New 2-loops results to the effective mixing angle

Ievgen Dubovyk

10th Annual Meeting of the Helmholtz Alliance "Physics at the Terascale"

in collaboration with:
T.Riemann (DESY/Zeuthen), J.Gluza (Silesian U. Katowice), A.Freitas (U. Pittsburg) and J.Usovitsch (Humboldt-U. Berlin)

based on: Phys.Lett. B762 (2016) 184-189 and LL16 arXiv:1610.07059, arXiv:1607.07538

Outline

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- Numerical evaluation of multi-loop Feynman integrals
  - Sector Decomposition
  - MB representation for Feynman integrals
  - Computation of MB integrals in Minkowskian region
- Results
- Summary

$e^+e^- \to (\gamma, Z) \to b\bar{b}$

e^+e^--annihilation into fermion pairs is described by a gauge invariant, unitary and analytic scattering amplitude:

$$\overline{\mathcal{M}}^0 \sim \frac{R}{s-\bar{s}_0} + S + (s-\bar{s}_0)\, S' + \cdots\,,$$

$$\bar{s}_0 = \overline{M}_Z^2 - i\overline{M}_Z\overline{\Gamma}_Z.$$

The residue R factorizes into initial- and final state vertex form factors, $V_\mu^{Ze^+e^-}$ and $V_\nu^{Zb\bar{b}}$, and Z-propagator corrections, $R_Z^{\mu\nu}$.

The unfolded Z-peak forward-backward asymmetry $A_{\rm FB}^{b\bar{b},0}$ and forward-backward left-right asymmetry $A_{\rm FB,LR}^{b\bar{b},0}$ can be written as

$$A_{\rm FB}^{b\bar{b},0} = \tfrac{3}{4} A_{\rm e} A_{\rm b}, \qquad\qquad A_{\rm FB,LR}^{b\bar{b},0} = \tfrac{3}{4} P_{\rm e} A_{\rm b},$$

where $P_{\rm e}$ is the electron polarization and A_b is the asymmetry parameter.

$$A_{\rm b} = \frac{2\,\Re e\,\frac{g_{\rm V}^b}{g_{\rm A}^b}}{1+\left(\Re e\,\frac{g_{\rm V}^b}{g_{\rm A}^b}\right)^2} \quad = \quad \frac{1-4|Q_b|\sin^2\theta_{\rm eff}^{\rm b}}{1-4|Q_b|\sin^2\theta_{\rm eff}^{\rm b}+8Q_b^2\left(\sin^2\theta_{\rm eff}^{\rm b}\right)^2}.$$

$$\sin^2\theta_{\rm eff}^{\rm b} = \frac{1}{4|Q_b|}\left(1-\Re e\,\frac{g_{\rm V}^b}{g_{\rm A}^b}\right).$$

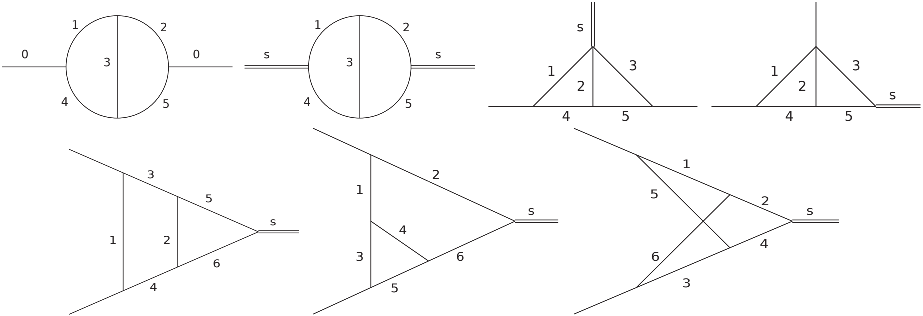
Technically, the calculation of $A_{\rm b}$ rests on the calculation of the vertex form factor $V_\mu^{Zb\bar{b}}$, whose vector and axial-vector components can be obtained using the projection operations

$$g_{\rm V}^b(k^2) = \frac{1}{2(2-D)k^2}\,{\rm Tr}[\gamma^\mu\,\not{p}_1\,V_\mu^{Zb\bar{b}}\,\not{p}_2],$$
$$g_{\rm A}^b(k^2) = \frac{1}{2(2-D)k^2}\,{\rm Tr}[\gamma_5\,\gamma^\mu\,\not{p}_1\,V_\mu^{Zb\bar{b}}\,\not{p}_2],$$

