Feynman Integrals
Mellin-Barnes representations
Sums

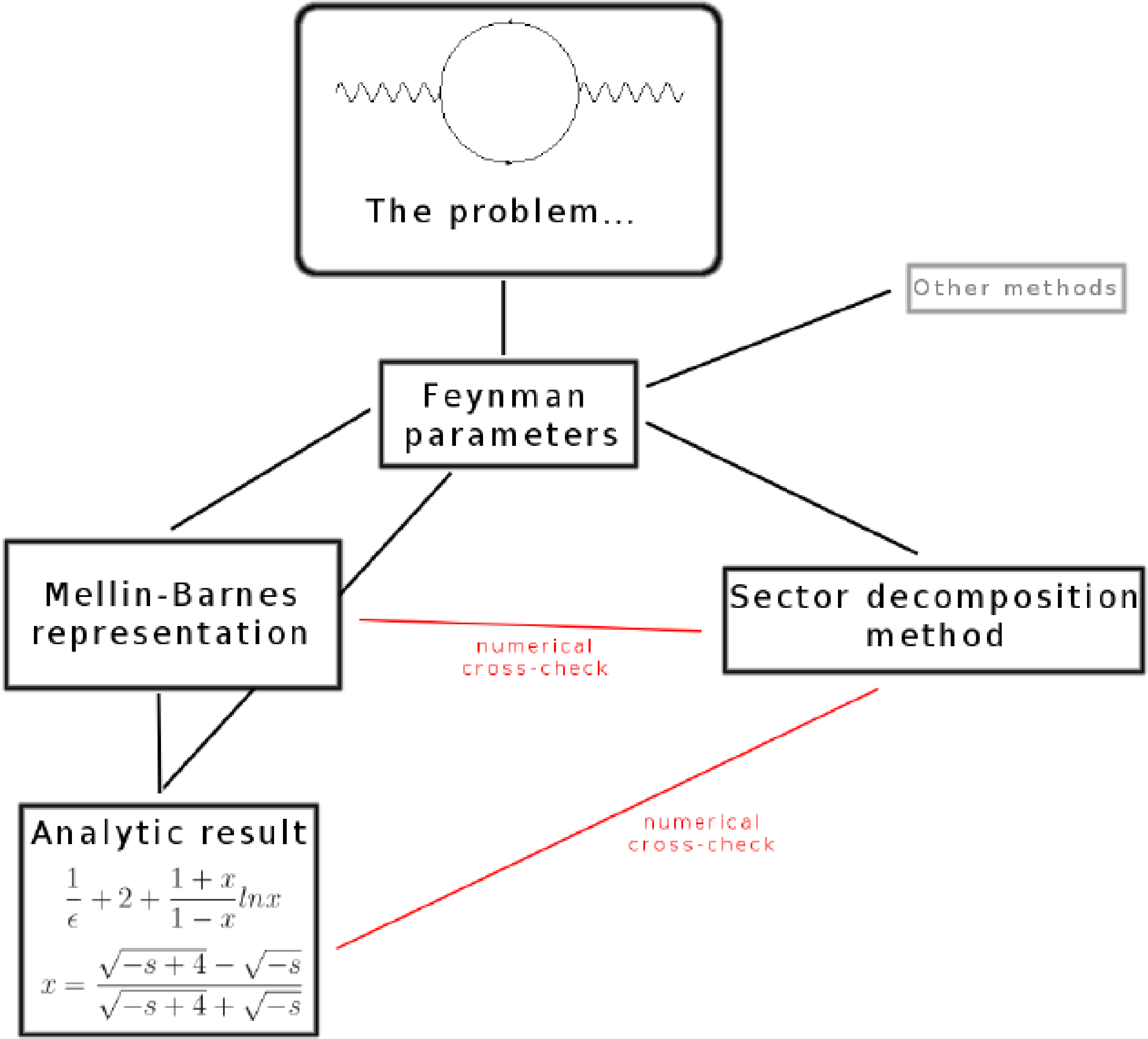
Computer Algebra and Particle Physics

The DESY CAPP School

30 March – 3 April, 2009, Zeuthen, Germany

Tord Riemann, DESY, Zeuthen

Exercises part: Janusz Gluza, Katowice



The problem...
Other methods
Feynman parameters
Mellin-Barnes representation
Sector decomposition method
numerical cross-check
Analytic result
$\frac{1}{\epsilon}+2+\frac{1+x}{1-x}\ln x$
$x=\frac{\sqrt{-s+4}-\sqrt{-s}}{\sqrt{-s+4}+\sqrt{-s}}$
numerical cross-check

