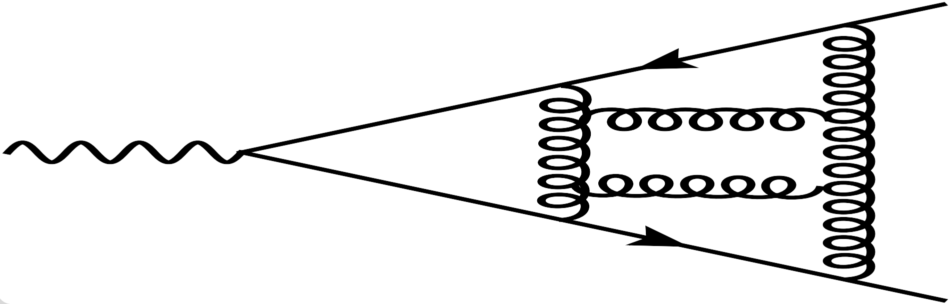
Four-loop form factors in QCD

Radcor 2017, St. Gilgen, Austria, September 24-29, 2017

Matthias Steinhauser | TTP Karlsruhe



Content

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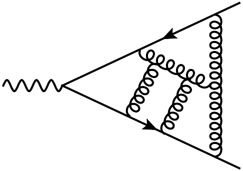
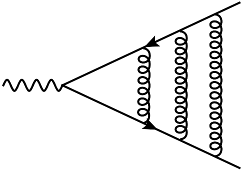
II. Massless form factor



III. Massive form factor



IV. Conclusions

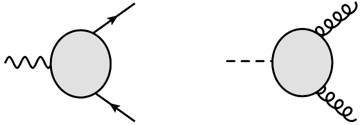
R. N. Lee, A. V. Smirnov, V. A. Smirnov, MS:
"The n_f^2 contributions to fermionic four-loop form factors"

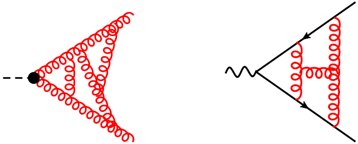
J. Henn, R. Lee, A. Smirnov, V. Smirnov, MS:
"Four-loop photon quark form factor and cusp anomalous dimension in the large-N_c limit of QCD"

J. Henn, A. Smirnov, V. Smirnov, MS:
"A planar four-loop form factor and cusp anomalous dimension in QCD"

J. Henn, A. Smirnov, V. Smirnov, MS:
"Massive three-loop form factor in the planar limit"

Quark and gluon form factor

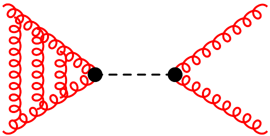
- 
- building block for Higgs production, Drell-Yan, heavy quark production, forward-backward asymmetry, . . .



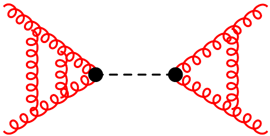
- IR poles $\leftrightarrow$ real radiation
- simplest objects in QCD with non-trivial IR poles
- wanted: all-order formulae for IR structure of gauge theories

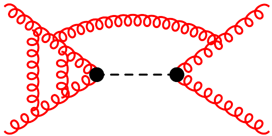
  [Catani'98; . . . ; Becher,Neubert'09; Gardi,Magnea'09]

Example: Higgs production at the LHC

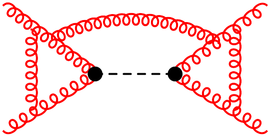


[Baikov,Chetyrkin,Smirnov,Smirnov, Steinhauser'09], [Gehrmann,Glover,Huber,Ikizlerli, Studerus'10]; [Lee,Smirnov'10]

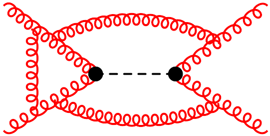




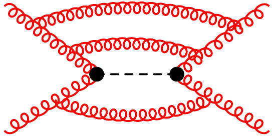
[Duhr,Gehrmann'13], [Li,Zhu'13], [Dulat,Mistlberger'14], [Duhr,Gehrmann,Jaquier'14]



[Anastasiou,Duhr,Dulat,Herzog, Mistlberger'13], [Kilgore'13]



[Anastasiou,Duhr,Dulat,Furlan,Gehrmann, Herzog,Mistlberger'14], [Li,von Manteuffel,Schabinger,Zhu'14]

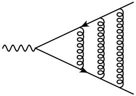


[Anastasiou,Duhr,Dulat,Mistlberger'13]

N^{3}LO: [Anastasiou,Duhr,Dulat,Herzog,Mistlberger'15]

[Anastasiou,Duhr,Dulat,Furlan,Gehrmann,Herzog,Lazopoulos,Mistlberger'16] (expansion around soft limit)

IR structure of massive form factor



- F: UV-renormalized massive form factor

- $F = Z\, F^{\rm finite}$

$$
\begin{aligned}
Z &= 1 + \frac{\alpha_s}{\pi}\left(-\frac{1}{2\epsilon}\Gamma^{(1)}_{\rm cusp}\right) \\
&\quad + \left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{\#}{\epsilon^2} - \frac{1}{4\epsilon}\Gamma^{(2)}_{\rm cusp}\right) \\
&\quad + \left(\frac{\alpha_s}{\pi}\right)^3\left(\frac{\#}{\epsilon^3} + \frac{\#}{\epsilon^2} - \frac{1}{6\epsilon}\Gamma^{(3)}_{\rm cusp}\right) + \ldots
\end{aligned}
$$

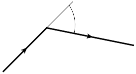
- $\Gamma^{(i)}_{\rm cusp} = \Gamma^{(i)}_{\rm cusp}(x)$

$$\frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$$

$\Gamma_{\rm cusp}$

HQET current $\bar{h}_{v_2}\Gamma h_{v_1}$
universal anomalous dimension:
$\gamma_w(\alpha_s)$
$w = v_1 \cdot v_2 = \cosh\theta$