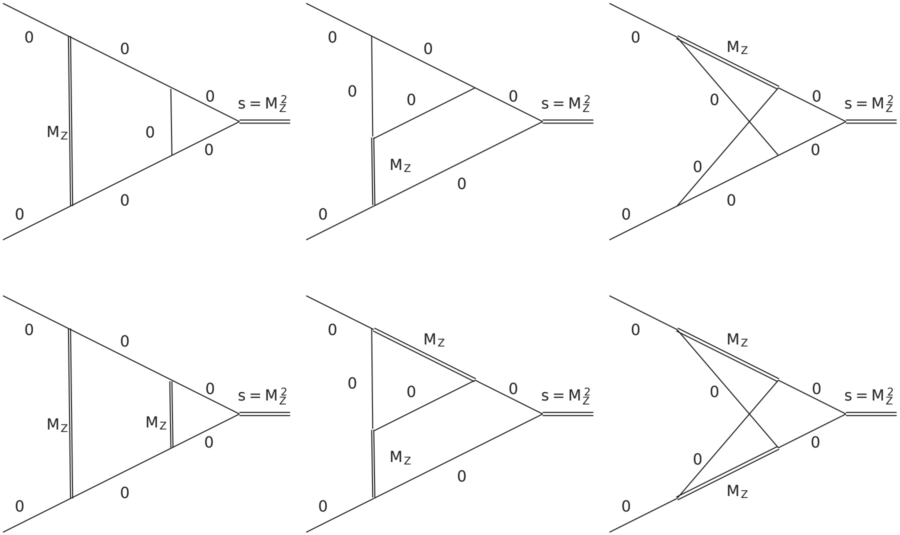
As a result, only scalar integrals remain after projection, but they may contain non-trivial combinations of scalar products in the numerator.
The experimental values for $A_{\rm b}$ and $\sin^2\theta_{\rm eff}^{\rm b}$ from a global fit to the LEP and SLC data are

$$A_{\rm b} = 0.899 \pm 0.013, \qquad \sin^2\theta_{\rm eff}^{\rm b} = 0.281 \pm 0.016.$$

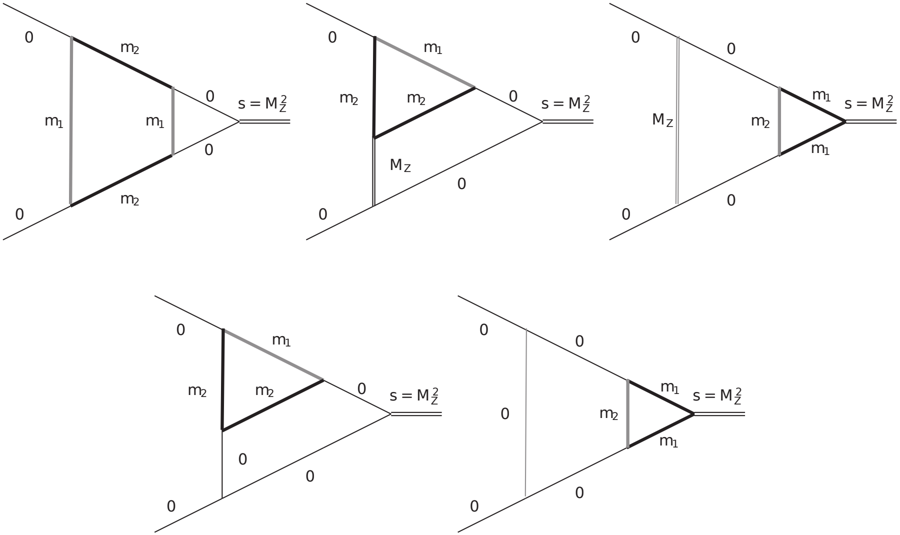
General structure of diagrams to be evaluated



The most difficult cases I



The most difficult cases II



Starting point

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j\, x_j^{n_j-1}\delta(1-\sum_{i=1}^{N} x_i)\frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The functions U and F are called graph or Symanzik polynomials.