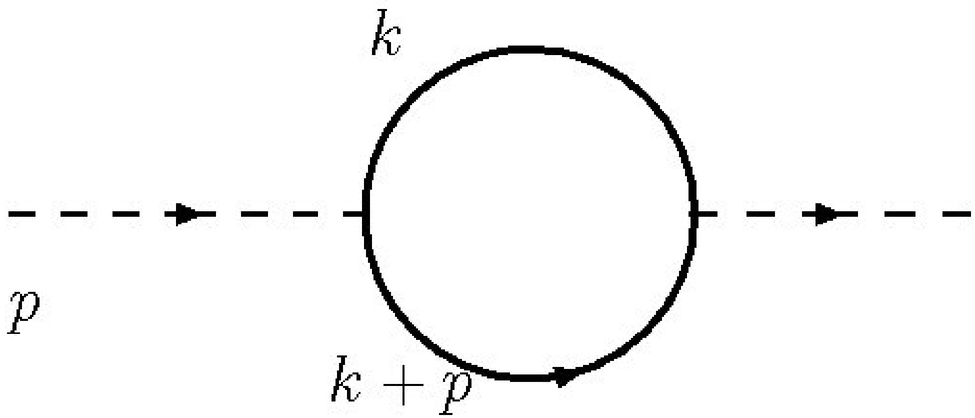
Layout: ***http://www.us.edu.pl/~gluza/capp2009/***

- *MB tools – link to the Mellin-Barnes software web page* (SE2l2m.nb,B5l2m2.nb,B5nf_0external.nb, PSLQ.nb)
- *IBPs (Tadpole.nb, SE_F.nb)*

$$G(1) = (-1)^{N_\nu} \frac{\Gamma\left(N_\nu - \frac{D}{2}L\right)}{\Gamma(\nu_1)\ldots\Gamma(\nu_N)} \int_0^1 \prod_{j=1}^{N} dx_j \; x_j^{\nu_j - 1} \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{U(x)^{N_\nu - D(L+1)/2}}{F(x)^{N_\nu - DL/2}}$$

