

Loop integrals, integration-by-parts and MATAD

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Outline

- I. Integration-by-parts
- II. MATAD
- III. Examples

I.

Integration-by-Parts (IBP)

A. Notation

- $$F(n_1, \dots, n_N) = \int \frac{d^d k_1 \cdots d^d k_l}{(-p_1^2 + m_1^2)^{n_1} \cdots (-p_N^2 + m_N^2)^{n_N}}$$

N — number of propagators

n_i — indices

l — number of loops

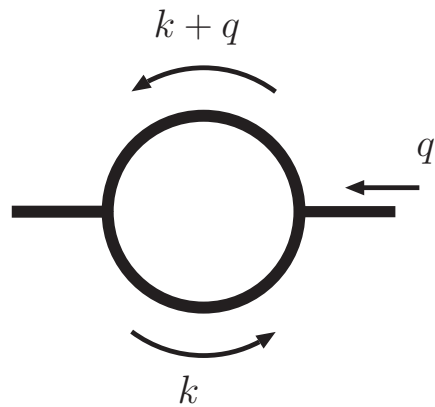
p_i — linear combinations of
loop momenta k_i and
external momenta q_i

- Dimensional Regularization

- $d = 4 - 2\epsilon$

- scaleless integrals = 0

B. Example: 1-loop self energy



$$= F(n_1, n_2) = \int d^d k \frac{1}{(-p_1^2 + m_1^2)^{n_1} (-p_2^2 + m_2^2)^{n_2}}$$

$$p_1 = k, \quad p_2 = k + q$$

special case:

● $m_1 = m_2 = 0: [n_1 = a, n_2 = b]$

$$I_q(a, b) = i\pi^{d/2} (-q^2)^{d/2 - a - b} \cdot \hat{I}(a, b)$$

$$\hat{I}(a, b) = \frac{\Gamma(a + b - d/2) \Gamma(d/2 - a) \Gamma(d/2 - b)}{\Gamma(a) \Gamma(b) \Gamma(d - a - b)}$$

Example: 1-loop self energy

$$\hat{I}(a, b) = \frac{\Gamma(a + b - d/2) \Gamma(d/2 - a) \Gamma(d/2 - b)}{\Gamma(a) \Gamma(b) \Gamma(d - a - b)}$$

- remarks:

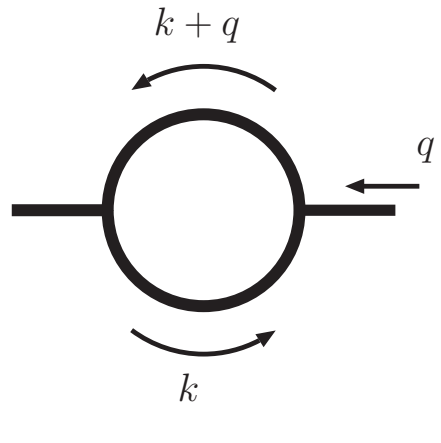
- valid for (almost) all a, b, d

- expansion in ϵ through $z\Gamma(z) = \Gamma(1 + z)$ and

$$\Gamma(1 + \epsilon) = 1 - \epsilon\gamma_E + \frac{\epsilon^2}{2} \left(\gamma_E^2 + \frac{\pi^2}{6} \right) + \dots$$

(γ_E is cancelled by our normalization)

Example: 1-loop self energy

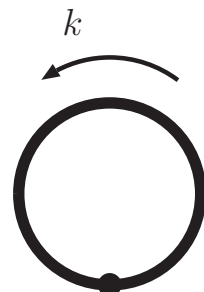


$$= F(n_1, n_2) = \int d^d k \frac{1}{(-p_1^2 + m_1^2)^{n_1} (-p_2^2 + m_2^2)^{n_2}}$$

$$p_1 = k, \quad p_2 = k + q$$

special case:

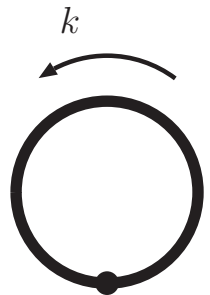
• $n_2 = 0$ [equivalently: $q = 0, m_1 = m_2$]



$$= F(n) = \int d^d k \frac{1}{(-k^2 + m^2)^n}$$

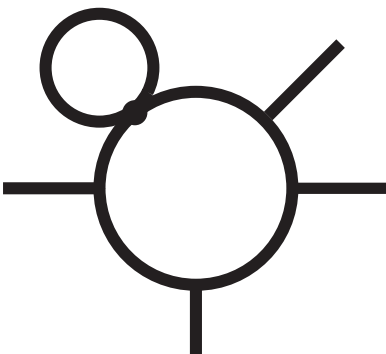
$$= i\pi^{d/2} (m^2)^{d/2-n} \frac{\Gamma(n-d/2)}{\Gamma(n)}$$

Example: 1-loop self energy


$$= F(n) = \int d^d k \frac{1}{(-k^2 + m^2)^n}$$
$$= i\pi^{d/2} (m^2)^{d/2-n} \frac{\Gamma(n-d/2)}{\Gamma(n)}$$

● note: massless tadpoles = 0 (scaleless integrals)

e.g.


$$= 0 \quad \text{in massless case}$$

C. Products and convolutions of 1-loop



→ factorization

$$= \int d^d l \frac{1}{(l^2)^{n_1+n_3} [(l-q)^2]^{n_4}} \underbrace{\int d^d k \frac{1}{(k^2)^{n_5} [(k+l)^2]^{n_2}}}_{\sim (l^2)^{d/2-n_5-n_2} \hat{I}(n_2, n_5)}$$

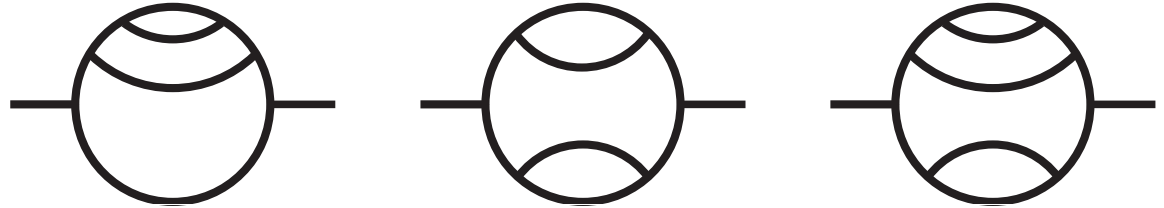
$$\left| a = n_1 + n_3 + n_2 + n_5 - d/2 \right| \sim \hat{I}(n_2, n_5) \int d^d l \frac{1}{(l^2)^a [(l-q)^2]^{n_4}}$$

$$\sim (q^2)^{d-\sum n_i} \cdot \hat{I}(n_1 + n_3 + n_2 + n_5 - d/2, n_4) \cdot \hat{I}(n_2, n_5)$$