Some remarks

Change of variables in Symanzik polynomials U and F is effective as:

- They are homogeneous in the Feynman parameters, U is of degree L, F is of degree $L+1$
- U is linear in each Feynman parameter. If all internal masses are zero, then also F is linear in each Feynman parameter
- In expanded form each monomial of U has coefficient $+1$

$$F = F_0 + U\sum_{i=1}^{N} x_i m_i^2$$

Sector decomposition

`SecDec` - https://secdec.hepforge.org/
`FIESTA` - http://science.sander.su/FIESTA.htm

Basic steps:

- integrating out δ-function - primary sector decomposition

$$G_l = \int_0^1 \prod_{j=1}^{N} \prod_{j \neq l} dt_j \, t_j^{n_j-1} \frac{U_l(t)^{N_\nu - d(L+1)/2}}{F_l(t)^{N_\nu - dL/2}}$$

- iterated sector decomposition

$$G_{lk} = \int_0^1 \prod_{j=1}^{N} \prod_{j \neq l} dt_j \, t_j^{a_j - b_j \epsilon} \frac{U_{lk}(t)^{N_\nu - d(L+1)/2}}{F_{lk}(t)^{N_\nu - dL/2}}$$

$$U_{lk}(t) = 1 + u(t), \quad F_{lk}(t) = -s + f(t)$$

- ϵ-expansion

$$\int_0^1 dx \, x^{-1-b\epsilon} f(x) = \frac{f(0)}{b\epsilon} + \int_0^1 dx \, x^{-b\epsilon} \frac{f(x) - f(0)}{x}$$

MB representation for Feynman integrals

Examples, description, links to basic tools and literature:
http://us.edu.pl/gluza/ambre/
`MBtools` suite: `https://mbtools.hepforge.org/`

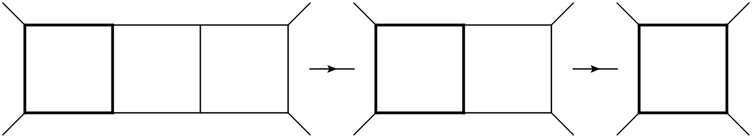
$$\frac{1}{(A_1+\ldots+A_n)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi i)^{n-1}}\int_{-i\infty}^{i\infty} dz_1\ldots dz_{n-1} \times \prod_{i=1}^{n-1} A_i^{z_i}\, A_n^{-\lambda-z_1-\ldots-z_{n-1}} \prod_{i=1}^{n-1}\Gamma(-z_i)\,\Gamma(\lambda+z_1+\ldots+z_{n-1})$$

There are several ways to apply general Mellin-Barnes relation to Feynman integrals:

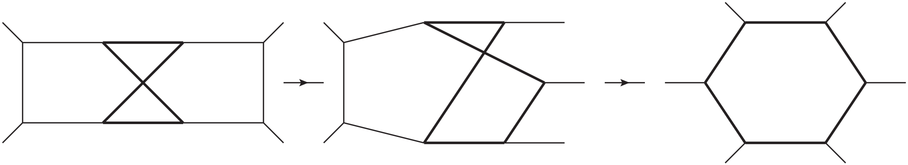
- iteratively to each subloop – loop-by-loop (LA) approach (`AMBREv1.3` & `AMBREv2.1`)
- in one step to the complete U and F polynomials – global (GA) approach (new `AMBREv3.1`)
- combination of the above methods – Hybrid approach (under development)

Limitations of LA approach

Planar case:



Non-planar case:



Limitations of GA approach

U polynomial for non-planar 3-loop box (64 terms)

```
x[1] x[2] x[4] + x[1] x[3] x[4] + x[2] x[3] x[4] + x[1] x[2] x[5] +
x[1] x[3] x[5] + x[2] x[3] x[5] + x[1] x[4] x[5] + x[2] x[4] x[5] +
x[2] x[4] x[6] + x[3] x[4] x[6] + x[2] x[5] x[6] + x[3] x[5] x[6] +
x[4] x[5] x[6] + x[2] x[4] x[7] + x[3] x[4] x[7] + x[2] x[5] x[7] +
x[3] x[5] x[7] + x[4] x[5] x[7] + x[1] x[2] x[8] + x[1] x[3] x[8] +
x[2] x[3] x[8] + x[1] x[4] x[8] + x[2] x[4] x[8] + x[2] x[6] x[8] +
x[3] x[6] x[8] + x[4] x[6] x[8] + x[2] x[7] x[8] + x[3] x[7] x[8] +
x[4] x[7] x[8] + x[1] x[2] x[9] + x[1] x[3] x[9] + x[2] x[3] x[9] +
x[2] x[4] x[9] + x[3] x[4] x[9] + x[1] x[5] x[9] + x[3] x[5] x[9] +
x[4] x[5] x[9] + x[2] x[6] x[9] + x[3] x[6] x[9] + x[5] x[6] x[9] +
x[2] x[7] x[9] + x[3] x[7] x[9] + x[5] x[7] x[9] + x[1] x[8] x[9] +
x[3] x[8] x[9] + x[4] x[8] x[9] + x[6] x[8] x[9] + x[7] x[8] x[9] +
x[1] x[2] x[10] + x[1] x[3] x[10] + x[2] x[3] x[10] +
x[1] x[4] x[10] + x[2] x[4] x[10] + x[2] x[6] x[10] +
x[3] x[6] x[10] + x[4] x[6] x[10] + x[2] x[7] x[10] +
x[3] x[7] x[10] + x[4] x[7] x[10] + x[1] x[9] x[10] +
x[3] x[9] x[10] + x[4] x[9] x[10] + x[6] x[9] x[10] + x[7] x[9] x[10]
```