[Falk,Georgi,Grinstein,Wise'90]



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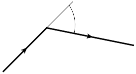

IR behaviour of QCD
soft-gluon exchange
between heavy quarks
⇨ $\Gamma_{\rm IR}(\alpha_s)$
$\widehat{=}$ form factor

$\Gamma_{\rm cusp}$

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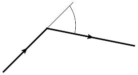

Vacuum average of
Wilson loop with cusp
$W \sim \langle 0|\mathrm{tr}\left[P\exp\left(i\oint_C \mathrm{d}x\cdot A(x)\right)\right]|0\rangle$
has add'l UV divergence
anomalous dimension:
$\Gamma_{\rm cusp}(\theta,\alpha_s)$

[Polyakov'80]

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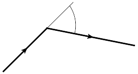

IR behaviour of QCD

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➪ $\gamma_w = \Gamma_{\rm IR} = \Gamma_{\rm cusp}$ [Korchemsky,Radyushkin'92]

$\Gamma_{\rm cusp}$

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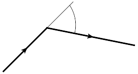

IR behaviour of QCD

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[Polyakov'80]

➪ $\gamma_w = \Gamma_{\rm IR} = \Gamma_{\rm cusp}$ [Korchemsky,Radyushkin'92]

high-energy limit:

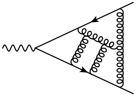
$\Gamma_{\rm cusp} \to C_F\gamma_{\rm cusp}\log(x) + \ldots$

$q^2/m^2 = -(1-x)^2/x$

➪ massless FF: $\gamma_{\rm cusp}$ from $1/\epsilon^2$ pole

[$\gamma_{\rm cusp}$: light-like cusp anom. dim.]

$\gamma_{\rm cusp}$: massless form factor



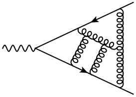
- $\log(F_q)|_{\text{pole part}} =$

$$\frac{\alpha_s}{4\pi}\left\{\frac{1}{\epsilon^2}\left[-\frac{1}{2}C_F\gamma^0_{\rm cusp}\right]+\frac{1}{\epsilon}\left[\gamma^0_q\right]\right\}$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^2\left\{\frac{1}{\epsilon^3}\left[\frac{3}{8}\beta_0 C_F\gamma^0_{\rm cusp}\right]+\frac{1}{\epsilon^2}\left[-\frac{1}{2}\beta_0\gamma^0_q-\frac{1}{8}C_F\gamma^1_{\rm cusp}\right]+\frac{1}{\epsilon}\left[\frac{\gamma^1_q}{2}\right]\right\}$$
$$+\ldots$$

$\gamma_{\rm cusp}$: light-like cusp anomalous dimension

γ_q: collinear anomalous dimension

$\gamma_{\rm cusp}$: massless form factor



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$$\frac{\alpha_s}{4\pi}\left\{\frac{1}{\epsilon^2}\left[-\frac{1}{2}C_F\gamma^0_{\rm cusp}\right]+\frac{1}{\epsilon}\left[\gamma_q^0\right]\right\}$$
$$+\left(\frac{\alpha_s}{4\pi}\right)^2\left\{\frac{1}{\epsilon^3}\left[\frac{3}{8}\beta_0 C_F\gamma^0_{\rm cusp}\right]+\frac{1}{\epsilon^2}\left[-\frac{1}{2}\beta_0\gamma_q^0-\frac{1}{8}C_F\gamma^1_{\rm cusp}\right]+\frac{1}{\epsilon}\left[\frac{\gamma_q^1}{2}\right]\right\}$$
$$+\ldots$$

$\gamma_{\rm cusp}$: light-like cusp anomalous dimension

γ_q: collinear anomalous dimension

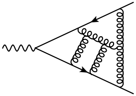
- RGE [Sudakov'54; Mueller'79; Collins'80; Sen'81]

$$-\frac{d}{d\log\mu^2}\log F_q = \frac{1}{2}\left[K\left(\alpha_s\right)+G\left(\alpha_s,q^2/\mu^2\right)\right]$$
$$\frac{d}{d\log\mu^2}K = -\frac{d}{d\log\mu^2}G = -C_F\gamma_{\rm cusp}$$

➪ predict poles of h.o. terms

[Mitov,Moch'01; Gluza,Mitov,Moch,Riemann'09; Ahmed,Henn,Steinhauser'17]

$\gamma_{\rm cusp}$: massless form factor



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$$+\ldots$$

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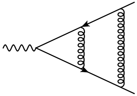
- RGE [Sudakov'54; Mueller'79; Collins'80; Sen'81]

$$-\frac{d}{d\log\mu^2}\log F_q = \frac{1}{2}\left[K(\alpha_s)+G\left(\alpha_s, q^2/\mu^2\right)\right]$$

- interesting question: Casimir scaling $\gamma_{\rm cusp,q} = \frac{C_F}{C_A}\gamma_{\rm cusp,g}$ at 4 loops ?
  Answer: No! [Boels,Huber,Yang'17; Moch,Ruijl,Ueda,Vermaseren,Vogt'17]
- 4-loop $\gamma_{\rm cusp,g}$ enters N^{3}LL resummations in Higgs production
  (see, e.g., [Bonvini,Marzani '14],...)

