NLO corrections with the OPP method

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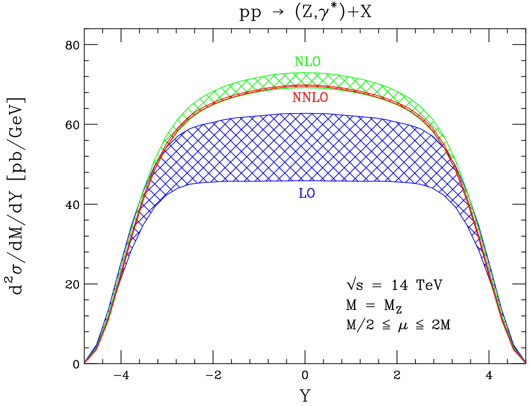
Outline

Introduction: LHC needs NLO

- The experimental programs of LHC require high precision predictions for multi-particle processes (also ILC of course)



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- The current need of precision goes beyond tree order. At LHC, most analyses require at least next-to-leading order calculations (NLO)
- As a result, a big effort has been devoted by several groups to the problem of an efficient computation of one-loop corrections for multi-particle processes!

NLO Wishlist Les Houches

[from G. Heinrich's Summary talk]

Wishlist Les Houches 2007
1. $pp \to V\,V + \text{jet}$
2. $pp \to t\bar{t}\,b\bar{b}$
3. $pp \to t\bar{t} + 2$ jets
4. $pp \to W\,W\,W$
5. $pp \to V\,V\,b\bar{b}$
6. $pp \to V\,V + 2$ jets
7. $pp \to V + 3$ jets
8. $pp \to t\bar{t}\,b\bar{b}$
9. $pp \to 4$ jets

Processes for which a NLO calculation is both desired and feasible

Will we "finish" in time for LHC?

What has been done? (2005-2007)

Some recent results $\rightarrow$ Cross Sections available

- $pp \rightarrow Z\,Z\,Z$ $pp \rightarrow t\bar{t}Z$ [Lazopoulos, Melnikov, Petriello]
- $pp \rightarrow H + 2$ jets [Campbell, et al., J. R. Andersen, et al.]
- $pp \rightarrow VV + 2$ jets via VBF [Bozzi, Jäger, Oleari, Zeppenfeld]
- $pp \rightarrow VV + 1$ jet [S. Dittmaier, S. Kallweit and P. Uwer]
- $pp \rightarrow t\bar{t} + 1$ jet [S. Dittmaier, P. Uwer and S. Weinzierl]
- $pp \rightarrow V\,V\,V$ [BOPP and Campanario et al.]

Mostly $2 \rightarrow 3$, very few $2 \rightarrow 4$ complete calculations.

- $e^+\ e^- \rightarrow 4$ fermions [Denner, Dittmaier, Roth]
- $e^+\ e^- \rightarrow H\,H\,\nu\,\bar{\nu}$ [GRACE group (Boudjema et al.)]
- $q\ \bar{q} \rightarrow t\bar{t}b\bar{b}$ [Bredenstein et al.]

This is NOT a complete list

(A lot of work has been done at NLO $\rightarrow$ calculations & new methods)

What has been done? 2009

- R. K. Ellis, K. Melnikov and G. Zanderighi, "Generalized unitarity at work: first NLO QCD results for hadronic W^+ 3jet production," arXiv:0901.4101 [hep-ph]

WHAT HAS BEEN DONE? 2009

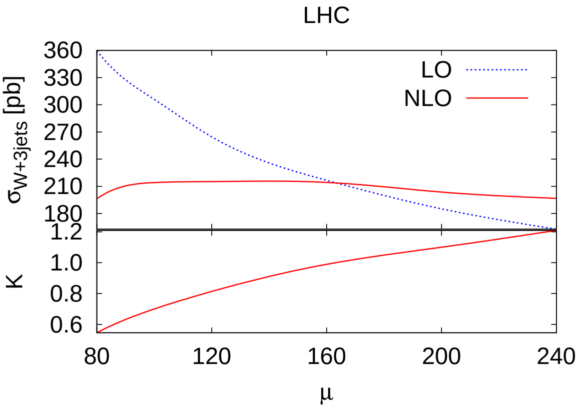

Figure 1: Inclusive W^++3 jet cross-section at the LHC and the K-factor defined as $K = \sigma_{\mathrm{NLO}}/\sigma_{\mathrm{LO}}$ as a function of the renormalization and factorization scales. Jets are defined with k_T algorithm with $R = 0.7$ and $p_T > 50$ GeV. Jet rapidities satisfy $|\eta| < 3$. The LO and NLO cross-sections are computed with CTEQ6L1 and CTEQ6M parton distributions, respectively.

What has been done? 2009

- C. F. Berger *et al.*, "Precise Predictions for $W + 3$ Jet Production at Hadron Colliders," arXiv:0902.2760 [hep-ph]

What has been done? 2009

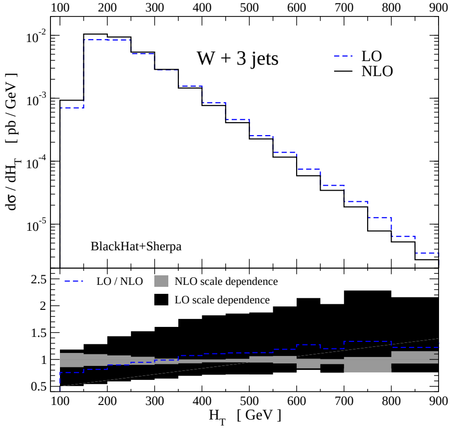

FIG. 3: The theoretical prediction for the H_T distribution in W + 3-jet production. The curves and bands are labeled as in fig. 2.

NLO TROUBLES

Problems arising in NLO calculations

- Large Number of Feynman diagrams
- Reduction to Scalar Integrals (or sets of known integrals)
- Numerical Instabilities (inverse Gram determinants, spurious phase-space singularities)
- Extraction of soft and collinear singularities (we need virtual and real corrections)

- Traditional Method: Feynman Diagrams & Passarino-Veltman Reduction:
    - general applicability major achievements
    - but major problem: not designed @ amplitude level

Methods available

- Traditional Method: Feynman Diagrams & Passarino-Veltman Reduction:
- Semi-Numerical Approach (Algebraic/Partly Numerical – Improved traditional) $\rightarrow$ Reduction to set of well-known integrals
- Numerical Approach (Numerical/Partly Algebraic) $\rightarrow$ Compute tensor integrals numerically
  - Ellis, Giele, Glover, Zanderighi;
  - Binoth, Guillet, Heinrich, Schubert;
  - Denner, Dittmaier; Del Aguila, Pittau;
  - Ferroglia, Passera, Passarino, Uccirati;
  - Nagy, Soper; van Hameren, Vollinga, Weinzierl;

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- Numerical Approach (Numerical/Partly Algebraic) $\rightarrow$ Compute tensor integrals numerically
- Analytic Approach (Twistor-inspired)
  $\rightarrow$ extract information from lower-loop, lower-point amplitudes
  $\rightarrow$ determine scattering amplitudes by their poles and cuts
  - ⋆ major advantage: designed to work @ amplitude level
  - ⋆ **quadruple and triple cuts major simplifications**
  - Bern, Dixon, Dunbar, Kosower, Berger, Forde;
  - Anastasiou, Britto, Cachazo, Feng, Kunszt, Mastrolia;

Methods available

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⋆ OPP Integrand-level reduction
combine: reduction@integrand + n-particle cuts

G. Ossola., C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007) – arXiv:hep-ph/0609007

and JHEP **0707** (2007) 085 – arXiv:0704.1271 [hep-ph]

R. K. Ellis, W. T. Giele and Z. Kunszt, JHEP **0803**, 003 (2008)

Any m-point one-loop amplitude can be written, before integration, as

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

A bar denotes objects living in $n = 4 + \epsilon$ dimensions

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2$$

$$\bar{q}^2 = q^2 + \tilde{q}^2$$

$$\bar{D}_i = D_i + \tilde{q}^2$$

External momenta p_i are 4-dimensional objects

The old "master" formula

$$
\begin{aligned}
\int A \;=\; & \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3) \\
+ & \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) C_0(i_0 i_1 i_2) \\
+ & \sum_{i_0<i_1}^{m-1} b(i_0 i_1) B_0(i_0 i_1) \\
+ & \sum_{i_0}^{m-1} a(i_0) A_0(i_0) \\
+ & \text{ rational terms}
\end{aligned}
$$

The old "master" formula

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

$$N(q) \to q^{\mu_1} \dots q^{\mu_m} \to g^{\mu_1 \mu_2} p_i^{\mu_3} \dots$$

G. Passarino and M. J. G. Veltman, "One Loop Corrections For E+ E- Annihilation Into Mu+ Mu- In The Weinberg Model," Nucl. Phys. B **160** (1979) 151.

THE OLD "MASTER" FORMULA

$$T^N_{\mu_1\ldots\mu_P}(p_1,\ldots,p_{N-1},m_0,\ldots,m_{N-1}) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq \frac{q_{\mu_1}\cdots q_{\mu_P}}{D_0 D_1\cdots D_{N-1}}$$

$$T^N_{\mu_1\ldots\mu_P}(p_1,\ldots,p_{N-1},m_0,\ldots,m_{N-1}) = \sum_{i_1,\ldots,i_P=0}^{N-1} T^N_{i_1\ldots i_P} p_{i_1\mu_1}\cdots p_{i_P\mu_P}.$$

$$\begin{aligned}
D_\mu &= \sum_{i=1}^{3} p_{i\mu} D_i,\\
D_{\mu\nu} &= g_{\mu\nu} D_{00} + \sum_{i,j=1}^{3} p_{i\mu} p_{j\nu} D_{ij},\\
D_{\mu\nu\rho} &= \sum_{i=1}^{3}(g_{\mu\nu}p_{i\rho} + g_{\nu\rho}p_{i\mu} + g_{\mu\rho}p_{i\nu}) D_{00i} + \sum_{i,j,k=1}^{3} p_{i\mu}p_{j\nu}p_{k\rho} D_{ijk},\\
D_{\mu\nu\rho\sigma} &= (g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}) D_{0000}\\
&\quad + \sum_{i,j=1}^{3} (g_{\mu\nu}p_{i\rho}p_{j\sigma} + g_{\nu\rho}p_{i\mu}p_{j\sigma} + g_{\mu\rho}p_{i\nu}p_{j\sigma}\\
&\qquad + g_{\mu\sigma}p_{i\nu}p_{j\rho} + g_{\nu\sigma}p_{i\mu}p_{j\rho} + g_{\rho\sigma}p_{i\mu}p_{j\nu}) D_{00ij}\\
&\quad + \sum_{i,j,k,l=1}^{3} p_{i\mu}p_{j\nu}p_{k\rho}p_{l\sigma} D_{ijkl}.
\end{aligned}$$

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W. L. van Neerven and J. A. M. Vermaseren, "Large Loop Integrals," Phys. Lett. B **137**, 241 (1984)

The derivation of the reduction formula starts as in ref. [1] with the Schouten identity which is a relation between five Levi-Civita tensors:

$$\epsilon^{p_1 p_2 p_3 p_4} Q_\mu = \epsilon^{\mu p_2 p_3 p_4} Q\cdot p_1 + \epsilon^{p_1 \mu p_3 p_4} Q\cdot p_2 + \epsilon^{p_1 p_2 \mu p_4} Q\cdot p_3 + \epsilon^{p_1 p_2 p_3 \mu} Q\cdot p_4 \,. \qquad (6)$$

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which yields the final formula for the scalar one-loop five-point function:

$$E_{01234}(w^2 - 4\Delta_4 m_0^2) = D_{1234}\,[2\Delta_4 - w\cdot(v_1 + v_2 + v_3 + v_4)]$$

$$+ D_{0234}\, v_1\cdot w + D_{0134}\, v_2\cdot w + D_{0124}\, v_3\cdot w + D_{0123}\, v_4\cdot w \,. \tag{19}$$

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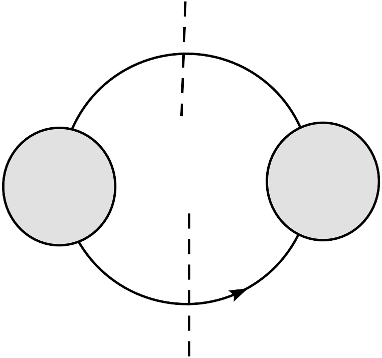
This method is completely different from the one used in ref. [3].

Unitarity



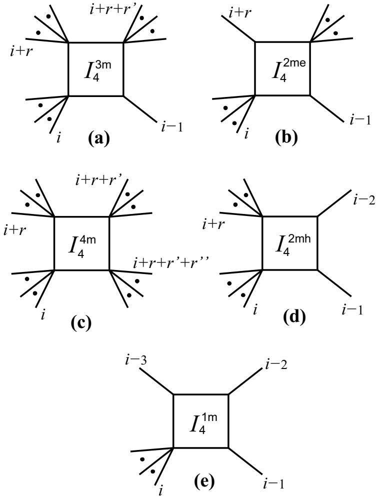
Started in 90's, mainly QCD, amplitude level (analytical results)
Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower,
[arXiv:hep-ph/9403226].
Gluing tree amplitudes plus colinear limits → extract coefficients

Unitarity

$$\mathcal{C} * \int A = \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3)\mathcal{C} * D_0(i_0 i_1 i_2 i_3) + \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2)\mathcal{C} * C_0(i_0 i_1 i_2) + \sum_{i_0<i_1}^{m-1} b(i_0 i_1)\mathcal{C} * B_0(i_0 i_1)$$

	Integral	Unique Function
a	$I_4^{0\mathrm{m}}(s,t)$	$\ln(-s)\ln(-t)$
b	$I_3^{1\mathrm{m}}(s)$	$\ln(-s)^2$
c	$I_3^{1\mathrm{m}}(t)$	$\ln(-t)^2$
d	$I_2(s)$	$\ln(-s)$
e	$I_2(t)$	$\ln(-t)$

Table 1: The set of integral functions that may appear in a cut-constructible massless four-point amplitude, together with the independent logarithms.