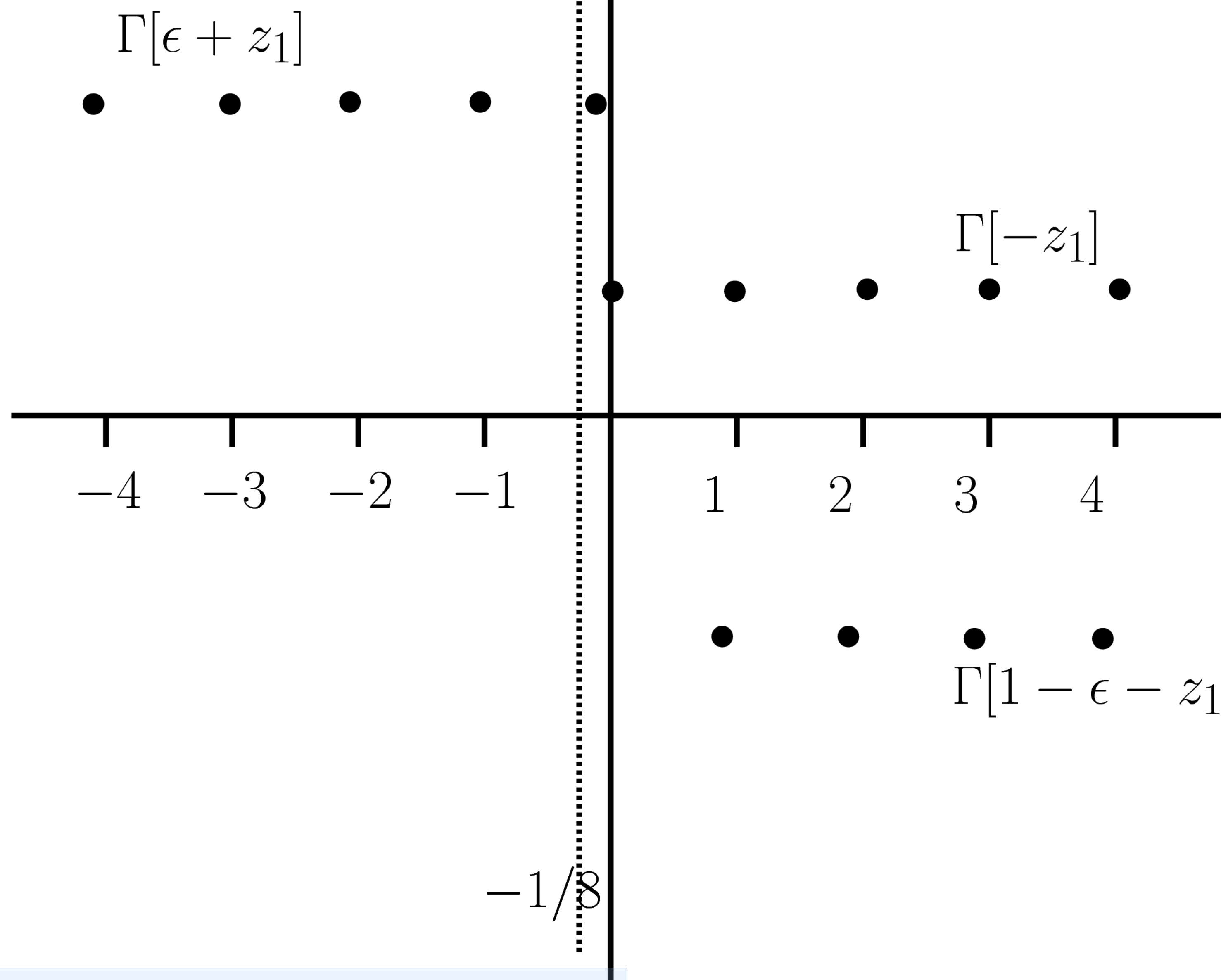
On red: in a tar file

Self-energy: SE2l2m.nb



Here we learn how:

- construct MB representation using AMBRE.m (http://www.us.edu.pl/~gluza/ambre)
- expand in ϵ using MB.m
- get approximate numerical results by summing up finite number of singularities in Gamma's both for large and small four-momenta p