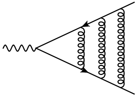
Known results

- massless form factor

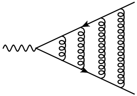


[Kramer,Lampe'87; Matsuura,van der Marck,van Neerven'88;

Harlander'00; Ravindran,Smith,van Neerven'05; Gehrmann,Huber,Maitre'05]



pole part: [Moch,Vermaseren,Vogt'05]
fermionic part: [Moch,Vermaseren,Vogt'05]
full: [Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09]
[Gehrmann,Glover,Huber,Ikizlerli,Studerus'10]; [Lee,Smirnov'10]



all n_f terms, large-N_c: [Henn,Smirnov,Smirnov,Steinhauser'16]
n_f^3 terms Higgs-gluon and photon-quark FF: [von Manteuffel,Schabinger'16]
full large-N_c: [Henn,Lee,Smirnov,Smirnov,Steinhauser'16]
complete n_f^2 terms: [Lee,Smirnov,Smirnov,Steinhauser'17]

$\mathcal{N} = 4$ SYM [Boels,Kniehl,Tarasov,Yang'12'16] numerical methods for MIs

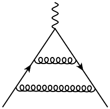
- $\gamma_{\rm cusp}$

[Boels,Huber,Yang]: $\mathcal{N} = 4$ SYM, numerically

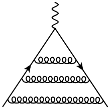
[Moch,Ruijl,Ueda,Vermaseren,Vogt'17] analytic and numerical results

Known results

- massive form factor



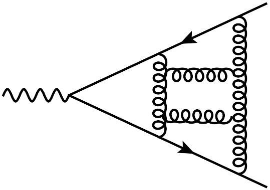
[Bernreuther,Bonciani,Gehrmann,Heinesch,Leineweber,Mastrolia,Remiddi'04'05]
$\mathcal{O}(\epsilon)$ [Gluza,Mitov,Moch,Riemann'09]



large-N_c: [Henn,Smrinov,Smirnov,Steinhauser'16]

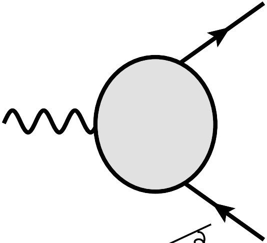
- $\Gamma_{\rm cusp}$ known to 3 loops [Korchemsky,Radyushkin'87], [Grozin,Henn,Korchemsky,Marquard'14'15]
  1 colour structure at 4 loops ($n_l(d_F^{abcd})^2$) for $\theta \to 0$: [Grozin,Henn,Stahlhofen'17]

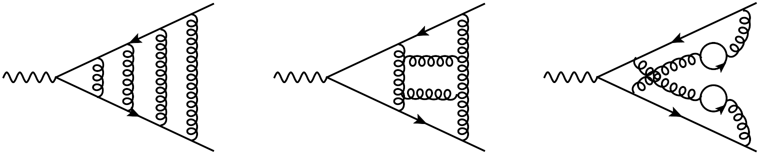
II. Massless form factor



F_q

$$F_q(q^2) = -\frac{1}{4(1-\epsilon)q^2}\mathrm{Tr}\left(\not{q}_2\,\Gamma_q^\mu \not{q}_1\,\gamma_\mu\right)$$



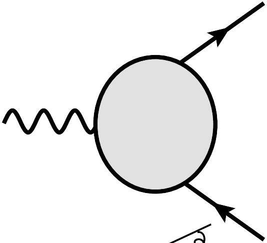


- All planar diagrams ➪ large-N_c

[Henn,Smirnov,Smirnov,Steinhauser'16; Henn,Lee,Smirnov,Smirnov,Steinhauser'16]

- All n_f^2 terms (planar and non-planar) [Lee,Smirnov,Smirnov,Steinhauser'17]

F_q



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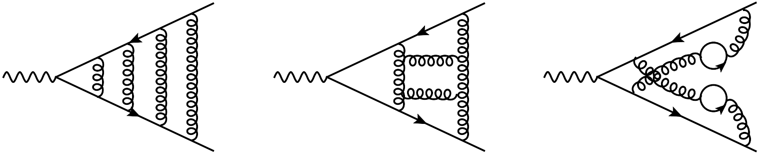

- All planar diagrams ⇨ large-N_c

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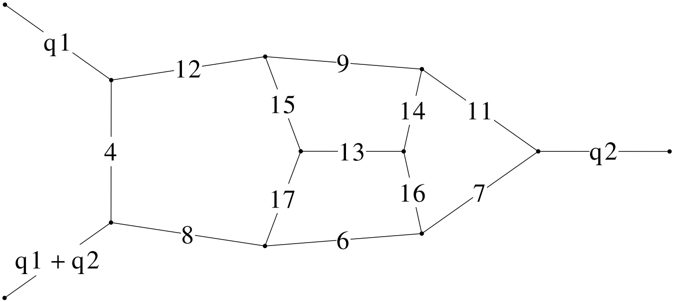
- All n_f^2 terms (planar and non-planar) [Lee,Smirnov,Smirnov,Steinhauser'17]

1. Reduction to master integrals
2. Compute master integrals

Planar: reduction to master integrals

- FIRE [Smirnov] $\bigoplus$ LiteRed [Lee]
- 38 planar integral families


q1
12
9
15
14
11
4
13
q2
17
16
7
8
6
q1 + q2

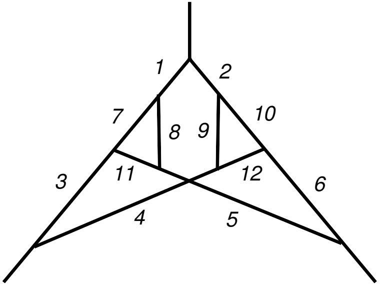
Planar: reduction to master integrals

- FIRE [Smirnov] $\oplus$ LiteRed [Lee]
- 38 integral families
- 12 + 6 = 18 indices
- $\mathcal{O}(1\,000\,000)$ integrals for $\xi = 0$ (Feynman gauge)
- $\mathcal{O}(3\,000\,000)$ integrals ξ^1 terms (fermionic part only)
- reduction time: $\mathcal{O}$(months)
- tsort [Pak,Smirnov] ⇨ $\boxed{\text{99 MIs}}$

n_f^2: reduction to master integrals

[Lee,Smirnov,Smirnov,Steinhauser'17]

- FIRE [Smirnov] ⊕ LiteRed [Lee]
- 2 non-planar integral families



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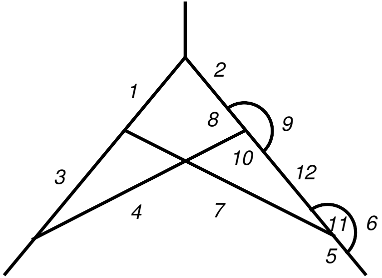

