MultivariateResidues: a Mathematica package for computing multivariate residues

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Introduction

- Describing higher order RADiative CORrections

  Many physical scales and/or hidden degrees

  of freedom (loops)

- Challenging evaluation of scattering amplitudes

  IBPs with many scales, two-loop integrand reduction, ...

- Requires multivariate tools

  Multivariate functional reconstruction, multivariate polynomial division, ...

- In this talk: Multivariate Residues

  Applications to

  Amplitudes from scattering equations
  Calculating master integral coefficients
  Leading singularities of Feynman integrals

Generalisation from univariate to multivariate residue

Univariate Residue

$$\omega = \frac{h(z)\mathrm{d}z}{f(z)}$$

The residue at a pole $p \in \mathbb{C}$ is

$$\operatorname*{Res}_{p}(\omega) = \frac{1}{2\pi i} \oint_{\gamma} \omega$$

with contour

$$\gamma = \{z \in \mathbb{C} : |z - p| = \epsilon\}$$

encircling the pole counterclockwise

Multivariate (Grothendieck)

$$\Omega = \frac{h(z)\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n}{f_1(z)\cdots f_n(z)}$$

The residue at a pole $p \in \mathbb{C}^n$ is

$$\operatorname*{Res}_{\{f_1,\ldots,f_n\},\, p}(\Omega) = \frac{1}{(2\pi i)^n} \oint_{\Gamma} \Omega$$

with contour

$$\Gamma = \{z \in \mathbb{C}^n : |f_i(z)| = \epsilon_i\}$$

oriented by $\mathrm{d}(\arg f_1) \wedge \cdots \wedge \mathrm{d}(\arg f_n) \geq 0$

[Griffiths and Harris, *Principles of algebraic geometry* (1994)]

Number of denominator factors

A given n-form may contain *more* or *less* than n denominator factors.

- *Less* denominator factors than variables: residue definition not applicable. Consider the notion of *residual form* instead.

$$\Omega = \frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3}{z_1(z_1 + z_2 z_3)} \quad \longrightarrow \quad \widetilde{\Omega} \; = \; \operatorname*{Res}_{z_1=0} \Omega \; = \; \frac{\mathrm{d}z_2 \wedge \mathrm{d}z_3}{z_2 z_3}$$

- *More* denominator factors than variables: partition into n factors

$$\Omega = \frac{h(z)\,\mathrm{d}z_1 \wedge \mathrm{d}z_2}{\phi_1(z)\,\phi_2(z)\,\phi_3(z)} \equiv \frac{h(z)\,\mathrm{d}z_1 \wedge \mathrm{d}z_2}{f_1(z)\,f_2(z)}$$

Three partitions: $\{f_1, f_2\} = \{\phi_1\,,\,\phi_2\phi_3\}\,,\;\{\phi_2\,,\,\phi_3\phi_1\}\,,\;\{\phi_3\,,\,\phi_1\phi_2\}$

Special cases of multivariate residues

- *Factorizable* residue has $f_i(z) = f_i(z_i)$. Evaluated as a product of univariate residues

$$\operatorname*{Res}_{\{f_1,\ldots,f_n\},\,p}(\Omega) = \frac{1}{(2\pi i)^n} \oint_{\Gamma_1} \frac{\mathrm{d}z_1}{f_1(z_1)} \cdots \oint_{\Gamma_n} \frac{\mathrm{d}z_n}{f_n(z_n)} h(z)$$

- *Non-degenerate* residue has a non-vanishing Jacobian

$$\mathrm{Jac}(p) \equiv \det_{i,j}\left(\frac{\partial f_i}{\partial z_j}\right)\Big|_{z=p} \neq 0$$

and evaluates to

$$\operatorname*{Res}_{\{f_1,\ldots,f_n\},p}(\Omega) = \frac{1}{(2\pi i)^n} \oint_{|w_i|\leq\epsilon_i} \frac{h\big(f^{-1}(w)\big)\,\mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_n}{\mathrm{Jac}(p)\, w_1 \cdots w_n} = \frac{h(p)}{\mathrm{Jac}(p)}$$

- What about *degenerate* residues?

Degenerate multivariate residue

Theorem (Transformation formula)

For $g_i(z) = \sum_{j=1}^{n} A_{ij}(z) f_j(z)$*, with locally holomorphic* $A_{ij}(z)$*, we have*

$$\operatorname*{Res}_{\{f_1,\ldots,f_n\},p}\left(\frac{h(z)\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n}{f_1(z)\cdots f_n(z)}\right) = \operatorname*{Res}_{\{g_1,\ldots,g_n\},p}\left(\frac{h(z)\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n}{g_1(z)\cdots g_n(z)}\,\det A(z)\right)$$

[Griffiths and Harris, *Principles of algebraic geometry* (1994)]

- The idea is to find univariate $g_i(z) = g_i(z_i)$.
- Compute the *factorized residue* as product of univariate residues.
- In practice, obtain $g_i(z_i)$ as the first element in the lexicographically-ordered Gröbner basis for $\{f_1, \ldots, f_n\}$ with the variable ordering $z_{i+1} \succ z_{i+2} \succ \cdots \succ z_n \succ z_1 \succ z_2 \cdots \succ z_i$.