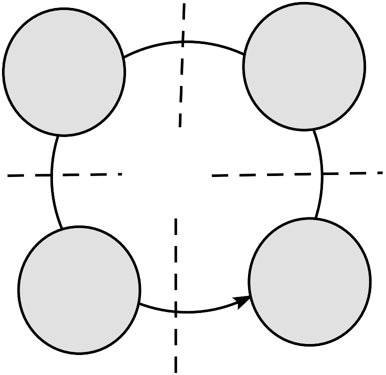
i+r+r'
i+r
I_4^{3m}
i
(a)
i−1
i+r
I_4^{2me}
i
(b)
i−1
i+r+r'
i+r
I_4^{4m}
i+r+r'+r''
i
(c)
i−2
i+r
I_4^{2mh}
i
(d)
i−1
i−3
i−2
I_4^{1m}
i
(e)
i−1

	Integral	Unique Function
a	$I^{3\mathrm{m}}_{4:r,r';i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r+r'}^{[n-r-r'-1]})$
b	$I^{2\mathrm{m}\,e}_{4:r;i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r+1}^{[n-r-2]})$
c	$I^{4\mathrm{m}}_{4:r,r',r'';i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r+r'}^{[r'']})$
d	$I^{2\mathrm{m}\,h}_{4:r;i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r}^{[n-r-1]})$
e	$I^{1\mathrm{m}}_{4;i}$	$\ln(-t_i^{[r]})\ln(-t_i^{[r+1]})$
f	$I^{3\mathrm{m}}_{3:r,r';i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r}^{[r']})$

Table 2: Following the ordering shown and taking large $t_i^{[r]}$ makes the proof of uniqueness of the cuts straightforward.

Quadruple cuts

R. Britto, F. Cachazo and B. Feng, [arXiv:hep-th/0412103].
Quadruple cut with complex momenta $\rightarrow d(i_0 i_1 i_2 i_3)$



OPP "MASTER" FORMULA - I

General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

OPP "master" formula - II

$$N(q) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) \qquad\qquad \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} [c(i_0 i_1 i_2) \qquad\qquad] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i$$
$$+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) \qquad\qquad \right] \prod_{i \neq i_0, i_1}^{m-1} D_i + \sum_{i_0}^{m-1} [a(i_0) \qquad\qquad] \prod_{i \neq i_0}^{m-1} D_i$$

- The quantities $d(i_0 i_1 i_2 i_3)$ are the coefficients of 4-point functions with denominators labeled by i_0, i_1, i_2, and i_3.
- $c(i_0 i_1 i_2)$, $b(i_0 i_1)$, $a(i_0)$ are the coefficients of all possible 3-point, 2-point and 1-point functions, respectively.

OPP "master" formula - II

$$N(q) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i$$
$$+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i\neq i_0}^{m-1} D_i$$

The quantities $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ are the "spurious" terms

- They still depend on q (integration momentum)
- They should vanish upon integration

What is the explicit expression of the spurious term?

Spurious Terms - I

Following F. del Aguila and R. Pittau, arXiv:hep-ph/0404120

- Express any q in $N(q)$ as

$$q^\mu = -p_0^\mu + \sum_{i=1}^4 G_i\, \ell_i^\mu \ , \ {\ell_i}^2 = 0$$

$$k_1 = \ell_1 + \alpha_1 \ell_2\,, \quad k_2 = \ell_2 + \alpha_2 \ell_1\,,\ k_i = p_i - p_0$$
$${\ell_3}^\mu = < \ell_1|\gamma^\mu|\ell_2]\,,\ {\ell_4}^\mu = < \ell_2|\gamma^\mu|\ell_1]$$

- The coefficients G_i either reconstruct denominators D_i

$\rightarrow$ They give rise to d, c, b, a coefficients

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$${\ell_3}^\mu =< \ell_1|\gamma^\mu|\ell_2]\,,\ {\ell_4}^\mu =< \ell_2|\gamma^\mu|\ell_1]$$

- The coefficients G_i either reconstruct denominators D_i or vanish upon integration

$\rightarrow$ They give rise to d, c, b, a coefficients
$\rightarrow$ They form the spurious $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ coefficients

Spurious Terms - II

- $\tilde{d}(q)$ term (only 1)

$$\tilde{d}(q) = \tilde{d}\, T(q)\,,$$

where $\tilde{d}$ is a constant (does not depend on q)

$$T(q) \equiv Tr[(\not{q} + \not{p}_0)\not{\ell}_1\not{\ell}_2\not{k}_3\gamma_5]$$

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- $\tilde{c}(q)$ terms (they are 6)

$$\tilde{c}(q) = \sum_{j=1}^{j_{max}} \left\{ \tilde{c}_{1j} [(q + p_0) \cdot \ell_3]^j + \tilde{c}_{2j} [(q + p_0) \cdot \ell_4]^j \right\}$$

In the renormalizable gauge, $j_{max} = 3$

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- $\tilde{c}(\mathrm{q})$ terms (they are 6)

$$\tilde{c}(q) = \sum_{j=1}^{j_{max}} \left\{ \tilde{c}_{1j}[(q+p_0)\cdot\ell_3]^j + \tilde{c}_{2j}[(q+p_0)\cdot\ell_4]^j \right\}$$

In the renormalizable gauge, $j_{max} = 3$

- $\tilde{b}(\mathrm{q})$ and $\tilde{a}(\mathrm{q})$ give rise to 8 and 4 terms, respectively

A simple example

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4}$$

A simple example

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4}$$

$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}$$

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- Melrose, Nuovo Cim. 40 (1965) 181
- G. Källén, J.Toll, J. Math. Phys. 6, 299 (1965)

A next to simple example

$$\int \frac{1}{D_0 D_1 D_2 D_3 \ldots D_{m-1}}$$

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$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left(\prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^+)} + \prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^-)} \right)$$

A next to simple example

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$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left(\prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^+)} + \prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^-)} \right)$$

A next to simple example

$$\begin{vmatrix} T_0^5 & -T_0^4(0) & -T_0^4(1) & -T_0^4(2) & -T_0^4(3) & -T_0^4(4) \\ 1 & Y_{00} & Y_{01} & Y_{02} & Y_{03} & Y_{04} \\ 1 & Y_{10} & Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ 1 & Y_{20} & Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ 1 & Y_{30} & Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ 1 & Y_{40} & Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{vmatrix} = 0,$$

Now we know the form of the spurious terms:

$$\begin{aligned} N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1}\left[d(i_0 i_1 i_2 i_3)+\tilde{d}(q;i_0 i_1 i_2 i_3)\right]\prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1}\left[c(i_0 i_1 i_2)+\tilde{c}(q;i_0 i_1 i_2)\right]\prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\ &+ \sum_{i_0<i_1}^{m-1}\left[b(i_0 i_1)+\tilde{b}(q;i_0 i_1)\right]\prod_{i\neq i_0,i_1}^{m-1} D_i + \sum_{i_0}^{m-1}\left[a(i_0)+\tilde{a}(q;i_0)\right]\prod_{i\neq i_0}^{m-1} D_i \end{aligned}$$

Our calculation is now reduced to an algebraic problem

Now we know the form of the spurious terms:

$$
\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
+ & \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

Our calculation is now reduced to an algebraic problem

Extract all the coefficients by evaluating N(q) for a set of values of the integration momentum q

General strategy

Now we know the form of the spurious terms:

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

Our calculation is now reduced to an algebraic problem

Extract all the coefficients by evaluating N(q) for a set of values of the integration momentum q

There is a very good set of such points: Use values of q for which a set of denominators D_i vanish $\rightarrow$ The system becomes "triangular": solve first for 4-point functions, then 3-point functions and so on

Example

$$
\begin{aligned}
N(q) &= d + \tilde{d}(q) + \sum_{i=0}^{3} \left[c(i) + \tilde{c}(q;i)\right] D_i + \sum_{i_0<i_1}^{3} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] D_{i_0} D_{i_1} \\
&+ \sum_{i_0=0}^{3} \left[a(i_0) + \tilde{a}(q; i_0)\right] D_{i\neq i_0} D_{j\neq i_0} D_{k\neq i_0}
\end{aligned}
$$

We look for a q of the form $q^\mu = -p_0^\mu + x_i \ell_i^\mu$ such that

$$D_0 = D_1 = D_2 = D_3 = 0$$

$\rightarrow$ we get a system of equations in x_i that has two solutions $q_0^\pm$

Example

$$N(q) = d + \tilde{d}(q)$$

Our "master formula" for $q = q_0^{\pm}$ is:

$$N(q_0^{\pm}) = [d + \tilde{d}\; T(q_0^{\pm})]$$

$\longrightarrow$ solve to extract the coefficients d and $\tilde{d}$

Example

$$N(q) - d - \tilde{d}(q) = \sum_{i=0}^{3} [c(i) + \tilde{c}(q;i)] D_i + \sum_{i_0<i_1}^{3} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] D_{i_0} D_{i_1}$$
$$+ \sum_{i_0=0}^{3} [a(i_0) + \tilde{a}(q; i_0)] D_{i\neq i_0} D_{j\neq i_0} D_{k\neq i_0}$$

Then we can move to the extraction of c coefficients using

$$N'(q) = N(q) - d - \tilde{d}\,T(q)$$

and setting to zero three denominators (ex: $D_1 = 0$, $D_2 = 0$, $D_3 = 0$)

Example

$$N(q) - d - \tilde{d}(q) = [c(0) + \tilde{c}(q;0)]\, D_0$$

We have infinite values of q for which

$$D_1 = D_2 = D_3 = 0 \quad \text{and} \quad D_0 \neq 0$$

$\rightarrow$ Here we need 7 of them to determine $c(0)$ and $\tilde{c}(q;0)$

- Let's go back to the integrand

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

- Insert the expression for $N(q) \rightarrow$ we know all the coefficients

$$N(q) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d + \tilde{d}(q) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} [c + \tilde{c}(q)] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i + \cdots$$

- Finally rewrite all denominators using

$$\frac{D_i}{\bar{D}_i} = \bar{Z}_i \,, \quad \text{with} \quad \bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i} \right)$$

Rational Terms - I

$$
\begin{aligned}
A(\bar{q}) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2} \bar{D}_{i_3}} \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2}} \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i_0} \bar{D}_{i_1}} \prod_{i \neq i_0, i_1}^{m-1} \bar{Z}_i \\
&+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i_0}} \prod_{i \neq i_0}^{m-1} \bar{Z}_i
\end{aligned}
$$

The rational part is produced, after integrating over $d^n q$, by the $\tilde{q}^2$ dependence in $\bar{Z}_i$

$$\bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i}\right)$$

Rational Terms - I

The "Extra Integrals" are of the form

$$I^{(n;2\ell)}_{s;\mu_1\cdots\mu_r} \equiv \int d^n q\, \tilde{q}^{2\ell} \frac{q_{\mu_1}\cdots q_{\mu_r}}{\bar{D}(k_0)\cdots\bar{D}(k_s)}\,,$$

where

$$\bar{D}(k_i) \equiv (\bar{q}+k_i)^2 - m_i^2\,,\, k_i = p_i - p_0$$

These integrals:

- **have dimensionality** $\mathcal{D} = 2(1+\ell-s)+r$
- **contribute only when** $\mathcal{D} \geq 0$**, otherwise are of** $\mathcal{O}(\epsilon)$

Rational Terms - II

Expand in D-dimensions ?

$$\bar{D}_i = D_i + \tilde{q}^2$$

Rational Terms - II

Expand in D-dimensions ?

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2)\right] \prod_{i\neq i_0,i_1}^{m-1} \bar{D}_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2)\right] \prod_{i\neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i
\end{aligned}
$$

Rational Terms - II

Expand in D-dimensions ?

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1}\left[d(i_0 i_1 i_2 i_3;\tilde{q}^2)+\tilde{d}(q;i_0 i_1 i_2 i_3;\tilde{q}^2)\right]\prod_{i\neq i_0,i_1,i_2,i_3}^{m-1}\bar{D}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1}\left[c(i_0 i_1 i_2;\tilde{q}^2)+\tilde{c}(q;i_0 i_1 i_2;\tilde{q}^2)\right]\prod_{i\neq i_0,i_1,i_2}^{m-1}\bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1}\left[b(i_0 i_1;\tilde{q}^2)+\tilde{b}(q;i_0 i_1;\tilde{q}^2)\right]\prod_{i\neq i_0,i_1}^{m-1}\bar{D}_i \\
&+ \sum_{i_0}^{m-1}\left[a(i_0;\tilde{q}^2)+\tilde{a}(q;i_0;\tilde{q}^2)\right]\prod_{i\neq i_0}^{m-1}\bar{D}_i+\tilde{P}(q)\prod_{i}^{m-1}\bar{D}_i
\end{aligned}
$$

$$m_i^2 \to m_i^2-\tilde{q}^2$$

Rational Terms - II

Polynomial dependence on $\tilde{q}^2$

$$b(ij;\tilde{q}^2) = b(ij) + \tilde{q}^2 b^{(2)}(ij)\,, \quad c(ijk;\tilde{q}^2) = c(ijk) + \tilde{q}^2 c^{(2)}(ijk)\,.$$

Rational Terms - II

Polynomial dependence on $\tilde{q}^2$

$$b(ij;\tilde{q}^2) \;=\; b(ij) + \tilde{q}^2 b^{(2)}(ij)\,, \quad c(ijk;\tilde{q}^2) \;=\; c(ijk) + \tilde{q}^2 c^{(2)}(ijk)\,.$$

$$\int d^n\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j} \;=\; -\frac{i\pi^2}{2}\left[m_i^2 + m_j^2 - \frac{(p_i-p_j)^2}{3}\right] + \mathcal{O}(\epsilon)\,,$$

$$\int d^n\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j\bar{D}_k} \;=\; -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon)\,, \quad \int d^n\bar{q}\,\frac{\tilde{q}^4}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} \;=\; -\frac{i\pi^2}{6} + \mathcal{O}(\epsilon)\,.$$

Rational Terms - II

Furthermore, by defining

$$\mathcal{D}^{(m)}(q,\tilde{q}^2) \equiv \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3;\tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3;\tilde{q}^2) \right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} \bar{D}_i \,,$$

the following expansion holds

$$\mathcal{D}^{(m)}(q,\tilde{q}^2) = \sum_{j=2}^{m} \tilde{q}^{(2j-4)} d^{(2j-4)}(q) \,,$$

where the last coefficient is independent on q

$$d^{(2m-4)}(q) = d^{(2m-4)} \,.$$

Rational Terms - II

In practice, once the 4-dimensional coefficients have been determined, one can redo the fits for different values of $\tilde{q}^2$, in order to determine $b^{(2)}(ij)$, $c^{(2)}(ijk)$ and $d^{(2m-4)}$.