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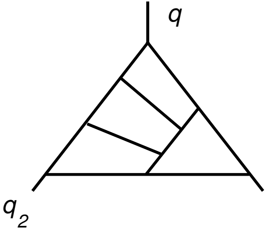
- $\boxed{\text{26 + 40 MIs} \Rightarrow \text{24 non-planar MIs}}$

Computation of MIs



$$q^2 \neq 0$$

$$q_2^2 = (q_2 + q)^2 = 0$$

idea 1:
- $q_2^2 \neq 0 \Rightarrow x = \frac{q_2^2}{q^2}$ [Henn,Smirnov,Smirnov'14]

  consider system of differential equations in x
- boundary conditions for $x = 1 \Rightarrow$ integrals "simple"
- get result for $x = 0$

idea 2: use canonical basis where differential equations have the form

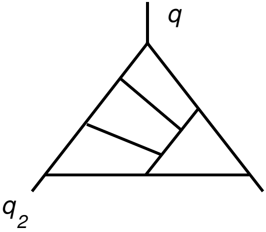
$$g'(x,\epsilon) = \epsilon\, A(x) \cdot g(x,\epsilon)$$ [Henn'13; Henn'14]

$$A(x) = \frac{a}{x} + \frac{b}{x-1}$$ [Lee'14] [Gituliar,Magerya'16; Meyer'16; Prausa'17]

solution: iterated integrals $\Rightarrow$ harmonic polylogarithms (HPLs)

[Remiddi,Vermaseren'99][Maitre'05]

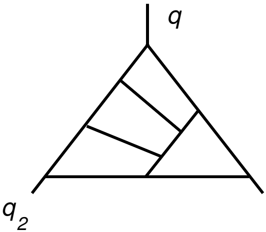
Computation of MIs: more details



➪ 76 MIs

$q_2^2 \neq 0$ ➪ 332 MIs

Computation of MIs: more details



➪ 76 MIs

$q_2^2 \neq 0$ ➪ 332 MIs

primary basis

canonical basis

$$f(x,\epsilon) \xrightarrow{f=T\cdot g \text{ [Lee'14]}} \boxed{\begin{aligned} g(x,\epsilon) &= \textstyle\sum_{k=0}^{8} g_k(x)\epsilon^k \\ g'(x,\epsilon) &= \epsilon\, A(x)\cdot g(x,\epsilon) \\ A(x) &= \tfrac{a}{x} + \tfrac{b}{x-1} \end{aligned}}$$

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$$\downarrow$$

solve in terms of HPLs

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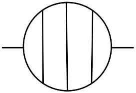
solve in terms of HPLs

boundary conditions for $x = 1$:



4-loop

2-point functions

[Baikov,Chetyrkin,Kühn'05'08;...; Lee,Smirnov,Smirnov'11]

Computation of MIs: more details

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canonical basis

$$f(x,\epsilon) \xrightarrow{f=T\cdot g \text{ [Lee'14]}}$$

$$\boxed{\begin{aligned} g(x,\epsilon) &= \textstyle\sum_{k=0}^{8} g_k(x)\epsilon^k \\ g'(x,\epsilon) &= \epsilon\, A(x)\cdot g(x,\epsilon) \\ A(x) &= \tfrac{a}{x} + \tfrac{b}{x-1} \end{aligned}}$$

$\downarrow$

solve in terms of HPLs
boundary conditions for $x=1$:

$\downarrow$

get $x=0$ result ("naive"):

1. $g_{x\to 0} = x^{\epsilon a} h(\epsilon)$ a: 332×332 matrix

➪ $x^{\epsilon a}$ is 332×332 matrix; each element is linear combination of $x^{k\epsilon}$ terms with $k\leq 0$

Computation of MIs: more details

primary basis

canonical basis

$$f(x,\epsilon) \xrightarrow{f=T\cdot g \text{ [Lee'14]}} g(x,\epsilon)$$

$\downarrow$

solve in terms of HPLs
boundary conditions for $x = 1$:

$\downarrow$

get $x = 0$ result (“naive”):

1. $g_{x\to 0} = x^{\epsilon a} h(\epsilon)$
2. expand HPLs for $x \to 0$

match 1. and 2.

➪ $h(\epsilon)$

➪ get $x^{0\epsilon}$ terms $\widehat{=}$ “naive”

Computation of MIs: more details

primary basis | canonical basis

$$f(x,\epsilon) \xrightarrow{f=T\cdot g \text{ [Lee'14]}} g(x,\epsilon)$$

$\downarrow$

solve in terms of HPLs
boundary conditions for $x = 1$:

$\downarrow$

get $x = 0$ result ("naive"):

1. $g_{x\to 0} = x^{\epsilon a} h(\epsilon)$
2. expand HPLs for $x \to 0$

match 1. and 2.

➪ $h(\epsilon)$

➪ get $x^{0\epsilon}$ terms $\widehat{=}$ "naive"