General structure of the MB integrals after expansion in ϵ

$$\frac{1}{(2\pi i)^r} \int_{c_1-i\infty}^{c_1+i\infty} \cdots \int_{c_r-i\infty}^{c_r+i\infty} \prod_i^r dz_i \, \mathbf{F}(Z,S) \frac{\prod_j \mathbf{G_j}(N_j)}{\prod_k \mathbf{G_k}(N_k)}$$

F depends on: Z – linear combinations of r complex variables z_i,
S – kinematic parameters and masses;

$\mathbf{G_i}$: Gamma and PolyGamma functions
N_i : linear combinations of z_i, e.g. $N_i = \sum_l \alpha_{il} z_l + \gamma_i$

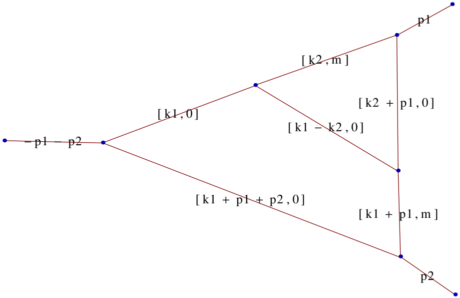
In practice F is a product of powers of S:

$$\mathbf{F} \sim \prod_k X_k^{\sum_i (\alpha_{ki} z_i + \gamma_k)}$$

$$\alpha_{ij}, \gamma_i \in \text{Integer}, \quad X = \left\{ -\frac{s}{m_1^2}, \frac{m_1^2}{m_2^2}, \frac{s}{t}, \ldots \right\}.$$

Where is the problem?

An Example:




```
{MBint[((-s/m^2)^(-z1) Gamma[-1-z1] Gamma[2+z1] Gamma[-1-z2]
 Gamma[z1-z2] Gamma[1+z2-z3]^2 Gamma[-z3] Gamma[1+z3]
Gamma[-z1+z3]^2 Gamma[-z2+z3])/(Gamma[-z1] Gamma[1+z1-z2]
           Gamma[1-z1+z3]), {{eps -> 0},
    {z1 -> -47/37, z2 -> -139/94, z3 -> -176/235}}]}
```

In Minkowskian region $s > 0$.
For $Zb\bar{b}$ vertex $s = M_Z^2 \rightarrow \left(-\frac{s}{m^2}\right)^{z_1} = (-1)^{z_1}$.

Steps for numerical integration (**MB.m** approach):

- real parametrization $z_i \to c_i + It_i, \quad t_i \in (-\infty, \infty)$
- transformation to finite integration interval (in our case $[0,1]$ to link with CUBA library)

$$t_i \to Log\left[\frac{x_i}{1-x_i}\right] \quad dt_i \to \frac{dx_i}{x_i(1-x_i)}$$

The potential problem: $(-1)^{-z1} \to e^{-\pi t_1}$ $t_1 \to -\infty$.
Let check cancelation of this factor (using Stirling's approximation):

$$\lim_{t\to\pm\infty} \Gamma(a+It) \sim e^{-\frac{\pi|t|}{2}} t^{a-\frac{1}{2}}$$

If we take a ray $t_1 = t,\ t_2 = 0,\ t_3 = 0$ and compute a limit for product of gamma functions we get the following:

$$\frac{\prod_j \mathbf{G_j}(N_j)}{\prod_k \mathbf{G_k}(N_k)} \sim e^{-\pi|t|} \frac{1}{t^{646/235}}$$

We have a complete cancelation of exponential factors when $t_1 \to -\infty$, so limit in general looks like $\frac{1}{t^{\alpha}}$.