[Euclidian momenta]

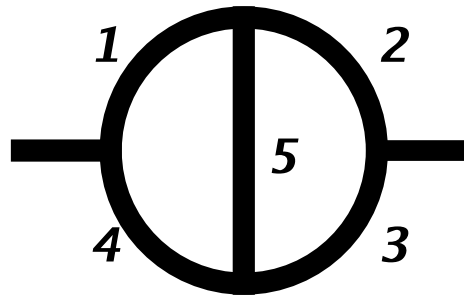
Products and convolutions of 1-loop (2)

- analogously:
(for $m = 0$)



convolutions of 1-loop self-energy diagrams

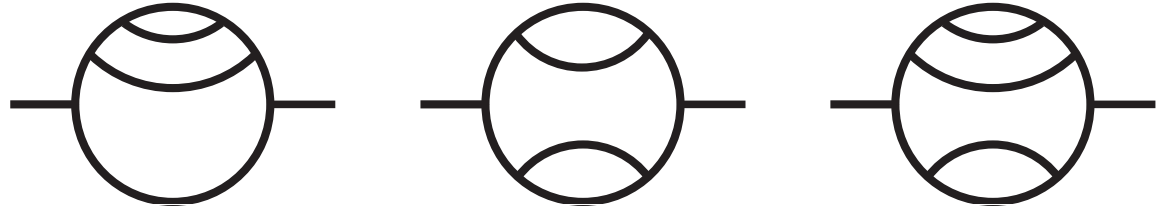
- however:
genuine 2-loop:



“ T_1 ”-topology

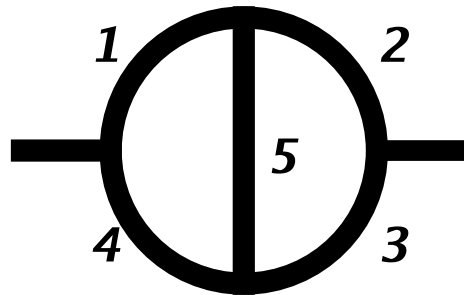
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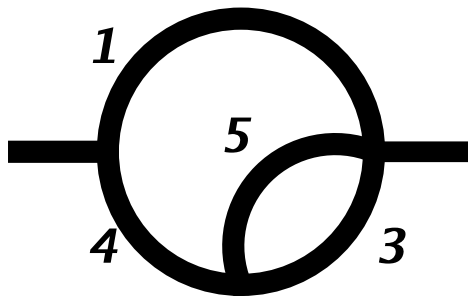
convolutions of 1-loop self-energy diagrams

- however:
genuine 2-loop:



“ T_1 ”-topology

but: shrinking **any** line \Rightarrow convolution of 1-loop!



D. IBP relations

Dimensional regularization:

- Integrals are finite
- Surface terms vanish

⇒

$$\int d^d k \frac{\partial}{\partial k_\mu} p_\mu f(k, p, \dots) = 0$$

p_μ : loop or external momentum

Example: $F(n) = \int d^d k \frac{1}{(-k^2 + m^2)^n}$

Consider:

$$0 = \int d^d k \frac{\partial}{\partial k_\mu} k_\mu \frac{1}{(-k^2 + m^2)^n}$$

D. IBP relations

Example: $F(n) = \int d^d k \frac{1}{(-k^2 + m^2)^n}$

$$0 = \int d^d k \frac{\partial}{\partial k_\mu} k_\mu \frac{1}{(-k^2 + m^2)^n}$$

$$= \int d^d k \frac{d}{(-k^2 + m^2)^n} + \frac{k_\mu (-n) (-2k^\mu)}{(-k^2 + m^2)^{(n+1)}}$$

=

=

D. IBP relations

Example: $F(n) = \int d^d k \frac{1}{(-k^2 + m^2)^n}$

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$$= \int d^d k \frac{d - 2n}{(-k^2 + m^2)^n} + \frac{2nm^2}{(-k^2 + m^2)^{(n+1)}}$$

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$$= \int d^d k \frac{d - 2n}{(-k^2 + m^2)^n} + \frac{2nm^2}{(-k^2 + m^2)^{(n+1)}}$$

$$= (d - 2n)F(n) + 2nm^2 F(n + 1)$$

$$\Leftrightarrow F(n) = -\frac{d - 2n + 2}{(2n - 2)m^2} F(n - 1)$$

\Leftrightarrow Any $F(n)$, $n > 1$ can be expressed by $F(1)$

$$F(n) = 0 \text{ if } n \leq 0$$

$$F(1) = \text{master integral}$$

D. IBP relations

Example: $F(n) = \int d^d k \frac{1}{(-k^2 + m^2)^n}$

$$F(n) = -\frac{d - 2n + 2}{(2n - 2)m^2} F(n - 1)$$

Raising and lowering indices:

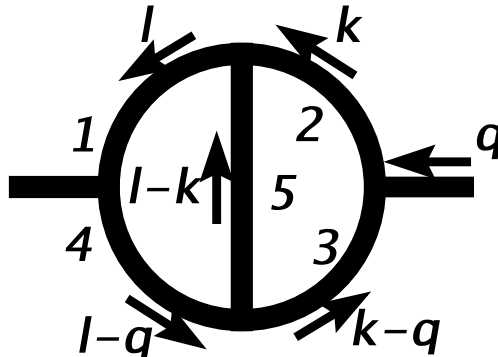
Define operators $\mathbf{1}^\pm, \mathbf{2}^\pm, \dots$:

$$\mathbf{I}^\pm F(n_1, \dots, n_I, \dots) = F(n_1, \dots, n_I \pm 1, \dots)$$



$$F(n) = -\frac{d - 2n + 2}{(2n - 2)m^2} \mathbf{1}^- F(n)$$

E. Topology T_1 (massless)



$$\begin{aligned}
 &= \int d^d l \int d^d k \frac{1}{[l^2]^{n_1} [k^2]^{n_2} [(k-q)^2]^{n_3}} \\
 &\quad \times \frac{1}{[(l-q)^2]^{n_4} [(l-k)^2]^{n_5}}
 \end{aligned}$$

act on integrand with

$$\frac{\partial}{\partial l_\mu} \cdot l_\mu, \quad \frac{\partial}{\partial l_\mu} \cdot k_\mu, \quad \frac{\partial}{\partial l_\mu} \cdot q_\mu, \quad \frac{\partial}{\partial k_\mu} \cdot k_\mu, \quad \frac{\partial}{\partial k_\mu} \cdot l_\mu, \quad \frac{\partial}{\partial k_\mu} \cdot q_\mu,$$

example:
$$\frac{\partial}{\partial l_\mu} \cdot l_\mu = d + l_\mu \cdot \frac{\partial}{\partial l_\mu}$$

$$l_\mu \cdot \frac{\partial}{\partial l_\mu} \frac{1}{[l^2]^{n_1}} = -2n_1 \frac{1}{[l^2]^{n_1}}$$