$$f(0,\epsilon) \xleftarrow{f=T\cdot g} $$

$\downarrow$

332 MIs

$\downarrow$

76 MIs

I_{99}

$$
\begin{aligned}
I_{99} \quad = \quad & e^{4\epsilon\gamma_E}\left(\frac{\mu^2}{-q^2}\right)^{4\epsilon}\left\{\frac{1}{\epsilon^7}\left[-\frac{1}{288}\right]+\frac{1}{\epsilon^6}\left[\frac{13}{576}\right]+\frac{1}{\epsilon^5}\left[-\frac{101}{576}-\frac{\pi^2}{48}\right]\right. \\
& +\frac{1}{\epsilon^4}\left[-\frac{17\zeta_3}{54}+\frac{5\pi^2}{36}+\frac{145}{96}\right]+\frac{1}{\epsilon^3}\left[\frac{1775\zeta_3}{432}-\frac{767\pi^4}{17280}-\frac{5\pi^2}{8}-\frac{1669}{144}\right] \\
& +\frac{1}{\epsilon^2}\left[-\frac{83}{72}\pi^2\zeta_3-\frac{21899\zeta_3}{864}-\frac{3659\zeta_5}{360}+\frac{31333\pi^4}{103680}+\frac{659\pi^2}{288}+\frac{11243}{144}\right] \\
& +\frac{1}{\epsilon}\left[-\frac{40231\zeta_3^2}{1296}+\frac{745\pi^2\zeta_3}{288}+\frac{18751\zeta_3}{144}+\frac{50191\zeta_5}{360}-\frac{277703\pi^6}{2177280}-\frac{14015\pi^4}{10368}\right. \\
& \left.-\frac{149\pi^2}{24}-\frac{22757}{48}\right] \\
& +\left[\frac{39173\zeta_3^2}{324}-\frac{77399\pi^4\zeta_3}{25920}+\frac{4013\pi^2\zeta_3}{432}-\frac{259559\zeta_3}{432}-\frac{568\pi^2\zeta_5}{45}-\frac{1123223\zeta_5}{1440}\right. \\
& \left.-\frac{2778103\zeta_7}{4032}+\frac{3129533\pi^6}{4354560}+\frac{28201\pi^4}{5760}+\frac{173\pi^2}{36}+\frac{382375}{144}\right] \\
& +\epsilon\left[\frac{4931 s_{8a}}{30}+\frac{2615}{144}\pi^2\zeta_3^2-\frac{276671\zeta_3^2}{2592}-\frac{2702413\zeta_5\zeta_3}{1080}+\frac{154037\pi^4\zeta_3}{31104}\right.
\end{aligned}
$$

$$
\begin{aligned}
I_{99} \quad = \quad & e^{4\epsilon\gamma_E}\left(\frac{}{-q^2}\right) \quad \left\{ \frac{}{\epsilon^7}\left[-\frac{}{288}\right] + \frac{}{\epsilon^6}\left[\frac{}{576}\right] + \frac{}{\epsilon^5}\left[-\frac{}{576} - \frac{}{48}\right] \right. \\
& + \frac{1}{\epsilon^4}\left[-\frac{17\zeta_3}{54} + \frac{5\pi^2}{36} + \frac{145}{96}\right] + \frac{1}{\epsilon^3}\left[\frac{1775\zeta_3}{432} - \frac{767\pi^4}{17280} - \frac{5\pi^2}{8} - \frac{1669}{144}\right] \\
& + \frac{1}{\epsilon^2}\left[-\frac{83}{72}\pi^2\zeta_3 - \frac{21899\zeta_3}{864} - \frac{3659\zeta_5}{360} + \frac{31333\pi^4}{103680} + \frac{659\pi^2}{288} + \frac{11243}{144}\right] \\
& + \frac{1}{\epsilon}\left[-\frac{40231\zeta_3^2}{1296} + \frac{745\pi^2\zeta_3}{288} + \frac{18751\zeta_3}{144} + \frac{50191\zeta_5}{360} - \frac{277703\pi^6}{2177280} - \frac{14015\pi^4}{10368} \right. \\
& \left. - \frac{149\pi^2}{24} - \frac{22757}{48}\right] \\
& + \left[\frac{39173\zeta_3^2}{324} - \frac{77399\pi^4\zeta_3}{25920} + \frac{4013\pi^2\zeta_3}{432} - \frac{259559\zeta_3}{432} - \frac{568\pi^2\zeta_5}{45} - \frac{1123223\zeta_5}{1440} \right. \\
& \left. - \frac{2778103\zeta_7}{4032} + \frac{3129533\pi^6}{4354560} + \frac{28201\pi^4}{5760} + \frac{173\pi^2}{36} + \frac{382375}{144}\right] \\
& + \epsilon\left[\frac{4931 s_{8a}}{30} + \frac{2615}{144}\pi^2\zeta_3^2 - \frac{276671\zeta_3^2}{2592} - \frac{2702413\zeta_5\zeta_3}{1080} + \frac{154037\pi^4\zeta_3}{31104} \right. \\
& - \frac{55327\pi^2\zeta_3}{432} + \frac{1100461\zeta_3}{432} + \frac{205\pi^2\zeta_5}{9} + \frac{155029\zeta_5}{48} + \frac{2732549\zeta_7}{1008} - \frac{665217829\pi^8}{1306368000} \\
& \left.\left. - \frac{131003\pi^6}{45360} - \frac{747929\pi^4}{51840} + \frac{2995\pi^2}{36} - \frac{2005247}{144}\right]\right\}
\end{aligned}
$$

$$
+ \frac{1}{\epsilon^2}\left[- \frac{83}{72}\pi^2\zeta_3 - \frac{21899\zeta_3}{864} - \frac{3659\zeta_5}{360} + \frac{31333\pi^4}{103680} + \frac{659\pi^2}{288} + \frac{11243}{144} \right]
$$

$$
+ \frac{1}{\epsilon}\left[- \frac{40231\zeta_3^2}{1296} + \frac{745\pi^2\zeta_3}{288} + \frac{18751\zeta_3}{144} + \frac{50191\zeta_5}{360} - \frac{277703\pi^6}{2177280} - \frac{14015\pi^4}{10368} \right.
$$

$$
\left. - \frac{149\pi^2}{24} - \frac{22757}{48} \right]
$$

$$
+ \left[\frac{39173\zeta_3^2}{324} - \frac{77399\pi^4\zeta_3}{25920} + \frac{4013\pi^2\zeta_3}{432} - \frac{259559\zeta_3}{432} - \frac{568\pi^2\zeta_5}{45} - \frac{1123223\zeta_5}{1440} \right.
$$

$$
\left. - \frac{2778103\zeta_7}{4032} + \frac{3129533\pi^6}{4354560} + \frac{28201\pi^4}{5760} + \frac{173\pi^2}{36} + \frac{382375}{144} \right]
$$