After transformation to finite region we get singular behavior of the integrand

$$\frac{1}{x\,Log[x]^{646/235}} \xrightarrow{x\to 0} \infty$$

The problem:

- factor $e^{\pm\pi t}$ from Minkowskian ivariants cancels $e^{-\pi|t|}$ from gamma functions in some direction, so limit in general looks like $\frac{1}{t^{\alpha}}$.
- after transformation to finite region in general the limit is $\frac{1}{xLog[x]^{\alpha}}$ and this makes numerical integration quite problematic*

*one should stress that this kind of sigularities is integratable when $\alpha > 1$

The (simple) solution: **another transformation**

$$t_i \to Tan[\pi(x_i - \frac{1}{2})] \quad dt_i \to \frac{\pi dx_i}{Cos[\pi(x_i - \frac{1}{2})]^2}$$

Now we have

$$\frac{Cos[\pi(x_i - \frac{1}{2})]^{176/235}}{Sin[\pi(x_i - \frac{1}{2})]^{646/235}} \xrightarrow{x\to 0} 0$$

Results (finite part):

```
*Analitical result: -1.199526183135566 + 5.567365907880696 I
**Numerical result: -1.199526183168498 + 5.567365907904922 I
MB(Vegas):          -1.199561086311856 + 5.569395048002913 I
MB(Cuhre):                              NaN
```

* arXiv:hep-ph/0401193

** $t_1 \to Tan[\ldots], \quad t_2, t_3 \to Log[\ldots].$

Further generalization:

- $\alpha < 1$?
- cancellation of the exponential damping factor take place more than in one direction?

MBnumerics

- Contour shifts – to restore α or even make the integral vanishing
- Contour deformations – to restore damping factors

For more details see the upcoming talk by J. Usovitsch.

Results

The Standard Model prediction for the effective weak mixing angle can be written as

$$\sin^2\theta_{\rm eff}^{\rm b} = \left(1 - \frac{M_{\rm W}^2}{M_{\rm Z}^2}\right)(1 + \Delta\kappa_{\rm b}),$$

For varying input parameters, the new result is best expressed in terms of a simple fitting formula,

$$\Delta\kappa_{\rm b}^{(\alpha^2,{\rm bos})} = k_0 + k_1 c_{\rm H} + k_2 c_{\rm t} + k_3 c_{\rm t}^2 + k_4 c_{\rm H} c_{\rm t} + k_5 c_{\rm W},$$

with

$$c_{\rm H} = \log\left(\frac{M_{\rm H}}{M_{\rm Z}} \times \frac{91.1876\ {\rm GeV}}{125.1\ {\rm GeV}}\right), \qquad c_{\rm t} = \left(\frac{m_{\rm t}}{M_{\rm Z}} \times \frac{91.1876\ {\rm GeV}}{173.2\ {\rm GeV}}\right)^2 - 1,$$

$$c_{\rm W} = \left(\frac{M_{\rm W}}{M_{\rm Z}} \times \frac{91.1876\ {\rm GeV}}{80.385\ {\rm GeV}}\right)^2 - 1.$$

Fitting this formula to the full numerical result, the coefficients are obtained as

$$k_0 = -0.98605 \times 10^{-4}, \quad k_1 = 0.3342 \times 10^{-4}, \quad k_2 = 1.3882 \times 10^{-4},$$
$$k_3 = -1.7497 \times 10^{-4}, \quad k_4 = -0.4934 \times 10^{-4}, \quad k_5 = -9.930 \times 10^{-4}.$$

$$\sin^2\theta^{\rm b}_{\rm eff} = s_0 + d_1 L_H + d_2 L_H^2 + d_3\Delta_\alpha + d_4\Delta_t + d_5\Delta_t^2 + d_6\Delta_t L_H + d_7\Delta_{\alpha_s} + d_8\Delta_t\Delta_{\alpha_s} + d_9\Delta_Z$$

with

$$L_H = \log\left(\frac{M_{\rm H}}{125.7\ \text{GeV}}\right), \qquad \Delta_t = \left(\frac{m_{\rm t}}{173.2\ \text{GeV}}\right)^2 - 1, \quad \Delta_Z = \frac{M_{\rm Z}}{91.1876\ \text{GeV}} - 1,$$

$$\Delta_\alpha = \frac{\Delta\alpha}{0.0059} - 1, \qquad \Delta_{\alpha_s} = \frac{\alpha_s}{0.1184} - 1.$$