Left-half: gives series
sutaible for small s

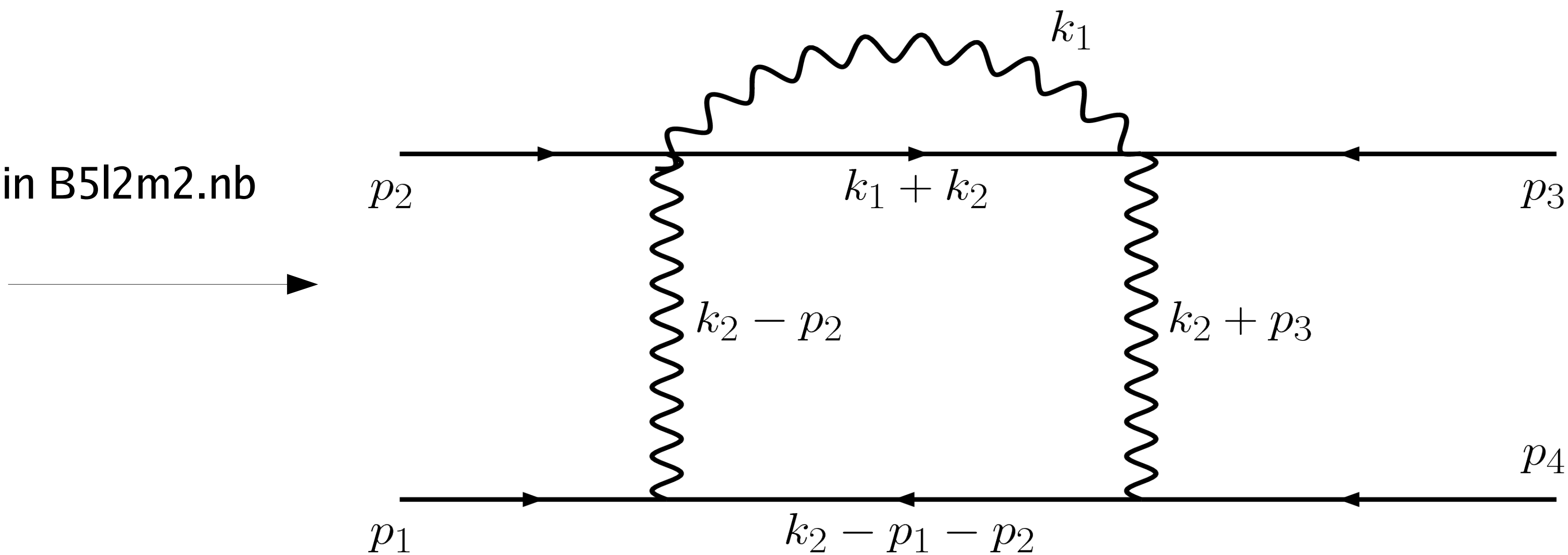
$$\int_{-i\infty}^{+i\infty} dz_1 \, (-s)^{-\epsilon} \left(-\frac{m^2}{s}\right)^{z_1} \frac{\Gamma[1-\epsilon-z_1]^2 \Gamma[-z_1] \Gamma[\epsilon+z_1]}{\Gamma[2-2\epsilon-z_1]},$$