$$
+ \epsilon\left[\frac{4931 s_{8a}}{30} + \frac{2615}{144}\pi^2\zeta_3^2 - \frac{276671\zeta_3^2}{2592} - \frac{2702413\zeta_5\zeta_3}{1080} + \frac{154037\pi^4\zeta_3}{31104} \right.
$$

$$
- \frac{55327\pi^2\zeta_3}{432} + \frac{1100461\zeta_3}{432} + \frac{205\pi^2\zeta_5}{9} + \frac{155029\zeta_5}{48} + \frac{2732549\zeta_7}{1008} - \frac{665217829\pi^8}{1306368000}
$$

$$
\left.\left. - \frac{131003\pi^6}{45360} - \frac{747929\pi^4}{51840} + \frac{2995\pi^2}{36} - \frac{2005247}{144} \right]\right\}
$$

$$
s_{8a} = \zeta_8 + \zeta_{5,3} \approx 1.0417850291827918834
$$

$$
\zeta_{m_1,\ldots,m_k} = \sum^{\infty} \sum^{i_1-1} \cdots \sum^{i_{k-1}-1} \prod^{k} \frac{\mathrm{sgn}(m_j)^{i_j}}{i_j^{|m_j|}}
$$

$$
\begin{aligned}
&- \frac{149\pi^2}{24} - \frac{22757}{48}\Bigg] \\
&+ \Bigg[\frac{39173\zeta_3^2}{324} - \frac{77399\pi^4\zeta_3}{25920} + \frac{4013\pi^2\zeta_3}{432} - \frac{259559\zeta_3}{432} - \frac{568\pi^2\zeta_5}{45} - \frac{1123223\zeta_5}{1440} \\
&- \frac{2778103\zeta_7}{4032} + \frac{3129533\pi^6}{4354560} + \frac{28201\pi^4}{5760} + \frac{173\pi^2}{36} + \frac{382375}{144}\Bigg] \\
&+ \epsilon\Bigg[\frac{4931 s_{8a}}{30} + \frac{2615}{144}\pi^2\zeta_3^2 - \frac{276671\zeta_3^2}{2592} - \frac{2702413\zeta_5\zeta_3}{1080} + \frac{154037\pi^4\zeta_3}{31104} \\
&- \frac{55327\pi^2\zeta_3}{432} + \frac{1100461\zeta_3}{432} + \frac{205\pi^2\zeta_5}{9} + \frac{155029\zeta_5}{48} + \frac{2732549\zeta_7}{1008} - \frac{665217829\pi^8}{1306368000} \\
&- \frac{131003\pi^6}{45360} - \frac{747929\pi^4}{51840} + \frac{2995\pi^2}{36} - \frac{2005247}{144}\Bigg]\Bigg\}
\end{aligned}
$$

$$
s_{8a} = \zeta_8 + \zeta_{5,3} \approx 1.0417850291827918834
$$

$$
\zeta_{m_1,\ldots,m_k} = \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1-1}\cdots\sum_{i_k=1}^{i_{k-1}-1}\prod_{j=1}^{k}\frac{\mathrm{sgn}(m_j)^{i_j}}{i_j^{|m_j|}}
$$

Pole part of $\log(F_q)$

$$
\begin{aligned}
\log(F_q)|_{\text{pole part}} = {}& \\
& \frac{\alpha_s}{4\pi}\left\{\frac{1}{\epsilon^2}\left[-\frac{1}{2}C_F\gamma^0_{\text{cusp}}\right]+\frac{1}{\epsilon}\left[\gamma^0_q\right]\right\} \\
& +\left(\frac{\alpha_s}{4\pi}\right)^2\left\{\frac{1}{\epsilon^3}\left[\frac{3}{8}\beta_0 C_F\gamma^0_{\text{cusp}}\right]+\frac{1}{\epsilon^2}\left[-\frac{1}{2}\beta_0\gamma^0_q-\frac{1}{8}C_F\gamma^1_{\text{cusp}}\right]+\frac{1}{\epsilon}\left[\frac{\gamma^1_q}{2}\right]\right\} \\
& +\left(\frac{\alpha_s}{4\pi}\right)^3\left\{\frac{1}{\epsilon^4}\left[-\frac{11}{36}\beta_0^2 C_F\gamma^0_{\text{cusp}}\right]+\frac{1}{\epsilon^3}\left[C_F\left(\frac{2}{9}\beta_1\gamma^0_{\text{cusp}}+\frac{5}{36}\beta_0\gamma^1_{\text{cusp}}\right)+\frac{1}{3}\beta_0^2\gamma^0_q\right]\right. \\
& \left.+\frac{1}{\epsilon^2}\left[-\frac{1}{3}\beta_1\gamma^0_q-\frac{1}{3}\beta_0\gamma^1_q-\frac{1}{18}C_F\gamma^2_{\text{cusp}}\right]+\frac{1}{\epsilon}\left[\frac{\gamma^2_q}{3}\right]\right\} \\
& +\left(\frac{\alpha_s}{4\pi}\right)^4\left\{\frac{1}{\epsilon^5}\left[\frac{25}{96}\beta_0^3 C_F\gamma^0_{\text{cusp}}\right]+\frac{1}{\epsilon^4}\left[C_F\left(-\frac{13}{96}\beta_0^2\gamma^1_{\text{cusp}}-\frac{5}{12}\beta_1\beta_0\gamma^0_{\text{cusp}}\right)-\frac{1}{4}\beta_0^3\gamma^0_q\right]\right. \\
& +\frac{1}{\epsilon^3}\left[C_F\left(\frac{5}{32}\beta_2\gamma^0_{\text{cusp}}+\frac{3}{32}\beta_1\gamma^1_{\text{cusp}}+\frac{7}{96}\beta_0\gamma^2_{\text{cusp}}\right)+\frac{1}{4}\beta_0^2\gamma^1_q+\frac{1}{2}\beta_1\beta_0\gamma^0_q\right] \\
& \left.+\frac{1}{\epsilon^2}\left[-\frac{1}{4}\beta_2\gamma^0_q-\frac{1}{4}\beta_1\gamma^1_q-\frac{1}{4}\beta_0\gamma^2_q-\frac{1}{32}C_F\gamma^3_{\text{cusp}}\right]+\frac{1}{\epsilon}\left[\frac{\gamma^3_q}{4}\right]\right\}
\end{aligned}
$$