Topology T_1 (massless)

$$\begin{aligned}
 & \text{Diagram} \equiv \int d^d l \int d^d k \frac{1}{[l^2]^{n_1} [k^2]^{n_2} [(k-q)^2]^{n_3}} \\
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 \end{aligned}$$

$$l_\mu \cdot \frac{\partial}{\partial l_\mu} \frac{1}{[l^2]^{n_1}} = -2n_1 \frac{1}{[l^2]^{n_1}}$$

$$l_\mu \cdot \frac{\partial}{\partial l_\mu} \frac{1}{[(l-q)^2]^{n_4}} = -2n_4 \frac{1}{[(l-q)^2]^{n_4+1}} \left\{ l^2 - l \cdot q \right\}$$

Topology T_1 (massless)

$$\begin{aligned}
 & \text{Diagram} \equiv \int d^d l \int d^d k \frac{1}{[l^2]^{n_1} [k^2]^{n_2} [(k-q)^2]^{n_3}} \\
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$$l_\mu \cdot \frac{\partial}{\partial l_\mu} \frac{1}{[(l-q)^2]^{n_4}} = -2n_4 \frac{1}{[(l-q)^2]^{n_4+1}} \left\{ l^2 + \frac{1}{2} [(l-q)^2 - l^2 - q^2] \right\}$$

Topology T_1 (massless)

$$\begin{aligned}
 & \text{Diagram} \equiv \int d^d l \int d^d k \frac{1}{[l^2]^{n_1} [k^2]^{n_2} [(k-q)^2]^{n_3}} \\
 & \quad \times \frac{1}{[(l-q)^2]^{n_4} [(l-k)^2]^{n_5}}
 \end{aligned}$$

$$l_\mu \cdot \frac{\partial}{\partial l_\mu} \frac{1}{[l^2]^{n_1}} = -2n_1 \frac{1}{[l^2]^{n_1}}$$

$$l_\mu \cdot \frac{\partial}{\partial l_\mu} \frac{1}{[(l-q)^2]^{n_4}} = -2n_4 \underbrace{\frac{1}{[(l-q)^2]^{n_4+1}}}_{4^+} \left\{ \underbrace{l^2}_{1^-} + \frac{1}{2} \left[\underbrace{(l-q)^2}_{4^-} - \underbrace{l^2}_{1^-} - q^2 \right] \right\}$$

$$\Rightarrow [d - 2n_1 - n_4 - n_5 + n_4(q^2 - 1^-)4^+ + n_5(2^- - 1^-)5^+] T_1 = 0$$

IBP identities for T_1

$$\left[\underbrace{d - 2n_1 - n_4 - n_5}_{\text{Diagram 1}} + \underbrace{n_4(q^2)}_{\text{Diagram 2}} - \underbrace{1^- 4^+}_{\text{Diagram 3}} + n_5(2^- - 1^-) 5^+ \right] T_1 = 0$$

task: combine **all** IBP identities such that

$T_1(n_1, n_2, n_3, n_4, n_5) \rightarrow$ “simpler” integrals, i.e.

- convolutions of 1-loop
- low values of n_i

IBP identities for T_1

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task: combine **all** IBP identities such that

$T_1(n_1, n_2, n_3, n_4, n_5) \rightarrow$ “simpler” integrals, i.e.

- convolutions of 1-loop
- low values of n_i

Exercise: compute IBP with “ $\frac{\partial}{\partial l_\mu} k_\mu$ ”

IBP identities for T_1

$$d - 2n_1 + n_4(-1 + (\boxed{q^2} - \mathbf{1}^-) \boxed{4}^+) + n_5(-1 + (-\mathbf{1}^- + \mathbf{2}^-) \mathbf{5}^+) = 0$$

$$n_1(-1 + (-q^2 + \mathbf{4}^-) \mathbf{1}^+) + n_4(1 + (q^2 - \mathbf{1}^-) \mathbf{4}^+) + (-\mathbf{1}^- + \mathbf{2}^- - \mathbf{3}^- + \mathbf{4}^-) n_5 \mathbf{5}^+ = 0$$

$$n_1(-1 + (-\mathbf{2}^- + \mathbf{5}^-) \mathbf{1}^+) + (\boxed{q^2} - \mathbf{1}^- - \mathbf{3}^- + \mathbf{5}^-) n_4 \boxed{4}^+ + n_5(1 + (-\mathbf{1}^- + \mathbf{2}^-) \mathbf{5}^+) = 0$$


$$n_2(-1 + (-\mathbf{1}^- + \mathbf{5}^-) \mathbf{2}^+) + (q^2 - \mathbf{2}^- - \mathbf{4}^- + \mathbf{5}^-) n_3 \mathbf{3}^+ + n_5(1 + (\mathbf{1}^- - \mathbf{2}^-) \mathbf{5}^+) = 0$$


$$n_2(-1 + (-q^2 + \mathbf{3}^-) \mathbf{2}^+) + n_3(1 + (q^2 - \mathbf{2}^-) \mathbf{3}^+) + (\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^- - \mathbf{4}^-) n_5 \mathbf{5}^+ = 0$$

$$d - 2n_2 + n_3(-1 + (q^2 - \mathbf{2}^-) \mathbf{3}^+) + n_5(-1 + (\mathbf{1}^- - \mathbf{2}^-) \mathbf{5}^+) = 0$$

(1) - (3):

$$\left[\underbrace{d - 2n_5 - n_1 - n_4}_{\text{Diagram 1}} - \underbrace{n_1(\mathbf{5}^- - \mathbf{2}^-) \mathbf{1}^+ - n_4(\mathbf{5}^- - \mathbf{3}^-) \mathbf{4}^+}_{\text{Diagram 2}} \right] T_1 = 0$$





F. Triangle rule

$$T(n_1, n_2, n_3, n_4, n_5) = \frac{1}{d - 2n_5 - n_1 - n_4} \times \left[n_1(5^- - 2^-)1^+ + n_4(5^- - 3^-)4^+ \right] T_1(n_1, n_2, n_3, n_4, n_5)$$