Example

Consider the following two-form, with a pole at $p = (0,0)$,

$$\Omega = \frac{z_1 \, \mathrm{d}z_1 \wedge \mathrm{d}z_2}{z_2(a_1 z_1 + a_2 z_2)(b_1 z_1 + b_2 z_2)} \equiv \frac{z_1 \, \mathrm{d}z_1 \wedge \mathrm{d}z_2}{\phi_1 \, \phi_2 \, \phi_3}$$

Three choices for combining denominator factors into two functions f_1, f_2:

$$\{f_1, f_2\} = \{\phi_1 \, , \, \phi_2\phi_3\} \, , \;\; \{\phi_2 \, , \, \phi_3\phi_1\} \, , \;\; \{\phi_3 \, , \, \phi_1\phi_2\}$$

Select $\{f_1, f_2\} = \{\phi_1 \, , \, \phi_2\phi_3\}$. Gröbner basis computation yields

$$\begin{pmatrix} g_1(z_1) \\ g_2(z_2) \end{pmatrix} = \begin{pmatrix} a_1 b_1 z_1^2 \\ z_2 \end{pmatrix} = \begin{pmatrix} -(a_1 b_2 + a_2 b_1) z_1 - a_2 b_2 z_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}$$

Apply transformation formula

$$\operatorname*{Res}_{\{f_1, f_2\}, \, p} \Omega = \operatorname*{Res}_{\{g_1, g_2\}, \, p} \frac{z_1 \det A \, \mathrm{d}z_1 \wedge \mathrm{d}z_2}{g_1(z_1) g_2(z_2)} = -\frac{1}{a_1 b_1} \operatorname*{Res}_{p} \frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2}{z_1 z_2} = -\frac{1}{a_1 b_1} \quad \square$$

Package: MultivariateResidues

https://bitbucket.org/kjlarsen/multivariateresidues/downloads/

```
<< "MultivariateResidues`"

| Residue of h/(f_1 ... f_n) at pole z=p
MultivariateResidue[h, {f1,...,fn}, {z1 -> p1,...,zn -> pn}]

| Equivalent to Residue in univariate case
Residue[f[z]/z, {z, 0}]
MultivariateResidue[f[z], {z}, {z -> 0}]

| Multivariate example
f[1] = z[2];
f[2] = (a[1] z[1] + a[2] z[2])(b[1] z[1] + b[2] z[2]);
MultivariateResidue[z[1], {f[1], f[2]}, {z[1]->0, z[2]->0}]
Out: -1/(a[1] b[1])
```