$\gamma_{\rm cusp}$

$$
\begin{aligned}
\gamma^0_{\rm cusp} &= 4\,, \\
\gamma^1_{\rm cusp} &= \left(\frac{268}{9} - \frac{4\pi^2}{3}\right) C_A - \frac{40 n_f}{9}\,, \\
\gamma^2_{\rm cusp} &= C_A^2 \left(\frac{88\zeta_3}{3} + \frac{44\pi^4}{45} - \frac{536\pi^2}{27} + \frac{490}{3}\right) \\
&\quad + n_f \left[C_A \left(-\frac{112\zeta_3}{3} + \frac{80\pi^2}{27} - \frac{836}{27}\right) + C_F \left(32\zeta_3 - \frac{110}{3}\right)\right] - \frac{16 n_f^2}{27}
\end{aligned}
$$

3 loops: [Vogt'00;Berger'02;Moch,Vermaseren,Vogt'04;...]

4 loops: complete n_f^2, rest large-N_c

$$\gamma^3_{\rm cusp} = +\left(\frac{128\pi^2\zeta_3}{9} + 224\zeta_5 - \frac{44\pi^4}{27} - \frac{16252\zeta_3}{27} + \frac{13346\pi^2}{243} - \frac{39883}{81}\right) N_c^2 n_f + \Big(-32\zeta_3^2 - \frac{176\pi^2\zeta_3}{9} + \frac{20992\zeta_3}{27} - 352\zeta_5 - \frac{292\pi^6}{315} + \frac{902\pi^4}{45} - \frac{44416\pi^2}{243} + \frac{84278}{81}\Big) N_c^3 + n_f^2\left[C_A\left(\frac{2240\zeta_3}{27} - \frac{56\pi^4}{135} - \frac{304\pi^2}{243} + \frac{923}{81}\right) + C_F\left(-\frac{640\zeta_3}{9} + \frac{16\pi^4}{45} + \frac{2392}{81}\right)\right] + \left(\frac{64\zeta_3}{27} - \frac{32}{81}\right) n_f^3$$

4 loops, n_f^3: [Gracey'04; Beneke,Braun'95; von Manteuffel,Schabinger'16]
4 loop, n_f^2: [Ruijl,Ueda,Vermaseren,Davies,Vogt'16; Moch,Ruijl,Ueda,Vermaseren,Vogt'17]
4 loop, numerical: [Moch,Ruijl,Ueda,Vermaseren,Vogt'17]

γ_q

n_f^2 complete, rest large-N_c

$$\gamma_q^3 = \left(-\frac{680\zeta_3^2}{9} - \frac{1567\pi^6}{20412} + \frac{83\pi^2\zeta_3}{9} + \frac{557\zeta_5}{9} + \frac{3557\pi^4}{19440} - \frac{94807\zeta_3}{972} + \frac{354343\pi^2}{17496} + \frac{145651}{1728}\right) N_c^3 n_f + \left(\frac{1175\zeta_3^2}{9} + \frac{82\pi^4\zeta_3}{45} - \frac{377\pi^2\zeta_3}{6} + \frac{867397\zeta_3}{972} + 24\pi^2\zeta_5 - 1489\zeta_5 + 705\zeta_7 + \frac{114967\pi^6}{204120} - \frac{59509\pi^4}{9720} - \frac{120659\pi^2}{17496} - \frac{187905439}{839808}\right) N_c^4$$
$$+ n_f^2 \left[\left(-\frac{64}{27}\pi^2\zeta_3 - \frac{7436\zeta_3}{243} + \frac{592\zeta_5}{9} - \frac{19\pi^4}{135} - \frac{41579\pi^2}{8748} + \frac{97189}{34992}\right) C_A C_F + \left(\frac{56\pi^2\zeta_3}{27} + \frac{2116\zeta_3}{81} - \frac{520\zeta_5}{9} + \frac{1004\pi^4}{1215} - \frac{493\pi^2}{81} - \frac{9965}{972}\right) C_F^2\right]$$
$$+ \left(-\frac{712\zeta_3}{243} - \frac{16\pi^4}{1215} - \frac{4\pi^2}{81} + \frac{18691}{6561}\right) C_F n_f^3$$

4 loops, n_f^3: [von Manteuffel,Schabinger'16]

$\log(F_q)$

n_f^2 complete, rest large-N_c

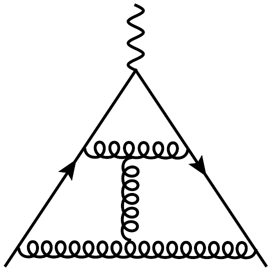
$$\log(F_q)|^{(4)}_{\text{large-}N_c\text{, finite part}} =$$

$$N_c^4 \Bigg(-14s_{8a} + 10\pi^2\zeta_3^2 - \frac{86647\zeta_3^2}{54} + 766\zeta_5\zeta_3 - \frac{251\pi^4\zeta_3}{6480} - \frac{57271\pi^2\zeta_3}{1296} + \frac{173732459\zeta_3}{23328}$$
$$+ \frac{1517\pi^2\zeta_5}{216} - \frac{881867\zeta_5}{1080} - \frac{36605\zeta_7}{288} + \frac{674057\pi^8}{5443200} - \frac{135851\pi^6}{77760} + \frac{386729\pi^4}{31104}$$
$$- \frac{429317557\pi^2}{839808} - \frac{54900768805}{6718464}\Bigg) + \ldots$$