→ recurrence relation

example: $n_1 = n_2 = n_3 = n_4 = n_5 = 1$

$$T_1(1, 1, 1, 1, 1) = \frac{1}{d - 4} \left[T_1(2, 1, 1, 1, 0) - T_1(2, 0, 1, 1, 1) + T_1(1, 1, 1, 2, 0) - T_1(1, 1, 0, 2, 1) \right]$$

$$\text{---} \bigcirc \text{---} = \frac{1}{\epsilon} \left[\text{---} \bigcirc \text{---} - \text{---} \bigcirc \bigcirc \text{---} \right] \sim 6\zeta(3)$$

Triangle rule

indices > 1 \rightarrow apply rec.-rel. repeatedly

$$T_1(1, 3, 1, 1, 1) \rightarrow T_1(2, 3, 1, 1, 0)$$

$$T_1(2, 2, 1, 1, 1) \rightarrow T_1(3, 2, 1, 1, 0)$$

$$T_1(1, 3, 1, 2, 0)$$

$$T_1(3, 1, 1, 1, 1) \rightarrow$$

$$T_1(4, 1, 1, 1, 0)$$

$$T_1(1, 3, 0, 2, 1)$$

$$T_1(2, 2, 1, 2, 0)$$

$$T_1(4, 0, 1, 1, 1)$$

$$T_1(2, 2, 0, 2, 1)$$

$$T_1(3, 1, 1, 2, 0)$$

$$T_1(3, 1, 0, 2, 1)$$

\rightarrow generates many terms

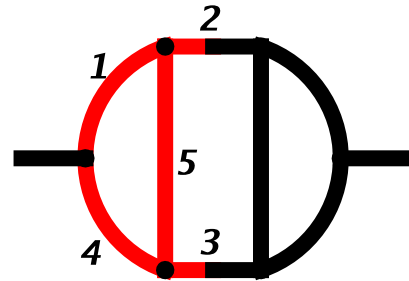
\rightarrow needs computer algebra!

$$\left[n_1(5^- - 2^-)1^+ + n_4(5^- - 3^-)4^+ \right]$$

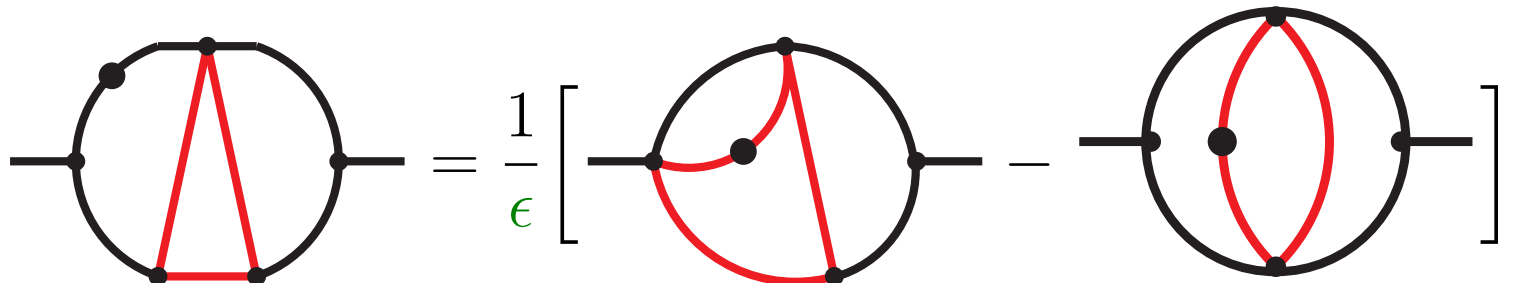
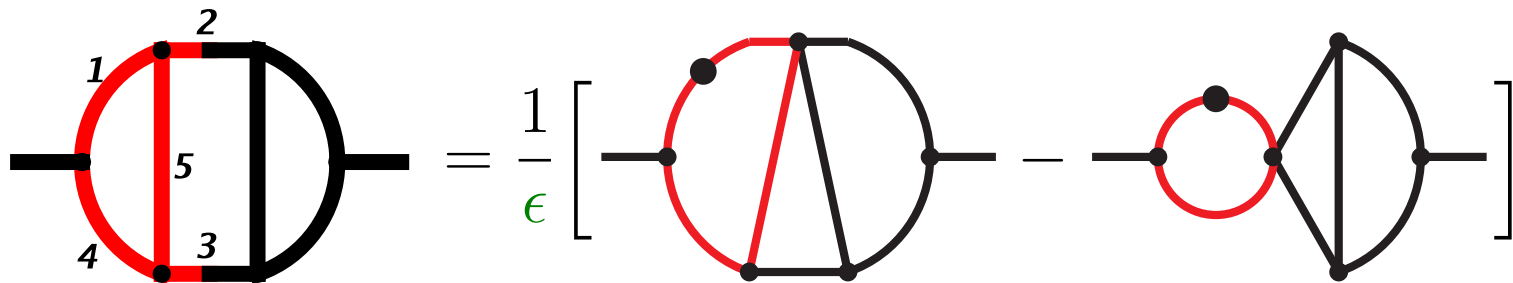
Triangle rule

observation: this rec.-relation applies to **any triangular sub-loop!**

e.g. 3-loop:

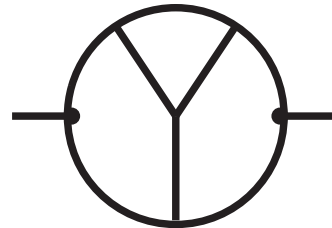


$$d - 2n_5 - n_1 - n_4 = n_1(5^- - 2^-)1^+ + n_4(5^- - 3^-)4^+$$



Triangle rule – exercise

Which convolutions of 1- and 2-loop integrals does the following integral reduce to?



G. Reduction to MIs

- “modern”: Laporta-, Gröbner-, Baikov-, . . . algorithm
- “old-fashioned” (but much more effective):
look (“by hand”) for combinations of IBP relations which systematically reduce complicated integrals to simpler ones

MINCER: massless 2-point function; 1, 2, 3 loops

[Larin, Tkachov, Vermaseren]

Grinder: HQET integrals; 1, 2, 3 loops

[Grozin]

MATAD: vacuum bubbles; 1, 2, 3 loops

[Steinhauser]

II. MATAD

(MAssive TADpoles)

[MATAD3 for FORM3]

MATAD: MAssive TADpoles

- written in FORM (MATAD3 \leftrightarrow FORM3)
- vacuum bubbles up to 3 loops
- propagators: mass M1 or massless
- zero external momentum
- final result up to (incl.) $\mathcal{O}(\epsilon^{3-l})$

MATAD: applications

- moments of $\Pi_\gamma(q^2) \Leftrightarrow m_c, m_b$
- ρ parameter: 3-loop corrections
- decoupling of α_s and m_q
- matching coefficient of eff. Lagrangian ($gg \rightarrow H$,
 $\bar{B} \rightarrow X_s \gamma$)
- $\Gamma(H \rightarrow \gamma\gamma)$
- ...