Program: MultivariateResidues

```
| Two independent methods
Method -> TransformationFormula (default)
Method -> QuotientRingDuality   (advanced)

| Outsource Groebner basis computations for performance
$MultiResUseSingular = True


| Restrictions:
|   Integrand is a rational function
|   Denominator has finitely many roots
|   Same number of denominator factors as integration variables
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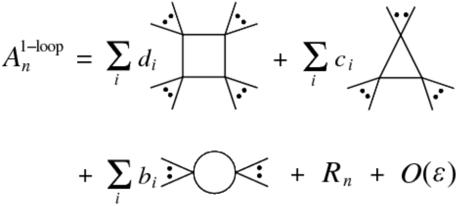
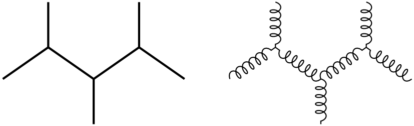
Applications

- Maximal unitarity: master integral coefficients



- Scattering equations: tree-level amplitudes



- Canonical masters: leading singularities of Feynman integrals

Application: Maximal unitarity

- Scattering amplitude decomposition [Bern,Dixon,Dunbar,Kosower, Nucl.Phys.B425 (1994) 217-260]

$$\mathcal{A} = \sum_k c_k I_k + \text{rational terms}$$

  Extract coefficients c_k from contour integrals.

- Example: coefficient $c_\Box$ of one-loop box integral [Britto,Cachazo,Feng, Nucl.Phys.B725 (2005) 275-305]

$$I_\Box = \int_{\mathbb{R}^D} \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{1}{\prod_{i=1}^4 p_i^2(\ell)} \,,$$

  obtained by replacing integration

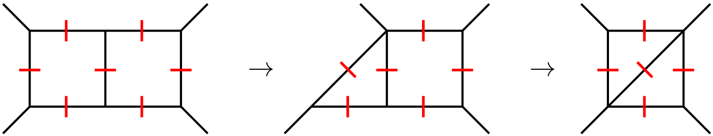
$$\mathbb{R}^D \;\longrightarrow\; T_\epsilon^4 = \{\ell \in \mathbb{C}^4 : |p_i^2(\ell)| = \epsilon\}$$

  (A non-degenerate multivariate residue.)

- At two-loops degenerate residues appear...

Application: Maximal unitarity

- Maximal cuts of double-box topology [Kosower,Larsen, PRD85 (2012) 045017]



- Slashed-box: $8-5=3$ complex variables (z_1, z_2, z_3)

  Typical degenerate residue

$$\underset{\{f_1,f_2,f_3\},\,(0,1,0)}{\mathrm{Res}}\left(\frac{\mathrm{d}z_1\wedge\mathrm{d}z_2\wedge\mathrm{d}z_3}{f_1\;f_2\;f_3}\right)=\frac{1+\chi}{\chi}\qquad(\chi=t/s)$$

  with $f_1=z_1(1-z_1-z_2),\ f_2=z_2z_3,\ f_3=(1-z_1-z_2-z_1\chi+z_1z_3\chi)$
- Tested package on all slashed-box residues (6395, $\sim$ 10 min)
- Ingredient to fixing integral coefficients (avoiding IBP reduction)

Application: Scattering equations

- Tree-level scattering amplitudes encoded in scattering equations

  [Cachazo,He,Yuan, PRD90 (2014) no.6, 065001]

$$\sum_{j=1,j\neq i}^{n} \frac{s_{ij}}{z_i - z_j} = 0\,, \quad i \in \{1,\ldots,n\}$$

  Amplitude localized to $(n-3)!$ solutions z_i

$$\mathcal{A} = \int \frac{\mathrm{d}^n z}{\mathrm{vol}\, SL(2,\mathbb{C})} \prod_i{}' \, \delta\Big(\textstyle\sum_{j\neq i} \frac{s_{ij}}{z_{ij}}\Big) \frac{E(\{p,\varepsilon,z\})}{z_{12}z_{23}\cdots z_{n1}}$$

- In practice, very complicated already for low multiplicity. For $n > 5$ solutions irrational, but sum of residues rational. [Weinzierl, JHEP1404 (2014) 092]

- Problem circumvented by computing global multivariate residue.

  [Søgaard,Zhang, PRD93 (2016) no.10, 105009]

Application: Scattering equations

Example: Five scalar amplitude in ϕ^3-theory [Søgaard,Zhang, PRD93 (2016) no.10, 105009]

$$h_1 = s_{12} + s_{13} z_3 + s_{14} z_4 = 0$$
$$h_2 = s_{123} z_3 + s_{124} z_4 + s_{134} z_3 z_4 = 0$$

Cachazo-He-Yuan representation ($z_1 \to \infty, z_2 \to 1, z_5 \to 0$):

$$\mathcal{A} = \oint_\Gamma \frac{\mathrm{d}z_3\,\mathrm{d}z_4}{h_1\,h_2} \frac{z_3(1-z_4)}{(1-z_3)(z_3-z_4)z_4} = \oint \frac{\mathrm{d}z_3\,\mathrm{d}z_4}{h_1\,h_2}\,\widetilde{N}(z_3,z_4)$$

