N)*den(1 + N)^2 - 206/3*S(R(2,1),2 + N)*den(2 + N) - 10/3*S(R(2,1),2 +
N)*den(2 + N)^2 + 760/3*S(R(2,1),N) + 140*S(R(2,1),N)*N + 50/3*S(R(2,1,1),1 + N)
*den(1 + N) + 26/3*S(R(2,1,1),2 + N)*den(2 + N) - 170/3*S(R(2,1,1),N) - 100/3*S(
R(2,1,1),N)*N - 12*S(R(2,1,1,1),N) + 4/3*S(R(2,1,1,1),N)*N + 38/3*S(R(2,1,2),N)
 - 2/3*S(R(2,1,2),N)*N - 182/3*S(R(2,2),1 + N)*den(1 + N) - 116/3*S(R(2,2),2 + N
)*den(2 + N) + 676/3*S(R(2,2),N) + 308/3*S(R(2,2),N)*N - 118/3*S(R(2,2,1),N) -
18*S(R(2,2,1),N)*N + 296/3*S(R(2,3),N) + 36*S(R(2,3),N)*N + 694/3*S(R(3),1 + N)*
den(1 + N) + 188/3*S(R(3),1 + N)*den(1 + N)^2 + 448/3*S(R(3),2 + N)*den(2 + N)
 + 80/3*S(R(3),2 + N)*den(2 + N)^2 - 1454/3*S(R(3),N) - 290*S(R(3),N)*N - 86/3*
S(R(3,1),1 + N)*den(1 + N) - 56/3*S(R(3,1),2 + N)*den(2 + N) + 440/3*S(R(3,1),N)
 + 164/3*S(R(3,1),N)*N - 10*S(R(3,1,1),N) - 10/3*S(R(3,1,1),N)*N + 80*S(R(3,2),N
) + 80/3*S(R(3,2),N)*N + 302/3*S(R(4),1 + N)*den(1 + N) + 194/3*S(R(4),2 + N)*
den(2 + N) - 434*S(R(4),N) - 556/3*S(R(4),N)*N - 8*S(R(4,1),N) - 8/3*S(R(4,1),N)
*N - 150*S(R(5),N) - 50*S(R(5),N)*N - 200*den(1 + N)^2 - 380*den(1 + N)^3 - 896/
3*den(1 + N)^4 - 100*den(1 + N)^5 - 100*den(2 + N)^2 - 190*den(2 + N)^3 - 448/3*
den(2 + N)^4 - 50*den(2 + N)^5 );
```

- Result for $\mathbf{I}(\mathbf{N})$ in the G-scheme

CAPP 2009 – the program

- Computer algebra systems
  - Introduction to FORM J. Vermaseren
  - New features of Mathematica R. Mertig
- Algorithms, software tools and development
  - Arbitrary precision numerics and the PSLQ algorithm D. Bailey
  - Mathematica and Symbolic Computing T. Hahn
  - Efficient computing in particle physics P. Uwer
- Scattering amplitudes and NLO calculations
  - Helicity amplitudes and HELAC/PHEGAS M. Worek
  - Feynman integrals and the OPP reduction C. Papadopoulos
  - Real parton emission and automated dipole subtraction K. Hasegawa
- Higher loops
  - Integrals, Mellin-Barnes representations, Sums J. Gluza, T. Riemann
  - Loop integrals, integration-by-parts and MATAD M. Steinhauser

Sponsors

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- Wolfram Research