A. Flowchart

read FD
translate notation
traces
expansion
Wick rotation
...



reduction
MIs
simple integrals



expansion in ϵ
print and save
result



FD



projectors
FRs
...

Input files:

- .dia file: 1 FORM fold/diagram
- GLOBAL: overall settings, for interaction, projectors, ...

“TREAT”-files

- declare file for user-defined variables

A. Flowchart

read FD
translate notation
traces



FD



projectors

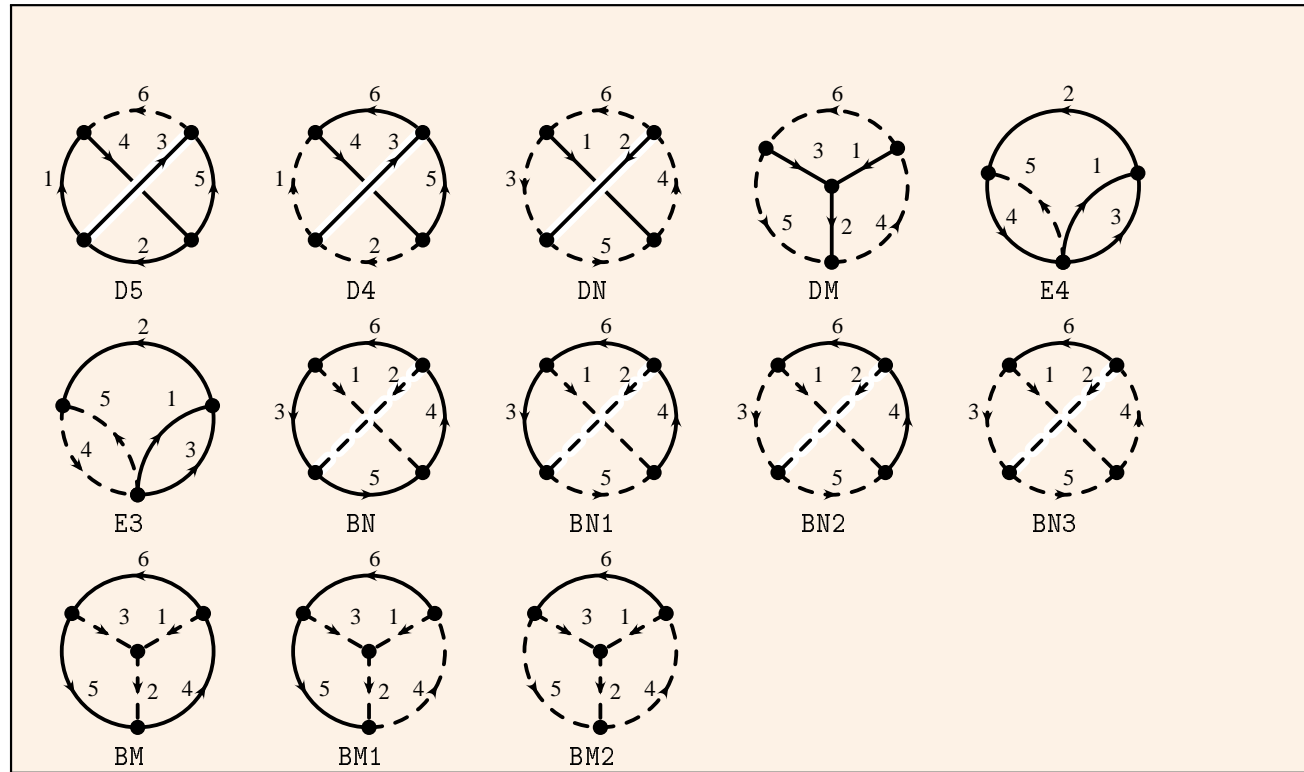
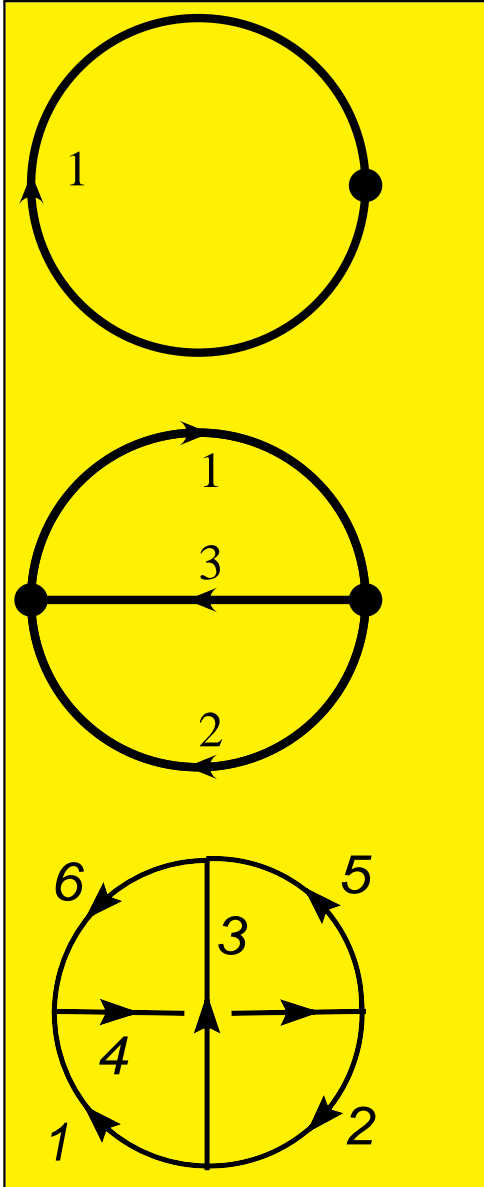
```
*--#[ GLOBAL :  
  #define LOWLIMM1 "-2"  
  #define GAUGE "xi"  
*--#] GLOBAL :  
*--#[ TREAT1 :  
  multiply, deno(3,-2)*( -d_(mu1,mu2)  
    + q1(mu1)*q1(mu2)/q1.q1 );  
  .sort  
*--#] TREAT1 :
```