```
Amp = MultivariateResidue[N, {h1, h2}, {z3,z4}, {GlobalResidue},
                                    Method->"QuotientRingDuality"]
```

$$\mathcal{A} = \frac{1}{s_{12}s_{34}} + \frac{1}{s_{12}s_{45}} + \frac{1}{s_{23}s_{51}} + \frac{1}{s_{23}s_{45}} + \frac{1}{s_{34}s_{51}}$$

- Multivariate residue efficiently calculates tree-level scattering amplitudes

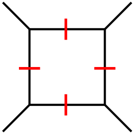
Application: Canonical basis of master integrals

- Existing tools
  - Fuchsia [Gituliar, Magerya, CPC219 (2017) 329-338]
  - epsilon [Prausa, CPC219 (2017) 361-376]
  - Canonica [Meyer (2017)]
  - ... (private codes)

- Initial idea: construct Feynman integrals with unit leading singularity

[Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka (2012)]

[Henn, PRL110 (2013) 251601]

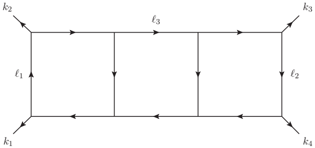


$$= \int \mathrm{d}^4\ell \; \delta(\ell^2)\delta((\ell+p_1)^2)\delta((\ell+p_{12})^2)\delta((\ell+p_{123})^2) \sim \frac{1}{s\,t}$$

- MultivariateResidues can calculate leading singularities as well

Application: Leading singularity of Feynman integral

- Example:



[Henn,Smirnov,Smirnov, JHEP1307 (2013) 128]

$$D_1 = \ell_1^2\,, \quad D_4 = (\ell_1 + k_1)^2\,, \quad D_7 = (\ell_2 - k_4)^2\,,$$
$$D_2 = \ell_2^2\,, \quad D_5 = (\ell_1 - k_2)^2\,, \quad D_8 = (\ell_3 + k_1 + k_2)^2\,,$$
$$D_3 = \ell_3^2\,, \quad D_6 = (\ell_2 + k_3)^2\,, \quad D_9 = (\ell_1 - \ell_3 - k_2)^2\,, \quad D_{10} = (\ell_3 - \ell_2 - k_3)^2$$

Master integrals: $\mathcal{I}[1]$, $\mathcal{I}[(\ell_1 + k_1 + k_4)^2]$, $\mathcal{I}[(\ell_3 - k_3)^2]$. Why?

- Parametrise loop momenta

$$\ell_1^\mu(\alpha_1,\ldots,\alpha_4) = \alpha_1 k_1^\mu + \alpha_2 k_2^\mu + \frac{\alpha_3}{2}\frac{\langle 2\,3\rangle}{\langle 1\,3\rangle}\langle 1|\sigma^\mu|2] + \frac{\alpha_4}{2}\frac{\langle 1\,3\rangle}{\langle 2\,3\rangle}\langle 2|\sigma^\mu|1]$$

$$\ell_2^\mu(\beta_1,\ldots,\beta_4) = \beta_1 k_3^\mu + \beta_2 k_4^\mu + \frac{\beta_3}{2}\frac{\langle 1\,4\rangle}{\langle 1\,3\rangle}\langle 3|\sigma^\mu|4] + \frac{\beta_4}{2}\frac{\langle 1\,3\rangle}{\langle 1\,4\rangle}\langle 4|\sigma^\mu|3]$$

$$\ell_3^\mu(\gamma_1,\ldots,\gamma_4) = \gamma_1 k_2^\mu + \gamma_2 k_3^\mu + \frac{\gamma_3}{2}\frac{\langle 3\,4\rangle}{\langle 2\,4\rangle}\langle 2|\sigma^\mu|3] + \frac{\gamma_4}{2}\frac{\langle 2\,4\rangle}{\langle 3\,4\rangle}\langle 3|\sigma^\mu|2]$$

Application: Leading singularity of Feynman integral

- On-shell constraints $(D_1 = \ldots = D_{10} = 0)$ have 14 solutions $\mathcal{S}_i$

[Badger,Frellesvig,Zhang, JHEP1208 (2012) 065]

$$\mathcal{I}[\mathcal{N}]_{\mathcal{S}_i} = \frac{\mathrm{Jac}}{(2\pi i)^{12}} \oint \frac{\mathrm{d}^4\alpha}{(2\pi)^4} \oint \frac{\mathrm{d}^4\beta}{(2\pi)^4} \oint \frac{\mathrm{d}^4\gamma}{(2\pi)^4} \mathcal{N} \prod_{j=1}^{10} \frac{1}{D_j(\alpha,\beta,\gamma)}$$

- Integrating out 10 on-shell constraints leaves 2-fold *degenerate* residues

$$\frac{1}{(2\pi i)^2} \oint \frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2 \; P(z_1, z_2)}{(1+z_1)(1+z_2)(1+z_1-(t/s)z_2)z_2}$$

[Søgaard,Zhang, JHEP1312 (2013) 008]

- With MultivariateResidues straightforward to calculate such residues.

  $\Rightarrow$ Integrals $\mathcal{I}[1]$, $\mathcal{I}[(\ell_1+k_1+k_4)^2]$, $\mathcal{I}[(\ell_3-k_3)^2]$ have constant leading singularities, so suitable candidates for canonical basis.

Conclusions

- Theory of multivariate residues
  - Residue depends on location of pole and denominator factors
  - Algorithm for computing degenerate residue
- Mathematica package MultivariateResidues
  - Two independent algorithms
  - Publicly available: https://bitbucket.org/kjlarsen/multivariateresidues/downloads/
- Applications include
  - Master integral coefficients with maximal unitarity
  - Tree-amplitudes from scattering equations
  - Leading singularities of Feynman integrals