$$\begin{aligned} \mathrm{R}_1 &= -\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0 i_1 i_2) \\ &- \frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0 i_1) \left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3} \right) . \end{aligned}$$

G. Ossola, C. G. Papadopoulos and R. Pittau,arXiv:0802.1876 [hep-ph]

Rational Terms - R_2

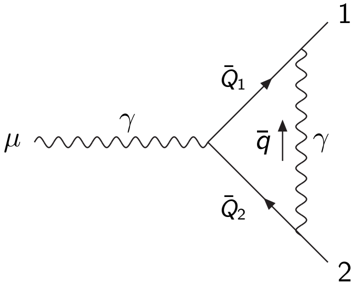
A different source of Rational Terms, called R_2, can also be generated from the ϵ-dimensional part of $N(q)$

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, \epsilon; q)$$

$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q} \frac{\tilde{N}(\tilde{q}^2, \epsilon; q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q}\,\mathcal{R}_2$$

$$\begin{aligned} \bar{q} &= q + \tilde{q}\,, \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}\,, \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}\,. \end{aligned}$$

New vertices/particles or GKM-approach



$$\bar{Q}_1 = \bar{q} + p_1 = Q_1 + \tilde{q}$$
$$\bar{Q}_2 = \bar{q} + p_2 = Q_2 + \tilde{q}$$

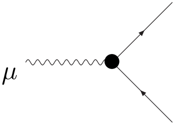
$$\bar{D}_0 = \bar{q}^2$$
$$\bar{D}_1 = (\bar{q} + p_1)^2$$
$$\bar{D}_2 = (\bar{q} + p_2)^2$$

$$\begin{aligned}
\bar{N}(\bar{q}) &\equiv e^3 \left\{ \bar{\gamma}_{\bar{\beta}} \, (\bar{\not{Q}}_1 + m_e) \, \gamma_\mu \, (\bar{\not{Q}}_2 + m_e) \, \bar{\gamma}^{\bar{\beta}} \right\} \\
&= e^3 \, \{ \gamma_\beta (\not{Q}_1 + m_e) \gamma_\mu (\not{Q}_2 + m_e) \gamma^\beta \\
&- \epsilon \, (\not{Q}_1 - m_e) \gamma_\mu (\not{Q}_2 - m_e) + \epsilon \tilde{q}^2 \, \gamma_\mu - \tilde{q}^2 \, \gamma_\beta \gamma_\mu \gamma^\beta \} \; ,
\end{aligned}$$

Rational Terms - R_2

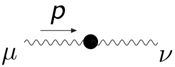
$$\int d^n\bar{q}\frac{\tilde{q}^2}{\bar{D}_0\bar{D}_1\bar{D}_2} = -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon)\,,$$
$$\int d^n\bar{q}\frac{q_\mu q_\nu}{\bar{D}_0\bar{D}_1\bar{D}_2} = -\frac{i\pi^2}{2\epsilon}g_{\mu\nu} + \mathcal{O}(1)\,,$$

$$\mathrm{R}_2 = -\frac{ie^3}{8\pi^2}\gamma_\mu + \mathcal{O}(\epsilon)\,,$$

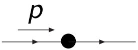


$$= -\frac{ie^3}{8\pi^2}\gamma_\mu$$

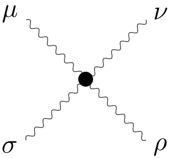
Rational counterterms



$$= -\frac{ie^2}{8\pi^2}\, g_{\mu\nu}\, (2m_e^2 - p^2/3)$$



$$= \frac{ie^2}{16\pi^2}(-\not{p} + 2m_e)$$



$$= \frac{ie^4}{12\pi^2}\,(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho})$$

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Evaluate scalar integrals

- massive integrals → FF [G. J. van Oldenborgh]
- massless+massive integrals → OneLOop [A. van Hameren]

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Cuttools

G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP **0803**, 042 (2008) [arXiv:0711.3596 [hep-ph]]

Properties of the master equation

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The $N \equiv N$ test

A tool to efficiently treat phase-space points with numerical instabilities

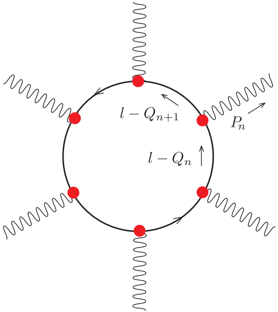
OPP "MASTER" FORMULA - I

General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

4-PHOTON AND 6-PHOTON AMPLITUDES

As an example we present 4-photon and 6-photon amplitudes (via fermionic loop of mass m_f)



Input parameters for the reduction:

- External momenta p_i
- Masses of propagators in the loop
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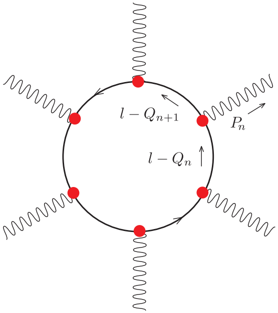

Input parameters for the reduction:

- External momenta $p_i \longrightarrow$ in this example massless, i.e. $p_i^2 = 0$
- Masses of propagators in the loop $\rightarrow$ all equal to m_f
- Polarization vectors $\longrightarrow$ various helicity configurations

Four Photons – Comparison with *Gounaris et al.*

$$\frac{F^f_{++++}}{\alpha^2 Q_f^4} = -8$$

Rational Part

$$
\begin{aligned}
\frac{F^f_{++++}}{\alpha^2 Q_f^4} &= -8 + 8\left(1+\frac{2\hat{u}}{\hat{s}}\right) B_0(\hat{u}) + 8\left(1+\frac{2\hat{t}}{\hat{s}}\right) B_0(\hat{t}) \\
&\quad - 8\left(\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} \qquad\qquad \right)\left[\hat{t}\,C_0(\hat{t}) + \hat{u}\,C_0(\hat{u})\right] \\
&\quad - 4\left[\qquad\qquad \frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} \qquad\qquad \right] D_0(\hat{t},\hat{u})
\end{aligned}
$$

Massless four-photon amplitudes

FOUR PHOTONS – COMPARISON WITH *Gounaris et al.*

$$\begin{aligned}\frac{F^{f}_{++++}}{\alpha^2 Q_f^4} &= -8 + 8\left(1+\frac{2\hat{u}}{\hat{s}}\right)B_0(\hat{u}) + 8\left(1+\frac{2\hat{t}}{\hat{s}}\right)B_0(\hat{t}) \\ &- 8\left(\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} - \frac{4m_f^2}{\hat{s}}\right)[\hat{t}C_0(\hat{t}) + \hat{u}C_0(\hat{u})] \\ &- 4\left[4m_f^4 - (2\hat{s}m_f^2 + \hat{t}\hat{u})\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} + \frac{4m_f^2\hat{t}\hat{u}}{\hat{s}}\right]D_0(\hat{t},\hat{u}) \\ &+ 8m_f^2(\hat{s}-2m_f^2)[D_0(\hat{s},\hat{t}) + D_0(\hat{s},\hat{u})]\end{aligned}$$

Massive four-photon amplitudes

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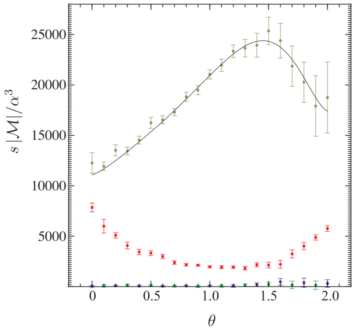
\end{aligned}
$$

Massive four-photon amplitudes

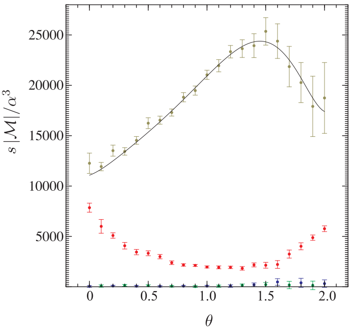
Results also checked for F^f_{+++-} and F^f_{++--}

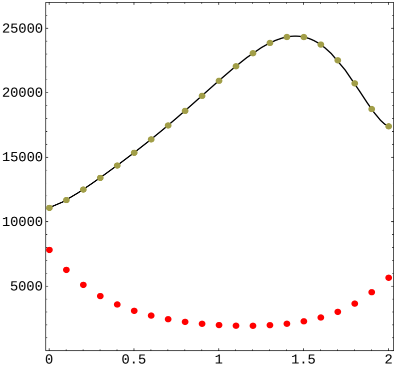
Massless case: $[+ + - - - -]$ and $[+ - - + + -]$



Plot presented by Nagy and Soper hep-ph/0610028
(also Binoth et al., hep-ph/0703311)

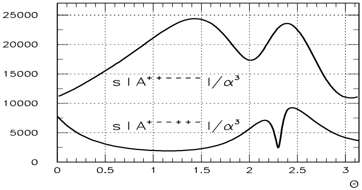
Massless case: $[++----]$ and $[+--++-]$

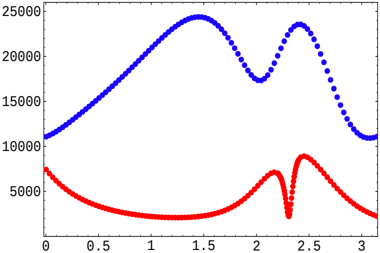




Analogous plot produced with OPP reduction

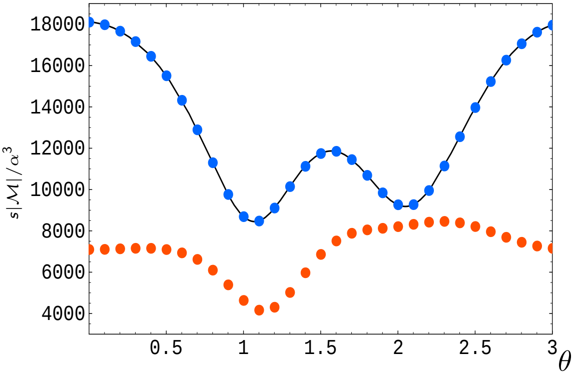
Massless case: $[++----]$ and $[++--+-]$





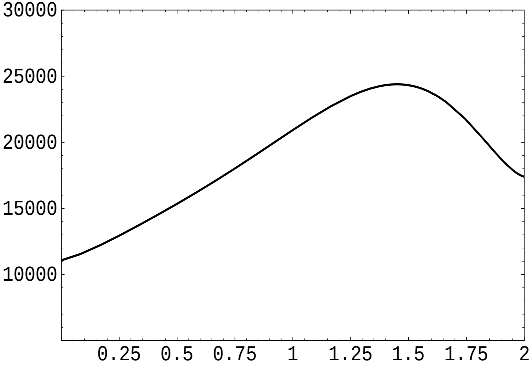
Same plot as before for a wider range of θ

Massless case: $[++----]$ and $[++--+-]$



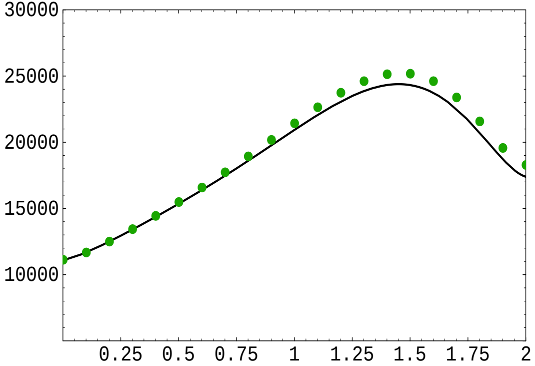
Same idea for a different set of external momenta

Six Photons with Massive Fermions



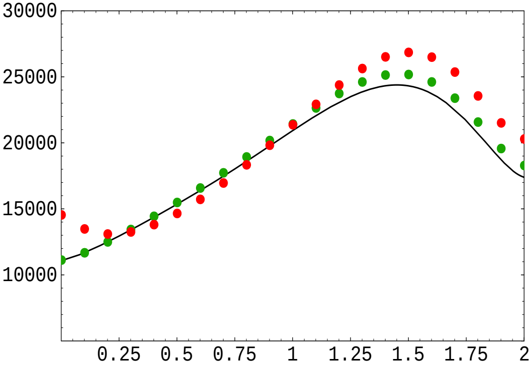
- Massless result [Mahlon]

Six Photons with Massive Fermions

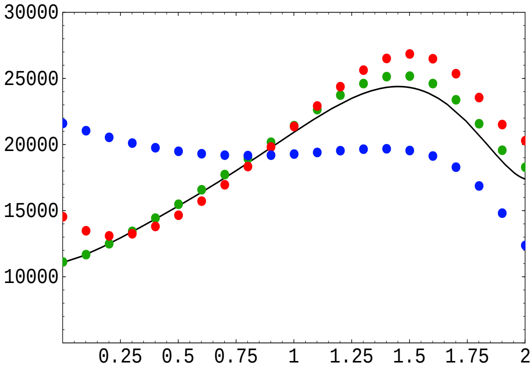


- Massless result [Mahlon]
- $m = 0.5$ GeV

Six Photons with Massive Fermions



- Massless result [Mahlon]
- $m = 0.5$ GeV
- $m = 4.5$ GeV



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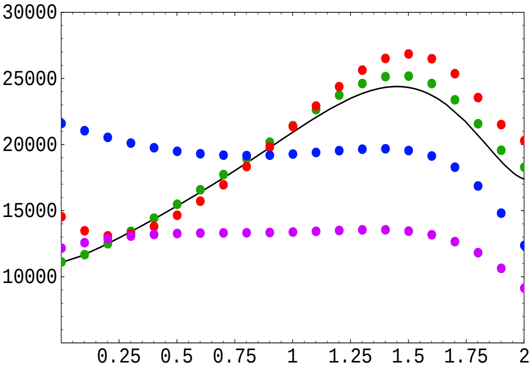
- $m = 12.0$ GeV

Six Photons with Massive Fermions



- Massless result [Mahlon]
- $m = 0.5$ GeV
- $m = 4.5$ GeV
- $m = 12.0$ GeV
- $m = 20.0$ GeV