expansion in ϵ
print and save
result

“TREAT”-files

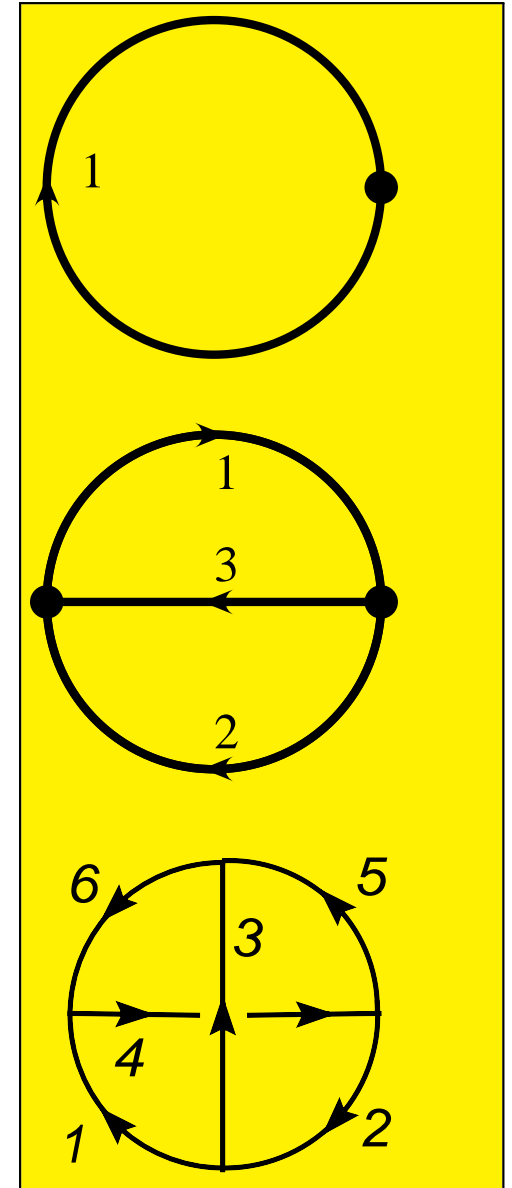
– declare file for user-defined
variables

B. Notation/Topologies



B. Notation

- masses: M_1, M_2, \dots
- $\epsilon M_1 = \left(\frac{\mu^2}{M_1^2}\right)^\epsilon, \dots$
- loop momenta: p_{11}, p_{12}, \dots
- external momenta: q_1, q_2, \dots
- massless scalar denominators:
 $1/p_{11} \cdot p_{11}, 1/p_{12} \cdot p_{12}, \dots$
- massive scalar denominators:
 $s_{11m_1}, s_{12m_1}, \dots; s_{12m_1} \equiv \frac{1}{M_1^2 - p_2^2}$
- internal indices: $\nu_{u1}, \nu_{u2}, \dots$
- external indices: $\mu_{u1}, \mu_{u2}, \dots$
- $\text{deno}(a, b) = \frac{1}{a + b\epsilon}$
- scalar denominator:



$\text{Den}(L, \#, \langle \text{momenta}, \text{masses} \rangle, \text{exp}, \langle \text{momenta}, \text{masses} \rangle), \#, L)$

B. Notation

- scalar denominator:

$\text{Den}(L, \#, \langle \text{momenta}, \text{masses} \rangle, \text{exp}, \langle \text{momenta}, \text{masses} \rangle), \#, L)$

example:

$$\text{Den}(L, 28, +p_{15}, p_{M1}, 28, L) \equiv \frac{1}{-p_5^2}$$

$$\text{Den}(L, 28, -p_{17}, p_{M1}, M1, p_{M1}, 28, L) \equiv \frac{1}{M_1^2 - p_7^2}$$

$$\text{Den}(L, 32, -p_{16}, p_{M1}, M1, p_{M1}, \text{exp}, +q_1, p_{Q1}, 32, L) \equiv \frac{1}{M_1^2 - (-p_6 + q_1)^2} \quad \text{expand in } q_1$$

p_{M1}, p_{Q1} : “powercount” variables; needed for expansion

- spinor lines: $FT1, FT2, \dots$

$$FT1(L, 32, +p_{15}, p_{M1}, M1, p_{M1}, 32, L) \equiv \frac{1}{M_1 - \gamma_\nu p_5^\nu}$$

$$FT1(L, 26, -p_{13}, p_{M1}, M1, p_{M1}, \text{exp}, +q_1, p_{Q1}, 26, L) \equiv \frac{1}{M_1 - \gamma_\nu (-p_3 + q_1)^\nu}$$

- $FT1(\mu_1) \equiv \gamma^{\mu_1}$

B. Notation

- gluon propagator:

$$Dg(\text{nu}13, \text{nu}14, L, 26, +p12, pM1, 26, L) \\ \equiv \frac{-g^{\nu_{13}\nu_{14}} - \text{'GAUGE'} p_2^{\nu_{13}} p_2^{\nu_{14}}}{-p_2^2}$$

- V_{gh} : ghost-gluon vertex
- V_{3g} : 3-gluon vertex
- V_{ggs}, D_{sig} : implementation of 4-gluon vertex
- other vertices: use "TREAT0"

B. Notation

- colour diagram: $f_{\text{qcd}} \dots$
 - fundamental, external: i_1, i_2, \dots
 - fundamental, internal: j_1, j_2, \dots
 - adjoint, external: $a(1), a(2), \dots$
 - adjoint, internal: $b(1), b(2), \dots$
 - $d_-(i_1, i_2) \equiv \delta^{i_1, i_2}$
 - $\text{GM}(a(1), i_1, i_2) \equiv \lambda_{i_1, i_2}^{a_1}$
 - $\text{prop}(a(1), a(2)) \equiv \delta^{a_1, a_2}$

More notation/convention

- imaginary unit i :

- $\frac{1}{M^2 - p^2} \xrightarrow{W.R.} \frac{1}{M^2 + p^2}$

- all vertices: $i \cdot (\dots)$

all propagators: $\frac{1}{i} \cdot (\dots)$

⇒ don't care about "i" of internal lines

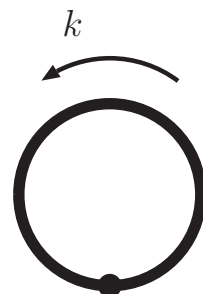
- note: $i \left(\frac{1}{i} \int d^d k_j \right)^{l=1,2,3} \xrightarrow{W.R.} i \left(\int d^d k_{j,E} \right)^{l=1,2,3}$
overall i (independent of l) is omitted