The best-fit numerical values for the coefficients are given by

$$\begin{aligned} s_0 = 0.232704, \quad & d_1 = 4.723\times 10^{-4}, & d_2 = 1.97\times 10^{-4}, \quad & d_3 = 2.07\times 10^{-2}, \\ & d_4 = -9.733\times 10^{-4}, & d_5 = 3.93\times 10^{-4}, \quad & d_6 = -1.38\times 10^{-4}, \\ & d_7 = 2.42\times 10^{-4}, & d_8 = -8.10\times 10^{-4}, \quad & d_9 = -0.664. \end{aligned}$$

Parameter	Value	Range
M_Z	91.1876 GeV	± 0.0042 GeV
Γ_Z	2.4952 GeV	
M_W	80.385 GeV	± 0.030 GeV
Γ_W	2.085 GeV	
M_H	125.1 GeV	± 5.0 GeV
m_t	173.2 GeV	± 4.0 GeV
α_s	0.1184	± 0.0050
$\Delta\alpha$	0.0590	± 0.0005

Table 1: *Reference values used in the numerical analysis.*

Order	Value $[10^{-4}]$	Order	Value $[10^{-4}]$
α	468.945	$\alpha_t^2\alpha_s$	1.362
$\alpha\alpha_s$	-42.655	α_t^3	0.123
$\alpha_t\alpha_s^2$	-7.074	α_{ferm}^2	3.866
$\alpha_t\alpha_s^3$	-1.196	α_{bos}^2	-0.986

Table 2: *Comparison of different orders of radiative corrections to $\Delta\kappa_b$, using the input parameters in Tab. 1.*

one-loop contributions	Akhundov:1985, Beenakker:1988
fermionic electroweak two-loop corrections	Awramik:2008
$\mathcal{O}(\alpha\alpha_s)$ QCD corrections	Djouadi:1987,Djouadi:1987,Kniehl:1989,Kniehl:1991, Fleischer:1992,Buchalla:1992,Czarnecki:1996
partial higher-order corrections of orders $\mathcal{O}(\alpha_t\alpha_s^2)$	Avdeev:1994,Chetyrkin:1995
$\mathcal{O}(\alpha_t\alpha_s^3)$	Schroder:2005,Chetyrkin:2006,Boughezal:2006
$\mathcal{O}(\alpha^2\alpha_t)$ and $\mathcal{O}(\alpha_t^3)$	vanderBij:2000,Faisst:2003

A.Freitas, PoS(LL2014)050

	M_W	Γ_Z	R_b	$\sin^2\theta^{\ell}_{eff}$
ILC exp. error	3…5 MeV	∼1 MeV	1.5×10^{-4}	1.3×10^{-5}
Current theory error	4 MeV	0.5 MeV	1.5×10^{-4}	4.5×10^{-5}
Projected theory error	1 MeV	0.2 MeV	$0.5 \ldots 1 \times 10^{-4}$	1.5×10^{-5}
Parametric error for ILC	2.6 MeV	0.5 MeV	$< 10^{-5}$	2×10^{-5}

Table 3: Projected experimental errors of ILC running at $\sqrt{s} \approx M_Z$ and $\sqrt{s} \approx 2M_W$ and current and expected future theory uncertainties for the SM prediction for several important electroweak precision observables. The future theory errors are estimated under the assumption that $\mathcal{O}(N_f^2\alpha\alpha_s)$, $\mathcal{O}(N_f\alpha\alpha_s)$, $\mathcal{O}(N_f^3\alpha^3)$ and $\mathcal{O}(N_f^2\alpha^3)$ corrections will become available. The parametric error describes the uncertainty of the SM prediction due to uncertainties of input parameters: $\delta m_t = 100$ MeV, $\delta\alpha_s = 0.001$ (from ILC), and $\delta M_Z = 2.1$ MeV (from LEP).

Summary

- The determination of the electroweak two-loop corrections to A_b and $\sin^2\theta_{eff}^{b}$ has been completed.
- Computed bosonic corrections is expectedly small and are about $\frac{1}{4}$ of the corresponting fermionic corrections. However, the anticipated measurements at a future accelerator of the ILC/FCC-ee/CEPC generation aim for an accuracy comparable to electroweak two-loop effects.
- New packages `AMBRE 3` and `MBnumerics` have been developed. The combination of these packages with numerical software `FIESTA`, `SecDec` and the `MBtools` suite will be sufficient for calculating the whole class of massive two-loop self-energy and vertex integrals in the Standard Model.
- Applications related to Drell-Yan processes at the LHC are also of the single-particle resonance type and may be envisaged with the technique developed here.