$pp \to VVV$ NLO

NLO corrections to tri-boson production

- $pp \to ZZZ$
- $pp \to W^+ZZ$
- $pp \to W^+W^-Z$
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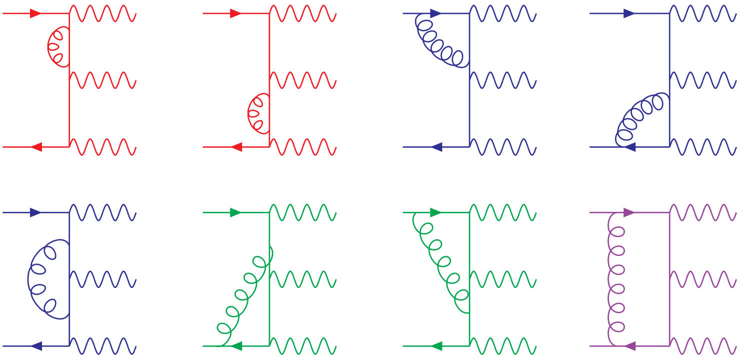
T. Binoth, G. Ossola, C. G. Papadopoulos and R. Pittau, arXiv:0804.0350 [hep-ph]

$pp \to ZZZ$ VIRTUAL CORRECTIONS

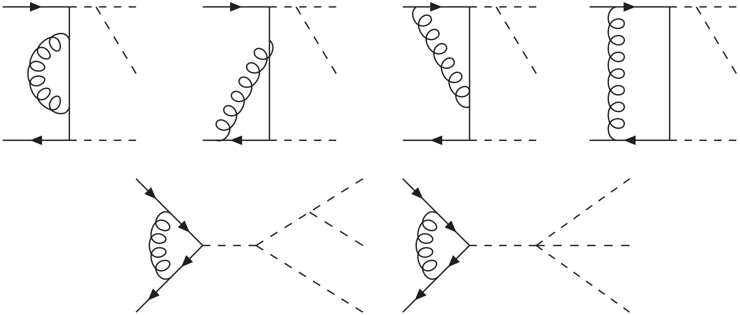
A. Lazopoulos, K. Melnikov and F. Petriello, [arXiv:hep-ph/0703273]



POLES $1/\epsilon^2$ AND $1/\epsilon$

$$\sigma^{\mathrm{NLO,virt}}|_{\mathrm{div}} = -C_F \frac{\alpha_s}{\pi} \frac{\Gamma(1+\epsilon)}{(4\pi)^{-\epsilon}} (s_{12})^{-\epsilon} \left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right) \sigma^{\mathrm{LO}}$$

$pp \longrightarrow WWZ$ VIRTUAL CORRECTIONS



Hankele and Zeppenfeld arXiv:0712.3544 [hep-ph]

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- Of course full agreement for the $1/\epsilon^2$ and $1/\epsilon$ terms
- An 'easy' agreement for all graphs with up to 4-point loop integrals
- A bit more work to uncover the differences in scalar function normalization that happen to show to order ϵ^2 thus influence only 5-point loop integrals.

$pp \to VVV$ VIRTUAL CORRECTIONS

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$$\begin{cases} -26.45706742815552 \\ -26.457067428165503661018557937723426 \end{cases}$$

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$$\begin{cases} -26.45706742815552 \\ -26.4570674281655036610185579377234 26 \end{cases}$$

Typical time: 10^4 times faster (for non-singular PS-points)

$pp \to VVV$ REAL CORRECTIONS

$$\sigma_{q\bar{q}}^{NLO} = \int\limits_{VVVg} \left[d\sigma_{q\bar{q}}^{R} - d\sigma_{q\bar{q}}^{A} \right] + \int\limits_{VVV} \left[d\sigma_{q\bar{q}}^{B} + d\sigma_{q\bar{q}}^{V} + \int\limits_{g} d\sigma_{q\bar{q}}^{A} + d\sigma_{q\bar{q}}^{C} \right]$$

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$$\mathcal{D}^{q_1 g_6, \bar{q}_2} = \frac{8\pi\alpha_s C_F}{2\tilde{x}\, p_1 \cdot p_6} \left(\frac{1+\tilde{x}^2}{1-\tilde{x}} \right) \left| \mathcal{M}_{q\bar{q}}^{B}(\tilde{p}_{16}, p_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5) \right|^2$$

$pp \to VVV$ REAL CORRECTIONS

$$\sigma^{NLO}_{q\bar{q}} = \int\limits_{VVVg} \left[d\sigma^R_{q\bar{q}} - d\sigma^A_{q\bar{q}} \right] + \int\limits_{VVV} \left[d\sigma^B_{q\bar{q}} + d\sigma^V_{q\bar{q}} + \int\limits_{g} d\sigma^A_{q\bar{q}} + d\sigma^C_{q\bar{q}} \right]$$

$$\mathcal{D}^{q_1 g_6, \bar{q}_2} = \frac{8\pi\alpha_s C_F}{2\tilde{x}\, p_1 \cdot p_6} \left(\frac{1+\tilde{x}^2}{1-\tilde{x}} \right) |\mathcal{M}^B_{q\bar{q}}(\tilde{p}_{16}, p_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5)|^2$$

$$\tilde{x} = \frac{p_1 \cdot p_2 - p_2 \cdot p_6 - p_1 \cdot p_6}{p_1 \cdot p_2}$$

$$\tilde{p}_{16} = \tilde{x}\, p_1 \quad , \quad K = p_1 + p_2 - p_6 \quad , \quad \tilde{K} = \tilde{p}_{16} + p_2$$

$$\Lambda^{\mu\nu} = g^{\mu\nu} - \frac{2(K^\mu + \tilde{K}^\mu)(K^\nu + \tilde{K}^\nu)}{(K+\tilde{K})^2} + \frac{2\tilde{K}^\mu K^\nu}{K^2}$$

$$\tilde{p}_j = \Lambda\, p_j$$

$pp \to VVV$ REAL CORRECTIONS

$$d\sigma^R_{q\bar{q}} - d\sigma^A_{q\bar{q}} = \frac{C_S}{N}\frac{1}{2s_{12}}\left[C_F\,|\mathcal{M}^R_{q\bar{q}}(\{p_j\}')|^2 - \mathcal{D}^{q_1 g_6,\bar{q}_2} - \mathcal{D}^{\bar{q}_2 g_6, q_1}\right] d\Phi_{VVVg}$$

$pp \longrightarrow VVV$ REAL CORRECTIONS

$$d\sigma^R_{q\bar{q}} - d\sigma^A_{q\bar{q}} = \frac{C_S}{N}\frac{1}{2s_{12}}\left[C_F\,|\mathcal{M}^R_{q\bar{q}}(\{p_j\}')|^2 - \mathcal{D}^{q_1 g_6,\bar{q}_2} - \mathcal{D}^{\bar{q}_2 g_6,q_1}\right] d\Phi_{VVVg}$$

$$d\sigma^C_{q\bar{q}} + \int_g d\sigma^A_{q\bar{q}} = \frac{\alpha_s C_F}{2\pi}\frac{\Gamma(1+\epsilon)}{(4\pi)^{-\epsilon}}\left(\frac{s_{12}}{\mu^2}\right)^{-\epsilon}\left[\frac{2}{\epsilon^2}+\frac{3}{\epsilon}-\frac{2\pi^2}{3}\right] d\sigma^B_{q\bar{q}}$$
$$+\frac{\alpha_s C_F}{2\pi}\int_0^1 dx\, \mathcal{K}^{q,q}(x)\, d\sigma^B_{q\bar{q}}(xp_1,p_2) + \frac{\alpha_s C_F}{2\pi}\int_0^1 dx\, \mathcal{K}^{\bar{q},\bar{q}}(x)\, d\sigma^B_{q\bar{q}}(p_1,xp_2)$$

$$\mathcal{K}^{q,q}(x) = \mathcal{K}^{\bar{q},\bar{q}}(x) = \left(\frac{1+x^2}{1-x}\right)_+ \log\left(\frac{s_{12}}{\mu_F^2}\right) + \left(\frac{4\log(1-x)}{1-x}\right)_+ + (1-x) - 2(1+x)\log(1-x)$$

$pp \to VVV$ REAL CORRECTIONS

$$\sigma_{gq}^{NLO} = \int\limits_{VVV} \left[\int\limits_{q} d\sigma_{gq}^{A} + d\sigma_{gq}^{C} \right] + \int\limits_{VVVq} \left[d\sigma_{gq}^{R} - d\sigma_{gq}^{A} \right]$$

$pp \to VVV$ REAL CORRECTIONS

$$\sigma_{gq}^{NLO} = \int_{VVV} \left[\int_{q} d\sigma_{gq}^{A} + d\sigma_{gq}^{C} \right] + \int_{VVVq} \left[d\sigma_{gq}^{R} - d\sigma_{gq}^{A} \right]$$

$$d\sigma_{gq}^{R} - d\sigma_{gq}^{A} = \frac{C_S}{N} \frac{1}{2s_{12}} \left[T_R |\mathcal{M}_{gq}^{R}|^2 - \mathcal{D}^{g_1 q_6, q_2} \right] d\Phi_{VVVq}$$

$pp \to VVV$ REAL CORRECTIONS

$$\sigma_{gq}^{NLO} = \int_{VVV} \left[\int_q d\sigma_{gq}^{A} + d\sigma_{gq}^{C} \right] + \int_{VVVq} \left[d\sigma_{gq}^{R} - d\sigma_{gq}^{A} \right]$$

$$d\sigma_{gq}^{R} - d\sigma_{gq}^{A} = \frac{C_S}{N} \frac{1}{2 s_{12}} \left[T_R |\mathcal{M}_{gq}^{R}|^2 - \mathcal{D}^{g_1 q_6, q_2} \right] d\Phi_{VVVq}$$

$$\mathcal{D}^{g_1 q_6, q_2} = \frac{8\pi\alpha_s\, T_R}{\tilde{x}\, 2\, p_1 \cdot p_6} [1 - 2\,\tilde{x}\,(1 - \tilde{x})] |\mathcal{M}_{q\bar{q}}^{B}(\tilde{p}_j)|^2$$

$$d\sigma^C_{gq} + \int_q d\sigma^A_{gq} = \frac{\alpha_s T_R}{2\pi} \int_0^1 dx\, \mathcal{K}^{g,q}(x)\, d\sigma^B_{q\bar{q}}(xp_1, p_2)$$

$$\mathcal{K}^{g,q}(x) = [x^2 + (1-x)^2] \log\left(\frac{s_{12}}{\mu_F^2}\right) + 2x(1-x) + 2[x^2 + (1-x)^2] \log(1-x)$$

$pp \to VVV$ REAL CORRECTIONS

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$$d\sigma(P_1, P_2) = \sum_{ab} \int dz_1 dz_2 f_a(z_1, \mu_F) f_b(z_2, \mu_F) d\sigma_{ab}(z_1 P_1, z_2 P_2)$$

$q\bar{q}$, $\bar{q}q$, gq, qg, $g\bar{q}$, $\bar{q}g$

$pp \to VVV$ REAL CORRECTIONS

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$q\bar{q}$, $\bar{q}q$, gq, qg, $g\bar{q}$, $\bar{q}g$

- check also with phase-space slicing method

$pp \rightarrow VVV$ NLO

- Virtual contributions obtained with `Cuttools`
- $O(100ms)$ per "event" $\rightarrow$ factor $O(10-10^2)$

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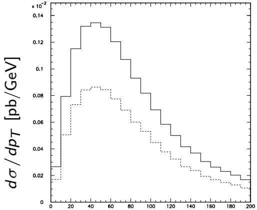
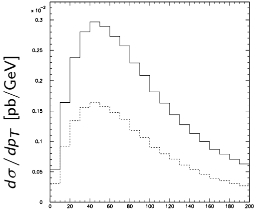
Process	scale μ	Born cross section [fb]	NLO cross section [fb]
ZZZ	$3M_Z$	9.7(1)	15.3(1)
WZZ	$2M_Z + M_W$	20.2(1)	40.4(2)
WWZ	$M_Z + 2M_W$	96.8(6)	185.5(8)
WWW	$3M_W$	82.5(5)	146.2(6)

Agreement with Hankele and Zeppenfeld.

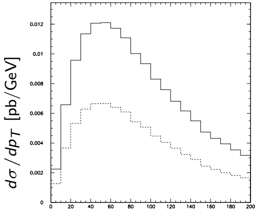
$pp \to VVV$ NLO



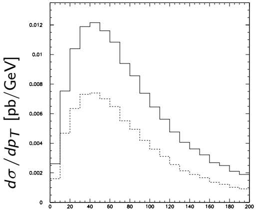
ZZZ



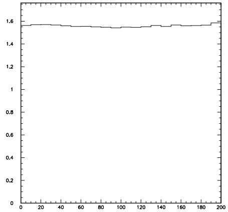
W^+ZZ



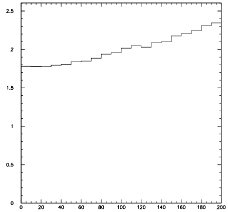
W^+W^-Z



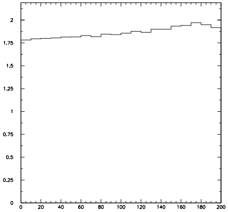
$W^+W^-W^+$



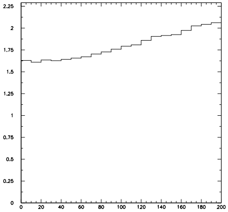
ZZZ



W^+ZZ

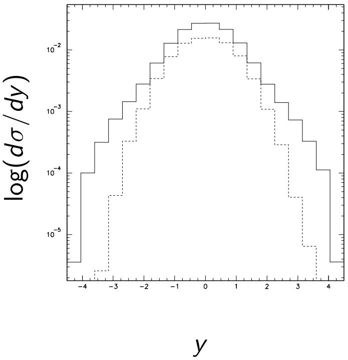


W^+W^-Z

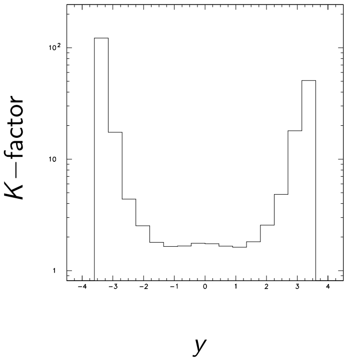


$W^+W^-W^+$

$pp \to VVV$ NLO



$\log(d\sigma/dy)$
10^{-2}
10^{-3}
10^{-4}
10^{-5}
−4 −3 −2 −1 0 1 2 3 4
y



K−factor
10^{2}
10
1
−4 −3 −2 −1 0 1 2 3 4
y

$pp \to VVV$ NLO

scale	σ_B	σ_{NLO}	K
$\mu = M/2$	82.7(5)	153.2(6)	1.85
$\mu = M$	81.4(5)	144.5(6)	1.77
$\mu = 2M$	81.8(5)	139.1(6)	1.70

scale	σ_B	σ_{NLO}	K
$\mu = M/2$	20.2(1)	43.0(2)	2.12
$\mu = M$	20.0(1)	39.7(2)	1.99
$\mu = 2M$	19.7(1)	37.8(2)	1.91