- γ_E

multiply with $(e^{\gamma_E \epsilon})^l \Rightarrow \gamma_E$ disappears from final result

- normalization of the integrals: $\int \frac{d^d k}{i\pi^{d/2}}$

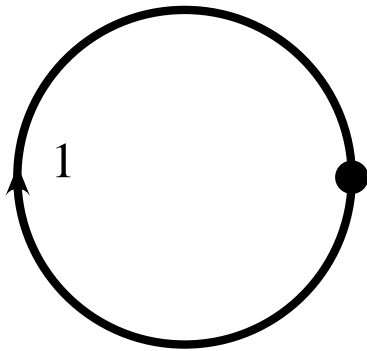
e.g.:



$$= e^{\gamma_E \epsilon} eM1 (M1^2)^{(2-n)} \frac{\Gamma(n - d/2)}{\Gamma(n)}$$

C. Examples

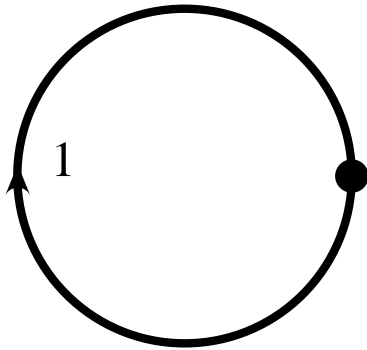
1-loop tadpole



```
*--#[ d111 :  
s11m1  
;  
#define INT1 "tad11"  
#define MASS1 "M1"  
*--#] d111 :  
  
*--#[ expd111 :  
multiply, eM1*pM1^4;  
#call denoexp{M1}  
#include matad.info # time  
*--#] expd111 :
```


C. Examples

1-loop tadpole

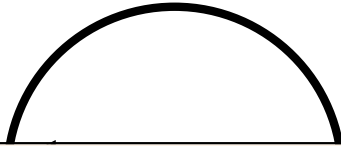


```
> cd calc3
```

```
> startform3 -S form.set -d CLASS=c_misc -d PROBLEM=test  
-d LOOPS=1 -d DIAFILE=test.dia  
-d RESDIR=results -d DIAGRAM=d111 generic/maindia.frm
```

C. Examples

1-loop tadpole

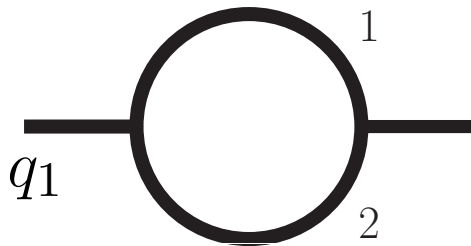


$$\begin{aligned} d1l1 = & \\ & + ep^{-1} * (- M1^2 * eM1) \\ & + ep * (- 1/2 * M1^2 * eM1 * z2 - M1^2 * eM1) \\ & + ep^2 * (- 1/2 * M1^2 * eM1 * z2 + 1/3 * M1^2 * eM1 * z3 - M1^2 * eM1) \\ & - M1^2 * eM1; \end{aligned}$$

```
-d RESDIR=results -d DIAGRAM=d1l1 generic/maindia.frm
```

C. Examples (2)

1-loop expanded in external momentum



```
*--#[ d112 :
```

```
Den(L, 11, p11, pM1, M1, pM1, 11, L) *
```

```
Den(L, 22, p11, pM1, M1, pM1, exp, q1, pQ1, 22, L)
```

```
;
```

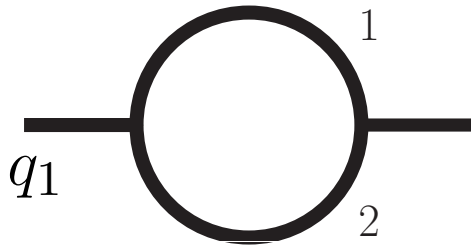
```
#define INT1 "tad11"
```

```
#define MASS1 "M1"
```

```
*--#] d112 :
```

C. Examples (2)

1-loop expanded in external momentum

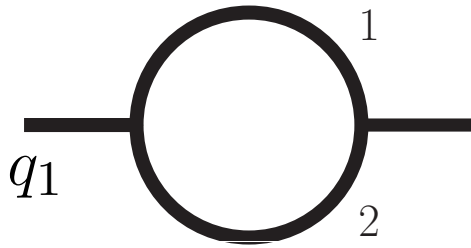


```
> cd calc3
```

```
> startform3 -S form.set -d CLASS=c.misc -d PROBLEM=test  
-d LOOPS=1 -d DIAFILE=test.dia  
-d RESDIR=results -d DIAGRAM=d112 generic/maindia.frm
```

C. Examples (2)

1-loop expanded in external momentum



Q1: Euclid. momentum

$$\begin{aligned} d1|2 = & \\ & + e p^{-1} * (e M1) \\ & + e p * (1/2 * e M1 * z^2 + 1/60 * Q1 . Q1^2 * M1^{-4} * e M1) \\ & + e p^2 * (- 1/3 * e M1 * z^3 - 1/12 * Q1 . Q1 * M1^{-2} * e M1 * z^2 + 1/120 * Q1 . Q1^2 * \\ & M1^{-4} * e M1 * z^2) \\ & - 1/6 * Q1 . Q1 * M1^{-2} * e M1 + 1/60 * Q1 . Q1^2 * M1^{-4} * e M1 ; \end{aligned}$$

D. (MIs) Symbols in MATAD

$$D6 = 6\zeta_3 - 17\zeta_4 - 4\zeta_2 \ln^2 2 + \frac{2}{3} \ln^4 2 + 16\text{Li}_4\left(\frac{1}{2}\right) - 4 \left[\text{Cl}_2\left(\frac{\pi}{3}\right) \right]^2$$

$$D5 = 6\zeta_3 - \frac{469}{27}\zeta_4 + \frac{8}{3} \left[\text{Cl}_2\left(\frac{\pi}{3}\right) \right]^2 - 16 \sum_{m>n>0} \frac{(-1)^m \cos(2\pi n/3)}{m^3 n} \approx -8.216859817508738062913398338601$$

$$D4 = 6\zeta_3 - \frac{77}{12}\zeta_4 - 6 \left[\text{Cl}_2\left(\frac{\pi}{3}\right) \right]^2$$

$$D3 = 6\zeta_3 - \frac{15}{4}\zeta_4 - 6 \left[\text{Cl}_2\left(\frac{\pi}{3}\right) \right]^2$$