$$Re[z_1] = -1/8$$

Right-half: gives series
suitable for large s

Two-loop box: B5l2m.nb

Notation like in B5l2m2.nb

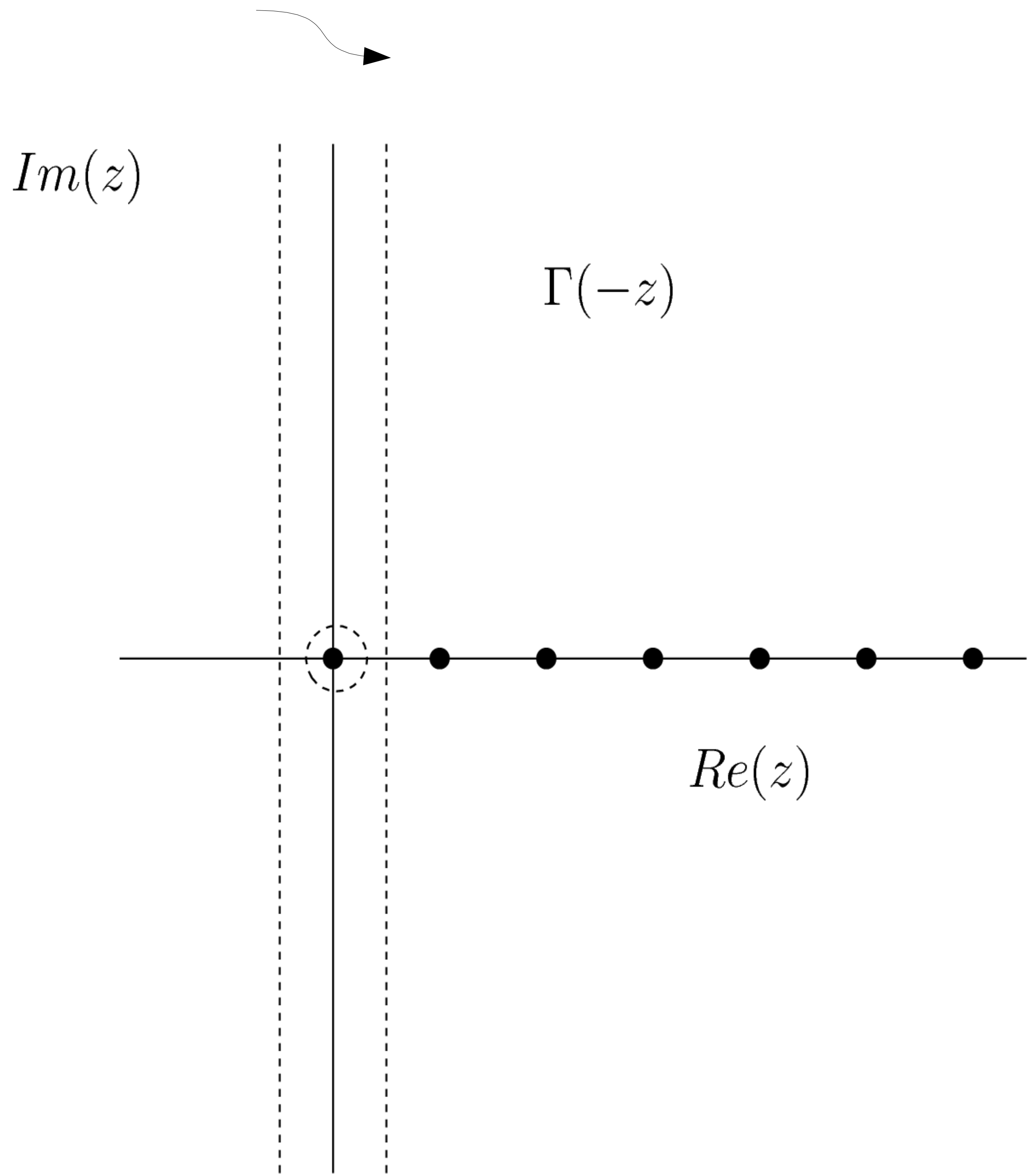


Here we learn how:

- construct MB representation using AMBRE.m beyond one-loop
- solve analytically MB integral by matching expanded MB integral in conformal variable y[t] to the general base which is assumed to be in terms of multiple Riemann's zeta functions, logs and polylogs, using mathematica
- What it means “Make analytic continuation”

```
(* shifting contours *)

:=
sim = Gamma[-z]
]=
Gamma[-z]

:=
Sum[-Residue[Gamma[-z], {z, n}], {n, 0, 100}] // N
]=
0.367879

:=
n1 = NIntegrate[
   1/(2 Pi) sim /. z → -1/20 + I y, {y, -10, 10}]
]=
0.367879 + 0. ⅈ

:=
n2 = NIntegrate[
   1/(2 Pi) sim /. z → 1/20 + I y, {y, -10, 10}]
]=
-0.632121 + 0. ⅈ

:=
n2 - n1
]=
-1. + 0. ⅈ

:=
Residue[sim, {z, 0}]
]=
-1
```

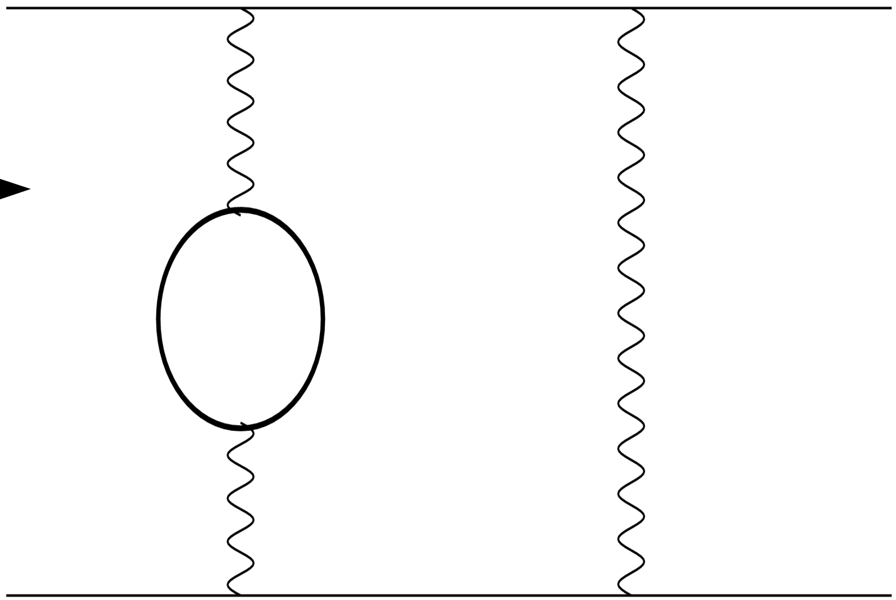
n2 = n1 + Residue[sim,{z,0}]



$Im(z)$
$\Gamma(-z)$
$Re(z)$

Two loop self-energy insertion QED diagram B5nf_0external.nb

All lines massless except 1-loop self-energy insertion



Question: what kind of singularities can be here?
Is 1/eps^3 singularity present?