- Still using Feynman Graphs, but a new (OPP) reduction approach

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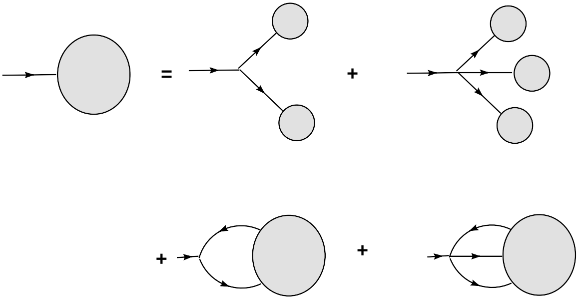
- Unitarity-like approach

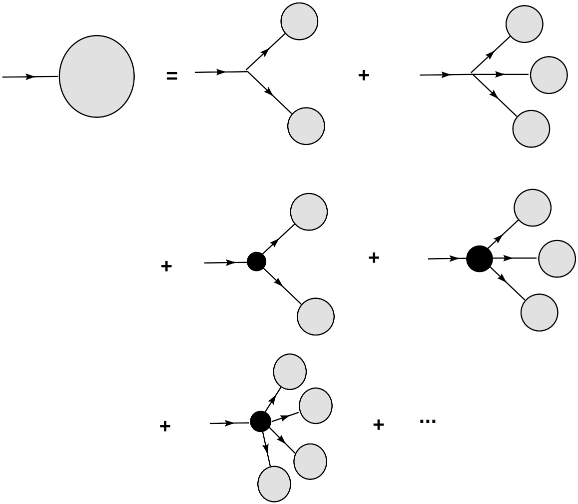
One Loop Amplitude Calculation

- Still using Feynman Graphs, but a new (OPP) reduction approach
- Unitarity-like approach
- Dyson-Schwinger recursion

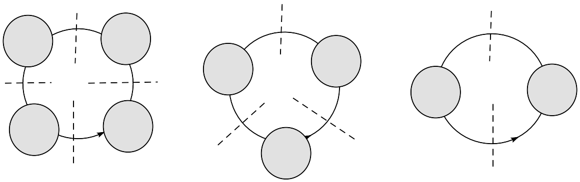
Amplitude calculation-I



Amplitude calculation-II



Amplitude calculation-III



HELAC - PHEGAS : automatic helicity amplitude calculation and parton level generation

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- **Reliable cross section computation and event generation for multiparticle processes, with $\sim$ 10-12 particles in the final state.**

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- – **HELAC:** A.Kanaki and C.G.Papadopoulos, CPC 132 (2000) 306, hep-ph/0002082.
  Matrix element computation algorithm, based on Dyson-Schwinger equations, including: EWK, QCD, fermion masses, reliable arithmetic, running couplings and masses . . ..

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  hep-ph/0012004 and Tokyo 2001,(CPP2001) Computational particle physics, p. 20-25

  T. Gleisberg, *et al.* Eur. Phys. J. C 34 (2004) 173

- Give the process

$$e^- e^+ \longrightarrow \mu\ \bar{\nu}_\mu\ u\ \bar{d}\ \gamma\ \gamma$$

- Give the process

$$e^- e^+ \to \mu \, \bar{\nu}_\mu \; u \; \bar{d} \, \gamma \, \gamma$$

- Define the cuts

$$E_i > E_0 \quad \cos\theta_i < c_0 \quad \cos\theta_{ij} < c_1$$

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$$e^- e^+ \to \mu\, \bar{\nu}_\mu\ u\, \bar{d}\, \gamma\, \gamma$$

- Define the cuts

$$E_i > E_0 \quad \cos\theta_i < c_0 \quad \cos\theta_{ij} < c_1$$

- The cross section is:

$$\sigma = 1.73(5)\ \text{fb} \ \ \text{at}\ \sqrt{s} = 500\ \text{GeV}$$

- **... plus any other kinematical distribution !**

Old **Feynman graphs $\longrightarrow$ computational cost $\sim n!$**

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New **Dyson-Schwinger $\longrightarrow$ computational cost $\sim 3^n$**

P.Draggiotis, R.H.Kleiss and C.G.Papadopoulos, Phys. Lett. B439 (1998) 157

F. Caravaglios and M. Moretti, Phys. Lett. B358 (1995) 332

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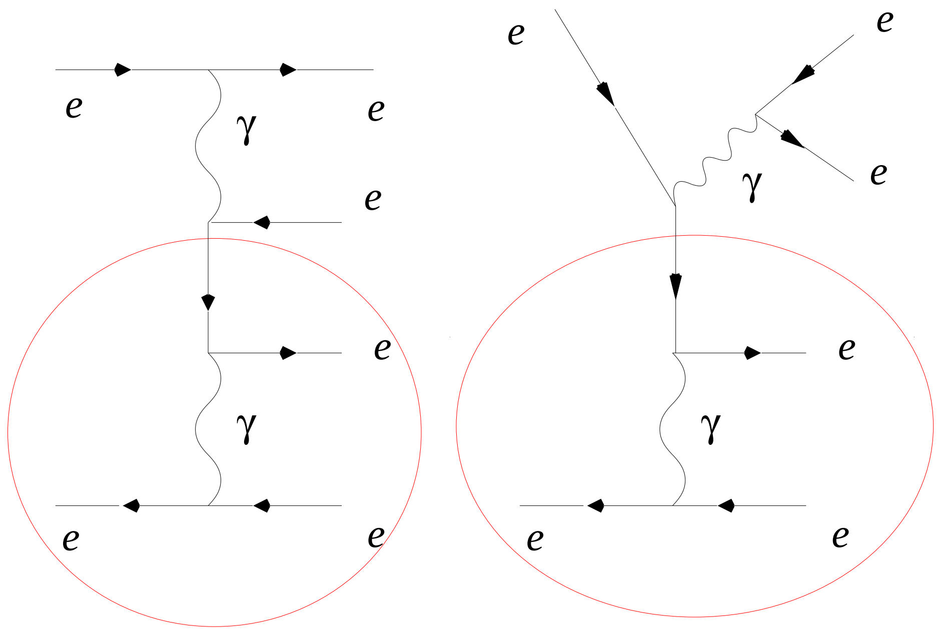
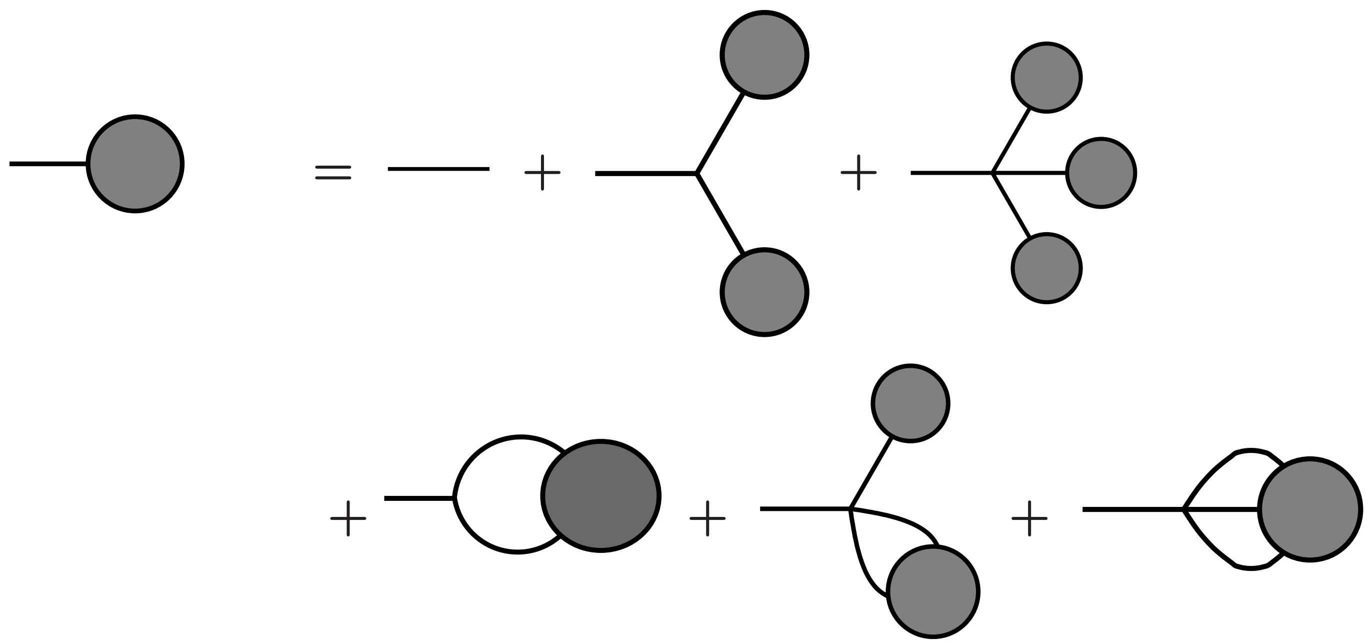
- ***Example:*** $e^- e^+ \to e^- e^+ e^- e^+$ **in QED:**



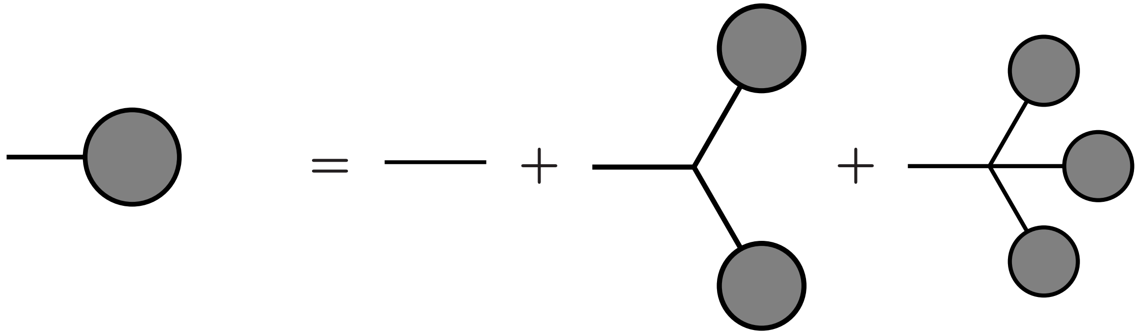
The Dyson-Schwinger recursion

- **Imagine a theory with 3- and 4- point vertices and just one field.**

  Then it is straightforward to write an equation that gives the amplitude for $1 \to n$



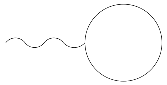
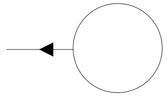
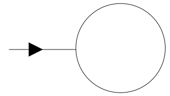
The Dyson-Schwinger recursion



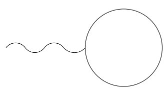
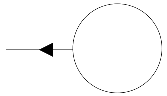
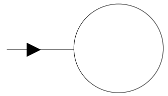
$$a(n) = \delta_{n,1} + \sum \frac{n!}{n_1!n_2!} a(n_1)a(n_2)\delta_{n_1+n_2,n}$$

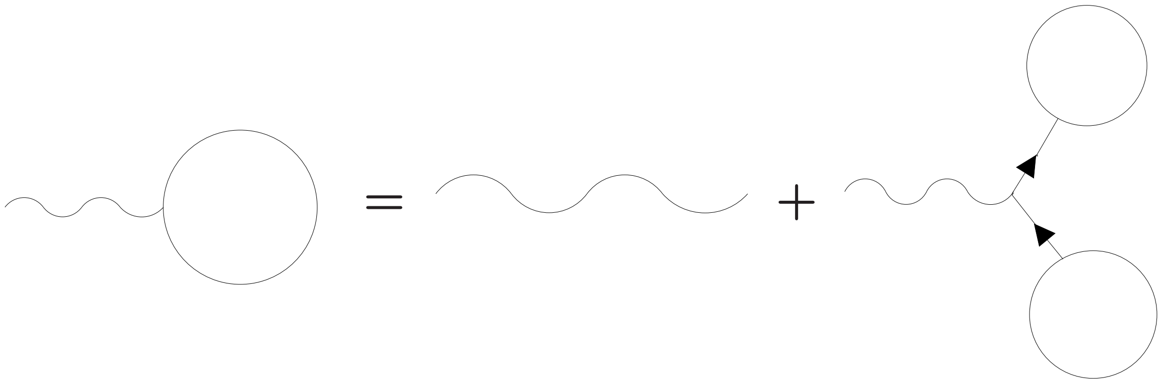
$$+\frac{n!}{n_1!n_2!n_3!} \sum a(n_1)a(n_2)a(n_3)\delta_{n_1+n_2+n_3,n}$$

⇒ **Systematic approach:**

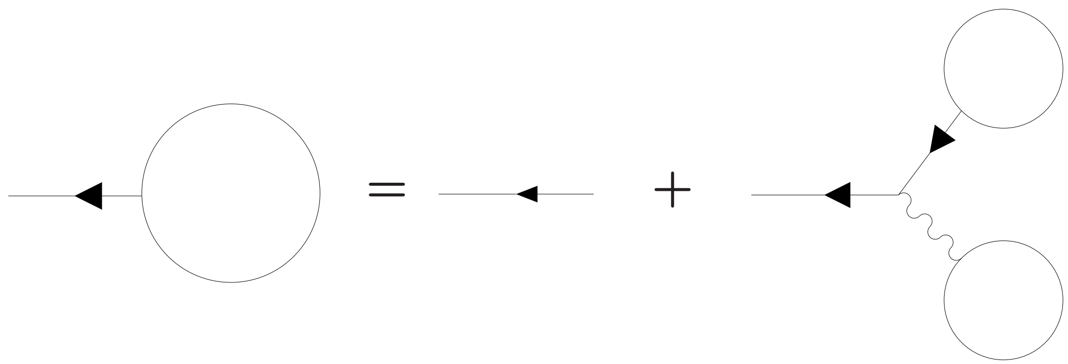
$b_\mu(P)$ =  $\psi(P)$ =  $\bar{\psi}(P)$ = 

⇒ **Systematic approach**:

$b_\mu(P) =$  $\psi(P) =$  $\bar{\psi}(P) =$ 



$$b^\mu(P) = \sum_{i=1}^{n} \delta_{P=p_i} b^\mu(p_i) + \sum_{P=P_1+P_2} (ig)\Pi^\mu_\nu \, \bar{\psi}(P_2)\gamma^\nu \psi(P_1)\epsilon(P_1, P_2)$$



$$\psi(P) = \sum_{i=1}^{n} \delta_{P=p_i} \psi(p_i) + \sum_{P=P_1+P_2} (ig) \not{\psi}(P_2) \frac{(\not{P}_1 + m)}{P_1^2 - m^2} \psi(P_1) \epsilon(P_1, P_2)$$

- **Dirac algebra simplification: 2-dim vs 4-dim and chiral representation, including $m_f \neq 0$.**
- **The sign factor:**

$$\epsilon(P_1, P_2) \longrightarrow \epsilon(m_1, m_2)$$

we define

$$\epsilon(m_1, m_2) = (-1)^{\chi(m_1, m_2)}$$

$$\chi(m_1, m_2) = \sum_{i=n}^{2} \hat{m}_{1i} \left(\sum_{j=1}^{i-1} \hat{m}_{2j} \right)$$

where hatted components are set to 0 if the corresponding external particle is a boson.