$$DM = 6\zeta_3 - \frac{11}{2}\zeta_4 - 4 \left[\text{Cl}_2\left(\frac{\pi}{3}\right) \right]^2$$

$$DN = 6\zeta_3 - 4\zeta_2 \ln^2 2 + \frac{2}{3} \ln^4 2 - \frac{21}{2}\zeta_4 + 16\text{Li}_4\left(\frac{1}{2}\right)$$

$$B4 = -4\zeta_2 \ln^2 2 + \frac{2}{3} \ln^4 2 - \frac{13}{2}\zeta_4 + 16\text{Li}_4\left(\frac{1}{2}\right)$$

$$E3 = -\frac{139}{3} - \frac{\pi\sqrt{3}\ln^2 3}{8} - \frac{17\pi^3\sqrt{3}}{72} - \frac{21}{2}\zeta_2 + \frac{1}{3}\zeta_3 + 10\sqrt{3}\text{Cl}_2\left(\frac{\pi}{3}\right) - 6\sqrt{3}\text{Im} \left[\text{Li}_3\left(\frac{e^{-i\pi/6}}{\sqrt{3}}\right) \right]$$

$$S2 = \frac{4}{9\sqrt{3}}\text{Cl}_2\left(\frac{\pi}{3}\right)$$

$$\text{OepS2} = -\frac{763}{32} - \frac{9\pi\sqrt{3}\ln^2 3}{16} - \frac{35\pi^3\sqrt{3}}{48} + \frac{195}{16}\zeta_2 - \frac{15}{4}\zeta_3 + \frac{57}{16}\zeta_4 + \frac{45\sqrt{3}}{2}\text{Cl}_2\left(\frac{\pi}{3}\right) - 27\sqrt{3}\text{Im} \left[\text{Li}_3\left(\frac{e^{-i\pi/6}}{\sqrt{3}}\right) \right]$$

$$\text{Tlep} = -\frac{45}{2} - \frac{\pi\sqrt{3}\ln^2 3}{8} - \frac{35\pi^3\sqrt{3}}{216} - \frac{9}{2}\zeta_2 + \zeta_3 + 6\sqrt{3}\text{Cl}_2\left(\frac{\pi}{3}\right) - 6\sqrt{3}\text{Im} \left[\text{Li}_3\left(\frac{e^{-i\pi/6}}{\sqrt{3}}\right) \right]$$

Literature

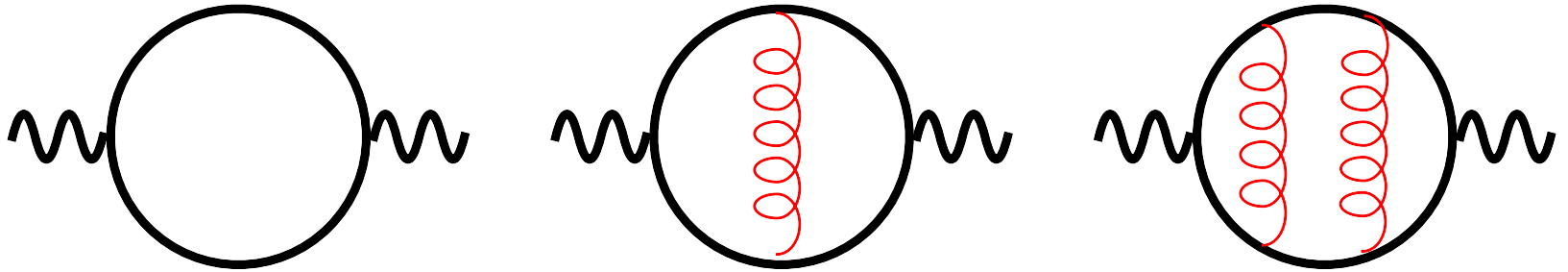
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- K. G. Chetyrkin and F. V. Tkachov, “Integration By Parts: The Algorithm To Calculate Beta Functions In 4 Loops,” *Nucl. Phys. B* **192** (1981) 159.
- D. J. Broadhurst, “Three Loop On-Shell Charge Renormalization Without Integration: Lambda- M_s (QED) To Four Loops,” *Z. Phys. C* **54**, 599 (1992).
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III. Examples

A. Scalar 3-loop integral

see: `problems/c_misc/test`

B. Photon polarization function



$$\begin{aligned}(-q^2 g_{\mu\nu} + q_\mu q_\nu) \Pi(q^2) &= i \int dx e^{iqx} \langle 0 | T j_\mu(x) j_\nu^\dagger(0) | 0 \rangle \\ &= \Pi^{(0)}(q^2) + \frac{\alpha_s}{4\pi} \Pi^{(1)}(q^2) + \left(\frac{\alpha_s}{4\pi}\right)^2 \Pi^{(2)}(q^2) \\ &= + \dots\end{aligned}$$

Projector:

$$q^2 \Pi(q^2) = \frac{1}{d-1} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \Pi^{\mu\nu}(q^2)$$

see: `problems/c_pi/pi`

Exercises I

1. Compute the ϵ expansion for the one-loop tadpole with index $n = 1, 2, 3, 4, \dots$ using
 - (a) the explicit analytic result,
 - (b) the recurrence relation.
2. Derive the triangle rule for the two-loop integral of type $T1$ where each line has a different mass.
3.
 - (a) Derive the recurrence relations for the massless two-loop integral of type $T1$.
 - (b) Program the triangle rule in order to compute the ϵ expansion of the integral $T1$ for arbitrary indices up to (including) terms of order ϵ^2 .
4.
 - (a) Derive the recurrence relations for the two-loop tadpole with 2 massive and 1 massless lines.
 - (b) Compare with explicit analytic result.

Exercises II

1. Compute
 - (a) one-loop tadpole,
 - (b) two-loop tadpole with 2 massive and 1 massless line
 - (c) three-loop tadpole with 4 lines where each of them connects vertex 1 with vertex 2; 2 massive and 2 massless lines,with `MATAD` for various choices of the indices. Compare with the analytic results.
2. Compute three-loop vacuum integrals (with six lines, “`tad3l`”) with various massive/massless combinations. Which constant appears in the final result? For which choice of indices is the result finite?
3. Check that the sum of the (two) singlet diagrams contributing to $\Pi(q^2)$ at three loops is zero. Why?
4. Compute $\Pi(q^2)$ for arbitrary gauge parameter ξ and check that ξ drops out in the sum of the bare diagrams.
5. Renormalize $\Pi(q^2)$ and compare with the result in the literature.
6. Compute the axial-vector, scalar and pseudo-scalar correlator.