Here we learn how:

- construct MB representation using AMBRE.m yet with different kinematics
- expand in ϵ using MB.m (analytic continuation in additional parameter is required)

PSLQ

- Sometimes in expansions we get scale independent integrals, which are after all just numbers
- See PSLQ.nb,
  http : // mathworld.wolfram.com/PSLQAlgorithm.html

Here we learn how:

Transform pure number into combination of irrationals using PSLQ algorithm (due to Bailey)

Integration by parts

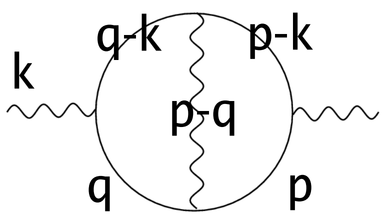
Here we learn how:

- Apply it manually to a simple case
- Use FIRE programm to get it automatically

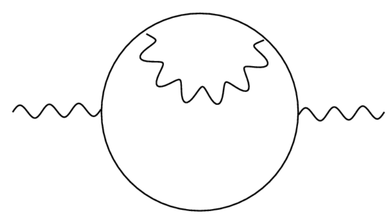
http://www-ttp.particle.uni-karlsruhe.de/~asmirnov/

Integration by parts

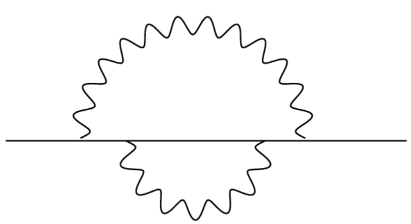
- Directory FIRE: Tadpole.nb (see T. Riemann lecture)
- For SE (an example from the original paper by Chetyrkin,Tkachov, NPB,1982)



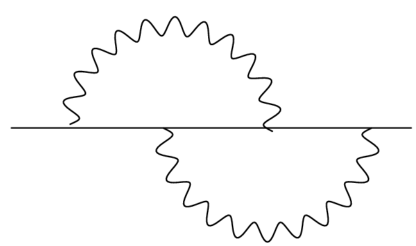
SE1



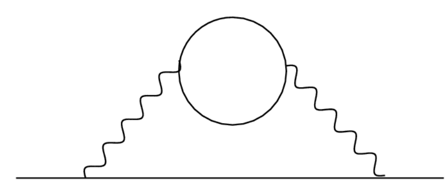
SE2



SE3



SE4

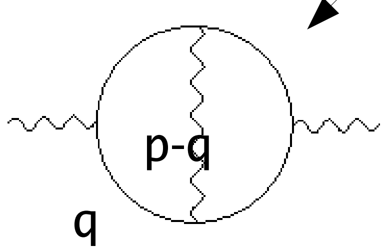


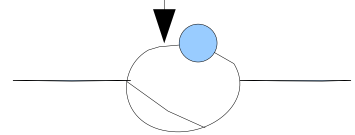
SE5

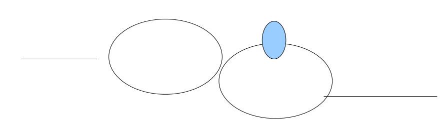
$$I = p^2(p-k)^2 q^2 (q-k)^2 (p-q)^2$$

$$\begin{aligned}
0 &= \frac{\partial}{\partial p^\mu} \frac{(p-q)^\mu}{p^2(p-k)^2q^2(q-k)^2(p-q)^2} \\
&= I[d - 2\frac{p(p-q)}{p^2} - 2\frac{(p-k)(p-q)}{(p-k)^2} - 2] \\
&= I[d - 4 + 2\frac{pq}{p^2} - 2\frac{(p-k)(p-q)}{(p-k)^2}] \\
&= I[d - 4 + 2\frac{\frac{1}{2}(q^2 + p^2 - (p-q)^2)}{p^2} \\
&- 2\frac{\frac{1}{2}((p-q)^2 + (p-q)^2 - (q-k)^2)}{(p-k)^2} - 2] \\
&= I[d - 4 + \frac{q^2}{p^2} + \frac{(q-k)^2}{(p-k)^2} - \frac{(p-q)^2}{p^2} - \frac{(p-q)^2}{(p-k)^2}]
\end{aligned}$$

$$I(d-4) = 2 \cdot I\left(\frac{q^2}{p^2} - \frac{(p-q)^2}{p^2}\right)$$







Prove it numerically!

See SE_F.nb