- **Full EWK theory, both Unitary and Feynman gauges.**

A. Denner, Fortsch. Phys. 41, 307 (1993).

HELAC

- Construction of the skeleton solution of the Dyson-Schwinger equations. At this stage only integer arithmetic is performed. This is part of the initialization phase.
- Dressing-up the skeleton with momenta, provided by `PHEGAS` and wave functions, propagators, n-point functions in general.
- Unitary and Feynman gauges implemented. Due to multi-precision arithmetic,tests of gauge invariance can be extended to arbitrary precision.
- All fermions masses can be non-zero.
- All Electroweak and QCD vertices are implemented, including Higgs and would-be Goldstone bosons.

Colour Configuration - EWK$\oplus$QCD

Colour Configuration - EWK$\oplus$QCD

- **Ordinary approach $SU(N)$-type**

$$\mathcal{A}^{a_1 \dots a_n} = \sum Tr(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}) \quad A(\sigma_1 \dots \sigma_n)$$

Colour Configuration - EWK⊕QCD

- **Ordinary approach $SU(N)$-type**

$$\mathcal{A}^{a_1 \dots a_n} = \sum Tr(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}) \quad A(\sigma_1 \dots \sigma_n)$$

$$\mathcal{C}_{ij} = \sum Tr(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}) Tr(T^{a_{\sigma'_1}} \dots T^{a_{\sigma'_n}})$$

Colour Configuration - EWK⊕QCD

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Quarks and gluons treated differently

Colour Configuration - EWK$\oplus$QCD

Colour Configuration - EWK$\oplus$QCD

- **New approach $U(N)$-type**

Each color-configuration amplitude is proportional to

$$D_i = \delta_{1,\sigma_i(1)}\delta_{2,\sigma_i(2)}\ldots\delta_{n,\sigma_i(n)}$$

where σ_i represents the i-th permutation of the set $1, 2, \ldots, n$.

Colour Configuration - EWK$\oplus$QCD

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- ⋆ **quarks** $1 \ldots n$
- ⋆ **antiquarks** $\sigma_i(1 \ldots n)$ **and**
- ⋆ **gluons** $= q\bar{q}$

Colour Configuration - EWK⊕QCD

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- ⋆ **antiquarks** $\sigma_i(1 \ldots n)$ **and**
- ⋆ **gluons** $= q\bar{q}$

$$\mathcal{C}_{ij} = \sum D_i D_j = N_c^{\alpha} \ , \ \ \alpha = \langle \sigma_1, \sigma_2 \rangle$$

Colour Configuration - EWK⊕QCD

- **New approach $U(N)$-type**

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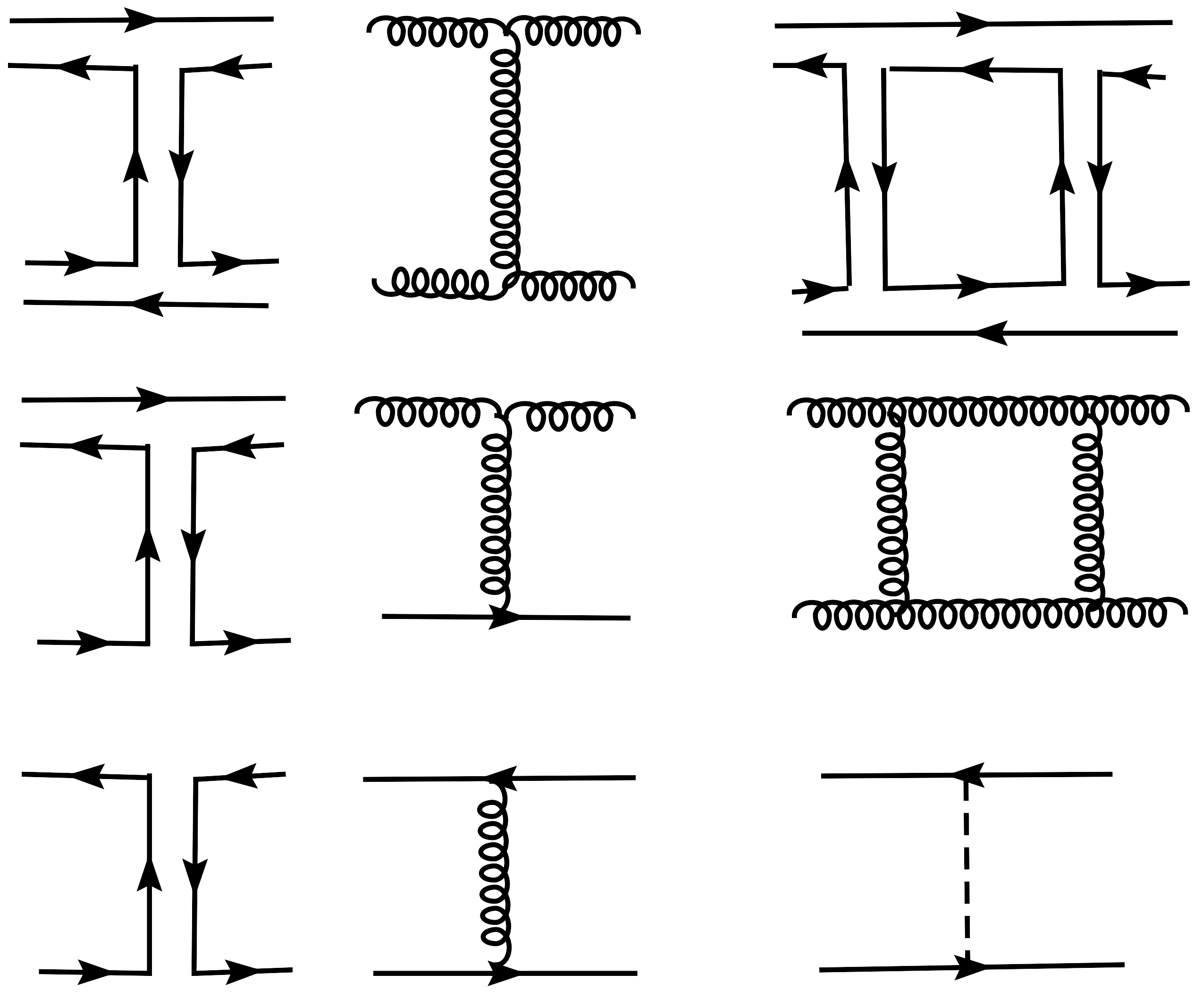
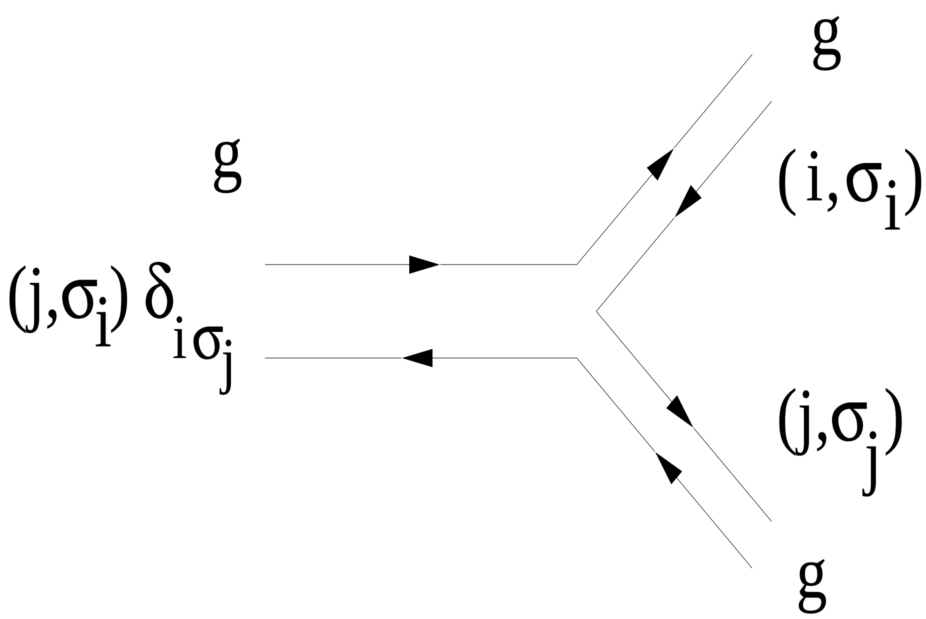
♠ **exact color treatment ⇒ low color charge**

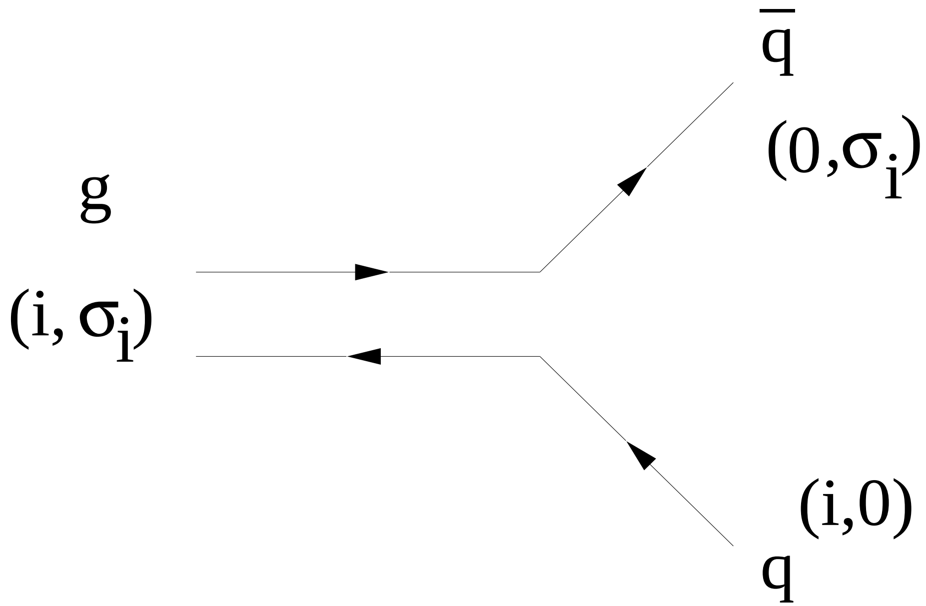
Problem: number of colour connection configurations: $\sim n!$ where n is the number of gluons or $q\bar{q}$ pairs. ⇒ Monte-Carlo over continuous colour-space.





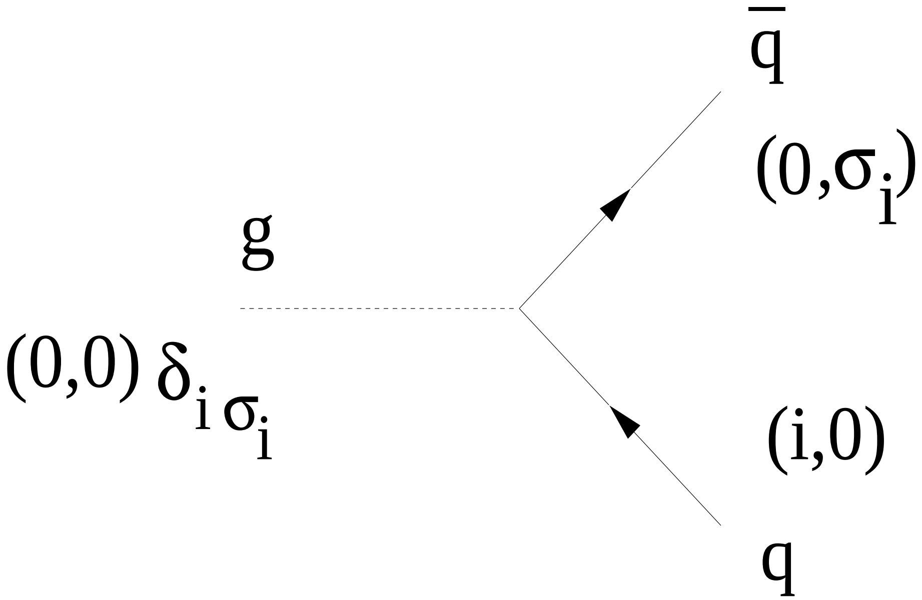
$$\sum f^{abc} t^a_{AB} t^b_{CD} t^c_{EF} = -\frac{i}{4}(\delta_{AD}\delta_{CF}\delta_{EB} - \delta_{AF}\delta_{CB}\delta_{ED})$$

$$\delta_{1\sigma_2}\delta_{2\sigma_3}\delta_{3\sigma_1}$$



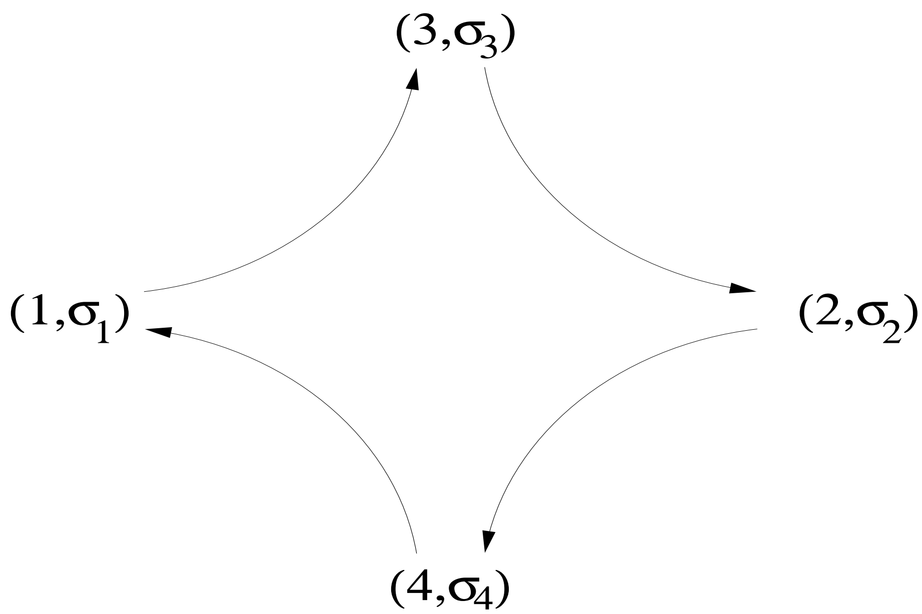
$$\sum t^a_{AB} t^b_{CD} = \frac{1}{2}(\delta_{AD}\delta_{CB} - \frac{1}{N_c}\delta_{AB}\delta_{AC})$$

$$\frac{1}{\sqrt{2}}$$



$$\sum t^{a}_{AB} t^{b}_{CD} = \frac{1}{2}\left(\delta_{AD}\delta_{CB} - \frac{1}{N_c}\delta_{AB}\delta_{AC}\right)$$

$$\frac{1}{\sqrt{2N_c}}$$



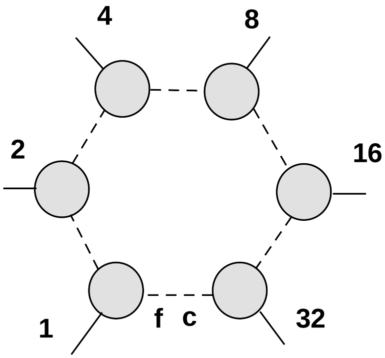
$$\delta_{1\sigma_3}\delta_{3\sigma_2}\delta_{2\sigma_4}\delta_{4\sigma_1}$$

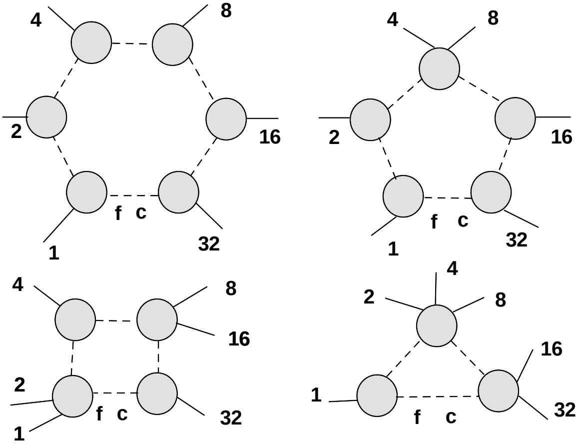
$$2g_{12}g_{34} - g_{13}g_{24} - g_{14}g_{23}$$

Current Status

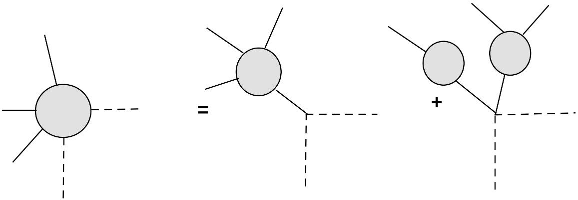
- **Single process mode: all SM processes. Only limitation memory and CPU cost ! to be judged by the user. Experience with as many as 10 particles in the final state.**
- **Summation over processes mode: all SM processes with fl_{ini} and fl_{fin} flavors for 'jets'. Only limitation memory and CPU cost ! to be judged by the user. Parallelism !**
- **Complete generation for pp and $p\bar{p}$ collisions, including all sub-processes. We do not exclude any processes!**
- **Interfacing with Pythia, including CKKW-like reweighting and use of `UPVETO` à la MLM.**
- **Extra version with HG^n and $H\gamma^n$ couplings**

HELAC 1-LOOP

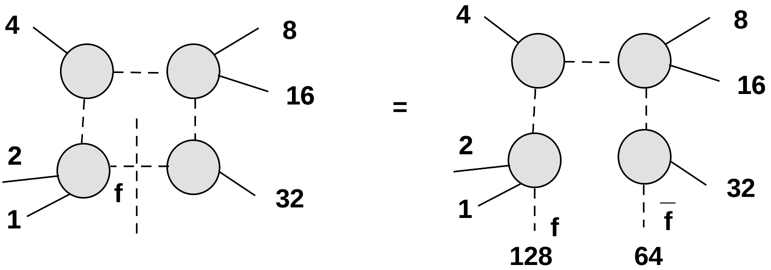



4
8
2
16
f c
1
32
4
8
2
16
f c
1
32
4
8
16
2
f c
1
32
4
2
8
16
1
f c
32

HELAC 1-LOOP



HELAC 1-LOOP



HELAC Color treatment

$$\mathcal{M}^{a_1,i_2,\ldots,i_k}_{j_2,\ldots,j_k} t^{a_1}_{i_1 j_1} \to \mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}$$

$$\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k} = \sum_\sigma \delta_{i_{\sigma_1},j_1} \delta_{i_{\sigma_2},j_2} \ldots \delta_{i_{\sigma_k},j_k} A_\sigma$$

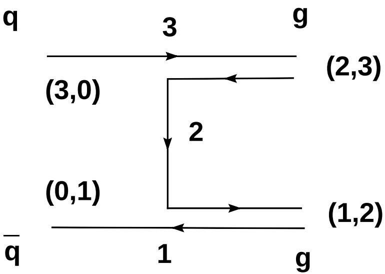
$$\sum_{\{i\},\{j\}} |\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}|^2$$

$$\sum_{\sigma,\sigma'} A^*_\sigma \mathcal{C}_{\sigma,\sigma'} A_{\sigma'}$$

$$\mathcal{C}_{\sigma,\sigma\prime} \equiv \sum_{\{i\},\{j\}} \delta_{i_{\sigma_1},j_1} \delta_{i_{\sigma_2},j_2} \ldots \delta_{i_{\sigma_k},j_k} \delta_{i_{\sigma'_1},j_1} \delta_{i_{\sigma'_2},j_2} \ldots \delta_{i_{\sigma'_k},j_k}$$

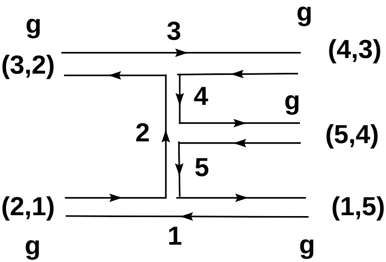
$$(x_1, y_1) \dots (x_n, y_n)$$

where y_i take the values $\{1, 2, \dots, n_l\}$ if i is a gluon or an outgoing quark (incoming anti-quark) otherwise $y_i = 0$, whereas x_i take the values $\{\sigma_1, \sigma_2, \dots, \sigma_{n_l}\}$ if i is a gluon or an incoming quark (outgoing anti-quark) otherwise $x_i = 0$. So for instance for a $q\bar{q} \to gg$ process, $n_l = 3$ and a possible color connection is given by $(3, 0)(0, 1)(1, 2)(2, 3)$



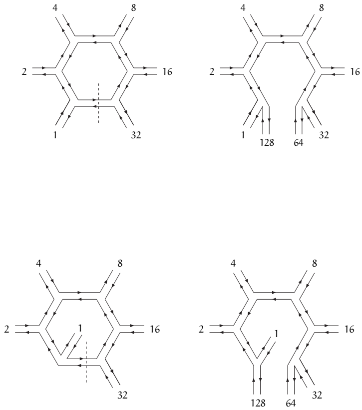
HELAC Color treatment

whereas for $gg \to ggg$, $n_l = 5$ and a possible color connection is given by $(2,1)(3,2)(4,3)(5,4)(1,5)$



$$\mathcal{C}_{\sigma,\sigma'} = N_c^{m(\sigma,\sigma')}$$

where $m(\sigma, \sigma')$ count the number of common cycles of the two permutations.


4
8
2
16
1
32
4
8
2
16
1
128
64
32
4
8
2
1
16
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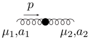
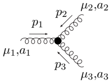
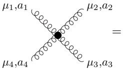
8
2
1
16
128
64
32



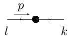
$$= \frac{ig^2 N_{col}}{48\pi^2}\,\delta_{a_1a_2}\left[\,\frac{p^2}{2}g_{\mu_1\mu_2} + \lambda_{HV}\left(g_{\mu_1\mu_2}p^2 - p_{\mu_1}p_{\mu_2}\right) + \frac{N_f}{N_{col}}\,(p^2 - 6\,m_q^2)\,g_{\mu_1\mu_2}\right]$$



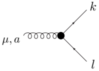
$$= -\frac{g^3 N_{col}}{48\pi^2}\,\left(\frac{7}{4} + \lambda_{HV} + 2\frac{N_f}{N_{col}}\right)\,f^{a_1a_2a_3}\,V_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)$$



$$\begin{aligned}= -\frac{ig^4 N_{col}}{96\pi^2}\sum_{P(234)}\Bigg\{\Bigg[\,&\frac{\delta_{a_1a_2}\delta_{a_3a_4} + \delta_{a_1a_3}\delta_{a_4a_2} + \delta_{a_1a_4}\delta_{a_2a_3}}{N_{col}}\\ &+4\,Tr(t^{a_1}t^{a_3}t^{a_2}t^{a_4} + t^{a_1}t^{a_4}t^{a_2}t^{a_3})\,(3+\lambda_{HV})\\ &-Tr(\{t^{a_1}t^{a_2}\}\{t^{a_3}t^{a_4}\})\,(5+2\lambda_{HV})\Bigg]\,g_{\mu_1\mu_2}g_{\mu_3\mu_4}\\ +12\frac{N_f}{N_{col}}Tr(t^{a_1}t^{a_2}t^{a_3}t^{a_4})&\left(\frac{5}{3}g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_2\mu_3}g_{\mu_1\mu_4}\right)\Bigg\}\end{aligned}$$

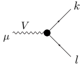


$$= \frac{ig^2}{16\pi^2}\,\frac{N_{col}^2-1}{2N_{col}}\delta_{kl}(-\not p + 2m_q)\,\lambda_{HV}$$

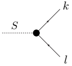


$$= \frac{ig^3}{16\pi^2}\,\frac{N_{col}^2-1}{2N_{col}}\,t^a_{kl}\gamma_\mu\,(1+\lambda_{HV})$$

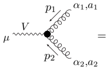
HELAC R2 TERMS



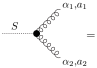
$$= -\frac{g^2}{16\pi^2}\frac{N_{col}^2-1}{2N_{col}}\,\delta_{kl}\gamma_\mu\,(v+a\gamma_5)\,(1+\lambda_{HV})$$



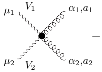
$$= -\frac{g^2}{8\pi^2}\frac{N_{col}^2-1}{2N_{col}}\,\delta_{kl}\,(c+d\gamma_5)\,(1+\lambda_{HV})$$



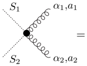
$$= a\,\frac{ig^2}{12\pi^2}\,\delta_{a_1a_2}\epsilon_{\mu\alpha_1\alpha_2\beta}\,(p_1-p_2)^\beta$$



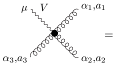
$$= c\,\frac{g^2}{8\pi^2}\,\delta_{a_1a_2}g_{\alpha_1\alpha_2}\,m_q$$



$$= -\frac{ig^2}{24\pi^2}\,\delta_{a_1a_2}(v_1v_2+a_1a_2)\,(g_{\mu_1\mu_2}g_{\alpha_1\alpha_2}+g_{\mu_1\alpha_1}g_{\mu_2\alpha_2}+g_{\mu_1\alpha_2}g_{\mu_2\alpha_1})$$



$$= \frac{ig^2}{8\pi^2}\,\delta_{a_1a_2}\,(c_1c_2-d_1d_2)\,g_{\alpha_1\alpha_2}$$

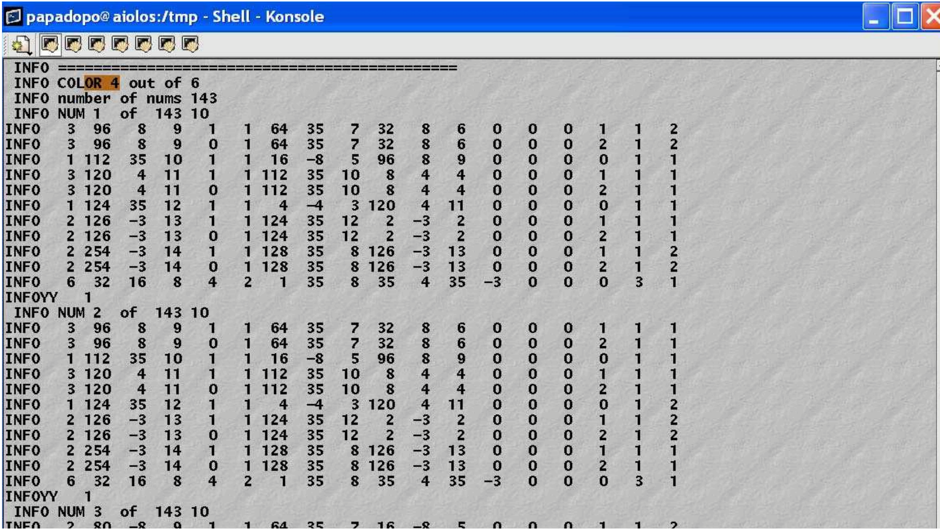


$$= -\frac{g^3}{24\pi^2}\,\{v\,Tr(t^{a_1}\{t^{a_2}t^{a_3}\})(g_{\mu\alpha_1}g_{\alpha_2\alpha_3}+g_{\mu\alpha_2}g_{\alpha_1\alpha_3}+g_{\mu\alpha_3}g_{\alpha_1\alpha_2})$$

$$-i9a\,[Tr(t^{a_1}t^{a_2}t^{a_3})-Tr(t^{a_1}t^{a_3}t^{a_2})]\,\epsilon_{\mu\alpha_1\alpha_2\alpha_3}\}$$

```
  kmax,ncc,imax 6 24 6 10
  INFO ==========================================
  INFO COLOR 1 out of 6
INFO    1  12  35   7   1   1   4  -4   3   8   4   4   0   0   0   0   1   2
INFO    1  48  35   8   1   1  16  -8   5  32   8   6   0   0   0   0   1   2
INFO    2  14  -3   9   1   1  12  35   7   2  -3   2   0   0   0   1   1   2
INFO    2  14  -3   9   0   1  12  35   7   2  -3   2   0   0   0   2   1   2
INFO    2  28  -8  10   1   1  12  35   7  16  -8   5   0   0   0   1   1   2
INFO    2  28  -8  10   0   1  12  35   7  16  -8   5   0   0   0   2   1   2
INFO    3  44   8  11   1   1  12  35   7  32   8   6   0   0   0   1   1   2
INFO    3  44   8  11   0   1  12  35   7  32   8   6   0   0   0   2   1   2
INFO    2  50  -3  12   1   1  48  35   8   2  -3   2   0   0   0   1   1   2
INFO    2  50  -3  12   0   1  48  35   8   2  -3   2   0   0   0   2   1   2
INFO    2  52  -4  13   1   1  48  35   8   4  -4   3   0   0   0   1   1   2
INFO    2  52  -4  13   0   1  48  35   8   4  -4   3   0   0   0   2   1   2
INFO    3  56   4  14   1   1  48  35   8   8   4   4   0   0   0   1   1   2
INFO    3  56   4  14   0   1  48  35   8   8   4   4   0   0   0   2   1   2
INFO    1  60  35  15   1   4   4  -4   3  56   4  14   0   0   0   0   1   2
INFO    1  60  35  15   2   4  16  -8   5  44   8  11   0   0   0   0   1   2
INFO    1  60  35  15   3   4  28  -8  10  32   8   6   0   0   0   0   1   2
INFO    1  60  35  15   4   4  52  -4  13   8   4   4   0   0   0   0   1   2
INFO    2  62  -3  16   1   3  12  35   7  50  -3  12   0   0   0   1   1   2
INFO    2  62  -3  16   0   3  12  35   7  50  -3  12   0   0   0   2   1   2
INFO    2  62  -3  16   2   3  48  35   8  14  -3   9   0   0   0   1   1   2
INFO    2  62  -3  16   0   3  48  35   8  14  -3   9   0   0   0   2   1   2
INFO    2  62  -3  16   3   3  60  35  15   2  -3   2   0   0   0   1   1   2
INFO    2  62  -3  16   0   3  60  35  15   2  -3   2   0   0   0   2   1   2
  INFO ==========================================
  INFO COLOR 2 out of 6
INFO    1  12  35   7   1   1   4  -4   3   8   4   4   0   0   0   0   1   1
INFO    1  48  35   8   1   1  16  -8   5  32   8   6   0   0   0   0   1   1
```

HELAC 1-LOOP




```
 INFO ==============================================
 INFO COLOR 4 out of 6
 INFO number of nums 143
 INFO NUM 1  of  143 10
INFO   3  96   8   9   1   1  64  35   7  32   8   6   0   0   0   1   1   2
INFO   3  96   8   9   0   1  64  35   7  32   8   6   0   0   0   2   1   2
INFO   1 112  35  10   1   1  16  -8   5  96   8   9   0   0   0   0   1   1
INFO   3 120   4  11   1   1 112  35  10   8   4   4   0   0   0   1   1   1
INFO   3 120   4  11   0   1 112  35  10   8   4   4   0   0   0   2   1   1
INFO   1 124  35  12   1   1   4  -4   3 120   4  11   0   0   0   0   1   1
INFO   2 126  -3  13   1   1 124  35  12   2  -3   2   0   0   0   1   1   1
INFO   2 126  -3  13   0   1 124  35  12   2  -3   2   0   0   0   2   1   1
INFO   2 254  -3  14   1   1 128  35   8 126  -3  13   0   0   0   1   1   2
INFO   2 254  -3  14   0   1 128  35   8 126  -3  13   0   0   0   2   1   2
INFO   6  32  16   8   4   2   1  35   8  35   4  35  -3   0   0   0   3   1
INFOYY    1
 INFO NUM 2  of  143 10
INFO   3  96   8   9   1   1  64  35   7  32   8   6   0   0   0   1   1   1
INFO   3  96   8   9   0   1  64  35   7  32   8   6   0   0   0   2   1   1
INFO   1 112  35  10   1   1  16  -8   5  96   8   9   0   0   0   0   1   1
INFO   3 120   4  11   1   1 112  35  10   8   4   4   0   0   0   1   1   1
INFO   3 120   4  11   0   1 112  35  10   8   4   4   0   0   0   2   1   1
INFO   1 124  35  12   1   1   4  -4   3 120   4  11   0   0   0   0   1   2
INFO   2 126  -3  13   1   1 124  35  12   2  -3   2   0   0   0   1   1   2
INFO   2 126  -3  13   0   1 124  35  12   2  -3   2   0   0   0   2   1   2
INFO   2 254  -3  14   1   1 128  35   8 126  -3  13   0   0   0   1   1   1
INFO   2 254  -3  14   0   1 128  35   8 126  -3  13   0   0   0   2   1   1
INFO   6  32  16   8   4   2   1  35   8  35   4  35  -3   0   0   0   3   1
INFOYY    1
 INFO NUM 3  of  143 10
INFO   2  80  -8   9   1   1  64  35   7  16  -8   5   0   0   0   1   1   2
```

HELAC 1-LOOP

papadopo@aiolos:/tmp - Shell - Konsole

```
 INFO NUM 127  of   143 15
INFO    1  48  35   9   1   1  16  -8   5  32   8   6   0   0   0   0   1   1
INFO    3 112   3  10   1   1  48  35   9  64   3   7   0   0   0   1   1   1
INFO    3 112   3  10   0   1  48  35   9  64   3   7   0   0   0   2   1   1
INFO    1  12  35  11   1   1   4  -4   3   8   4   4   0   0   0   0   1   1
INFO    1 240  35  12   1   1 128  -3   8 112   3  10   0   0   0   0  -1   1
INFO    2 242  -3  13   1   1 240  35  12   2  -3   2   0   0   0   1   1   1
INFO    2 242  -3  13   0   1 240  35  12   2  -3   2   0   0   0   2   1   1
INFO    3 248   4  14   1   1 240  35  12   8   4   4   0   0   0   1   1   1
INFO    3 248   4  14   0   1 240  35  12   8   4   4   0   0   0   2   1   1
INFO    1 252  35  15   1   2   4  -4   3 248   4  14   0   0   0   0   1   1
INFO    4 252  35  15   2   2  12  35  11 240  35  12   0   0   0   0   1   1
INFO    2 254  -3  16   1   2  12  35  11 242  -3  13   0   0   0   1   1   1
INFO    2 254  -3  16   0   2  12  35  11 242  -3  13   0   0   0   2   1   1
INFO    2 254  -3  16   2   2 252  35  15   2  -3   2   0   0   0   1   1   1
INFO    2 254  -3  16   0   2 252  35  15   2  -3   2   0   0   0   2   1   1
INFO    2  48  15   3   3   0   0   0   0   0   0   0   0   0   0   0   2   5
INFOYY     5
 INFO NUM 128  of   143 11
INFO    1  12  35   7   1   1   4  -4   3   8   4   4   0   0   0   0   1   1
INFO    1  48  35   8   1   1  16  -8   5  32   8   6   0   0   0   0   1   1
INFO    2  28  -8   9   1   1  12  35   7  16  -8   5   0   0   0   1   1   1
INFO    2  28  -8   9   0   1  12  35   7  16  -8   5   0   0   0   2   1   1
INFO    3  56   4  10   1   1  48  35   8   8   4   4   0   0   0   1   1   1
INFO    3  56   4  10   0   1  48  35   8   8   4   4   0   0   0   2   1   1
INFO    1  60  35  11   1   3   4  -4   3  56   4  10   0   0   0   0   1   1
INFO    4  60  35  11   2   3  12  35   7  48  35   8   0   0   0   0   1   1
INFO    1  60  35  11   3   3  28  -8   9  32   8   6   0   0   0   0   1   1
INFO   25  62  -3  12   1   1  60  35  11   2  -3   2   0   0   0   1   1   1
INFO   25  62  -3  12   0   1  60  35  11   2  -3   2   0   0   0   2   1   1
INFO    0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   1
INFOYY     1
 INFO NUM 129  of   143 12
INFO   23  12  35   7   1   1   4  -4   3   8   4   4   0   0   0   0   1   1
```

HELAC RESULTS

$pp \rightarrow t\bar{t}b\bar{b}$			
$u\bar{u} \rightarrow t\bar{t}b\bar{b}$			
	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
HELAC-1L	-2.347908989000179E-07	-2.082520105681483E-07	3.909384299635230E-07
$I(\epsilon)$	-2.347908989000243E-07	-2.082520105665445E-07	
$gg \rightarrow t\bar{t}b\bar{b}$			
HELAC-1L	-1.435108168334016E-06	-2.085070773763073E-06	3.616343483497464E-06
$I(\epsilon)$	-1.435108168334035E-06	-2.085070773651439E-06	

	p_x	p_y	p_z	E
$u(g)$	0	0	250	250
$\bar{u}(g)$	0	0	-250	250
t	12.99421901255723	-9.591511769543683	75.05543670827210	190.1845561691092
$\bar{t}$	53.73271578143694	-0.2854146459513714	17.68101382654795	182.9642163285034
b	-41.57664370692741	3.895531135098977	-91.94931862397770	100.9874727883170
$\bar{b}$	-25.15029108706678	5.981395280396083	-0.7871319108423604	25.86375471407044

HELAC RESULTS

$pp \to VVb\bar{b}$ and $pp \to VV+$ 2 jets			
$u\bar{u} \to W^+W^-b\bar{b}$			
	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
HELAC-1L	-2.493916939359002E-07	-4.885901774740355E-07	1.592538533368835E-07
$I(\epsilon)$	-2.493916939359001E-07	-4.885901774752593E-07	
$gg \to W^+W^-b\bar{b}$			
HELAC-1L	-2.686310592221201E-07	-6.078682316434646E-07	-2.431624440346638E-07
$I(\epsilon)$	-2.686310592221206E-07	-6.078682340168020E-07	

	p_x	p_y	p_z	E
$u(g)$	0	0	250	250
$\bar{u}(g)$	0	0	-250	250
W^+	22.40377113462118	-16.53704884550758	129.4056091248114	154.8819879118765
W^-	92.64238702192333	-0.4920930146078141	30.48443210132545	126.4095336206695
b	-71.68369328357026	6.716416578342183	-158.5329205583824	174.1159068988160
$\bar{b}$	-43.36246487297426	10.31272528177322	-1.357120667754454	44.59257156863792

HELAC RESULTS

	$pp \to V+$ 3 jets		
	$u\bar{d} \to W^+ggg$		
	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
HELAC-1L	-1.995636628164684E-05	-5.935610843551600E-05	-5.323285370666314E-05
$I(\epsilon)$	-1.995636628164686E-05	-5.935610843566534E-05	
	$u\bar{u} \to Zggg$		
HELAC-1L	-7.148261887172997E-06	-2.142170009323704E-05	-1.906378375774021E-05
$I(\epsilon)$	-7.148261887172976E-06	-2.142170009540120E-05	

	p_x	p_y	p_z	E
u	0	0	250	250
$\bar{d}$	0	0	-250	250
W^+	23.90724239064912	-17.64681636854432	138.0897548661186	162.5391101447744
g	98.85942812363483	-0.5251163702879512	32.53017998659339	104.0753327455388
g	-76.49423931754684	7.167141557113385	-169.1717405928078	185.8004692730082
g	-46.27243119673712	11.00479118171890	-1.448194259904179	47.58508783667868

HELAC RESULTS

$pp \to t\bar{t} +$ 2 jets			
$u\bar{u} \to t\bar{t}gg$			
	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
HELAC-1L	-6.127108113312741E-05	-1.874963444741646E-04	-3.305349683690902E-04
$I(\epsilon)$	-6.127108113312702E-05	-1.874963445081074E-04	
$gg \to t\bar{t}gg$			
HELAC-1L	-3.838786514961561E-04	-9.761168899507888E-04	-5.225385984750410E-04
$I(\epsilon)$	-3.838786514961539E-04	-9.761168898436521E-04	

	p_x	p_y	p_z	E
$u(g)$	0	0	250	250
$\bar{u}(g)$	0	0	-250	250
t	12.99421901255723	-9.591511769543683	75.05543670827210	190.1845561691092
$\bar{t}$	53.73271578143694	-0.2854146459513714	17.68101382654795	182.9642163285034
g	-41.57664370692741	3.895531135098977	-91.94931862397770	100.9874727883170
g	-25.15029108706678	5.981395280396083	-0.7871319108423604	25.86375471407044

HELAC RESULTS

$pp \to b\bar{b}b\bar{b}$			
$u\bar{u} \to b\bar{b}b\bar{b}$			
	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
HELAC-1L	-9.205269484951069E-08	-2.404679886692200E-07	-2.553568662778129E-07
$I(\epsilon)$	-9.205269484951025E-08	-2.404679886707971E-07	
$gg \to b\bar{b}b\bar{b}$			
HELAC-1L	-2.318436429821683E-05	-6.958360737366907E-05	-7.564212339279291E-05
$I(\epsilon)$	-2.318436429821662E-05	-6.958360737341511E-05	

	p_x	p_y	p_z	E
$u(g)$	0	0	250	250
$\bar{u}(g)$	0	0	-250	250
b	24.97040523056789	-18.43157602837212	144.2306511496888	147.5321146846735
$\bar{b}$	103.2557390255471	-0.5484684659584054	33.97680766420219	108.7035966213640
b	-79.89596300367462	7.485866671764871	-176.6948628845280	194.0630765341365
$\bar{b}$	-48.33018125244035	11.49417782256567	-1.512595929362970	49.70121215982584

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A generic NLO calculator *ante portas*

Tools 2009 ?

BlackHat

C.F. Berger, Z. Bern, L.J. Dixon, F. Febres Cordero, D. Forde, H. Ita, D.A. Kosower, D. Maitre, arXiv:0803.4180 [hep-ph]

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HELAC-1LOOP

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