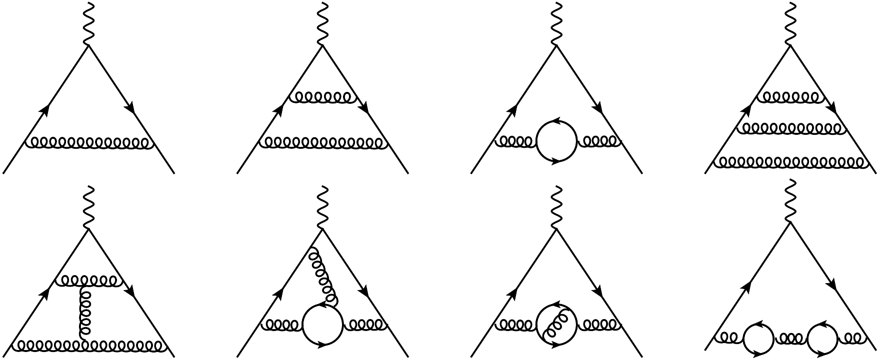
$$\log(F_q)|^{(4)}_{n_f^2, n_f^3\text{,finite part}} =$$

$$+ n_f^2 \Bigg[\Bigg(-\frac{1714\zeta_3^2}{3} - \frac{218\pi^2\zeta_3}{9}$$
$$+ \frac{2897315\zeta_3}{486} + \frac{150886\zeta_5}{135} - \frac{709\pi^6}{17010} - \frac{1861\pi^4}{2430} - \frac{5825827\pi^2}{7776} - \frac{3325501813}{279936}\Bigg) C_A C_F$$
$$+ \Bigg(\frac{2702\zeta_3^2}{3} + \frac{1820\pi^2\zeta_3}{27} - \frac{859249\zeta_3}{162} + \frac{1580\zeta_5}{3} + \frac{17609\pi^6}{17010} + \frac{1141\pi^4}{1620} + \frac{76673\pi^2}{243}$$
$$- \frac{25891301}{11664}\Bigg) C_F^2\Bigg] + \Bigg(-\frac{410}{243}\pi^2\zeta_3 - \frac{20828\zeta_3}{243} - \frac{2194\zeta_5}{135} + \frac{1661\pi^4}{2430} + \frac{145115\pi^2}{4374}$$
$$+ \frac{10739263}{17496}\Bigg) C_F n_f^3$$

III. Massive form factor



Massive photon quark form factor



$$\Gamma^\mu(q_1,q_2) = F_1(q^2)\gamma^\mu - \frac{i}{2m}F_2(q^2)\sigma^{\mu\nu}q_\nu$$

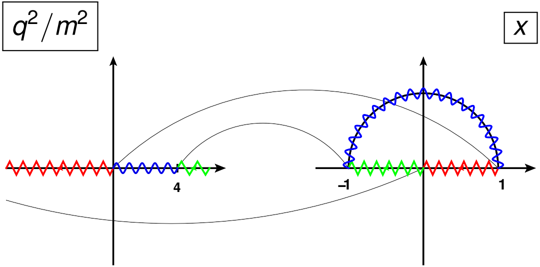
aim: $F_1(q^2)$ and $F_2(q^2)$ up to 3 loops, large-N_c limit
[1 loop less but massive quarks]

Computation of MIs

[Henn,Smirnov,Smirnov'16]

$$\frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$$

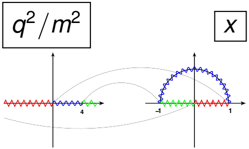
$x \to 0$	high energy:	$q^2 \to -\infty$
$x \to -1$	threshold:	$q^2 \to 4m^2$
$x \to 1$	low energy:	$q^2 \to 0$



Computation of MIs

[Henn,Smirnov,Smirnov'16]

$x \to 0$	high energy:	$q^2 \to -\infty$
$x \to -1$	threshold:	$q^2 \to 4m^2$
$x \to 1$	low energy:	$q^2 \to 0$



- system of diff. eqs. for MIs in x
- canonical basis:

$$g'(x,\epsilon) = \epsilon\, A(x) \cdot g(x,\epsilon)$$

$$A(x) = \frac{a_1}{x} + \frac{a_2}{1+x} + \frac{a_3}{1-x} + \frac{a_4}{1-x+x^2}$$

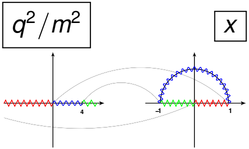
$$q^2 \to -\infty,\;\; q^2 \to 4m^2,\;\; q^2 \to 0,\;\; q^2 = m^2$$

new at 3 loops

Computation of MIs

[Henn,Smirnov,Smirnov'16]

$x \to 0$	high energy:	$q^2 \to -\infty$
$x \to -1$	threshold:	$q^2 \to 4m^2$
$x \to 1$	low energy:	$q^2 \to 0$



- canonical basis:

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new at 3 loops

- solution: iterative integrals
- $1 - x + x^2 = (x - r_1)(x - r_2)$

$$r_{1/2} = (1 \pm \sqrt{3}i)/2 = e^{\pm i\pi/3}$$ 6^{th} root of unity

⇨ Goncharov polylogarithms

[Goncharov'98]

$$G(\alpha_1,\ldots,\alpha_n;z) \;=\; \int_0^z \frac{\mathrm{d}t}{t-\alpha_1} G(\alpha_2,\ldots,\alpha_n;z)$$

Analytic result

... in terms of Goncharov polylogarithms up to transcendelity weight 6.
[Henn,Smirnov,Smirnov,Steinhauser'16]

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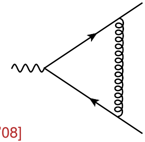

Analytic expansions in kinematical limits:

- $q^2 \to -\infty$, $x \to 0$ ➪ resummations
- $q^2 \to 0$, $x \to 1$ ➪ $(g-2)_{\mathrm{quark}}$ [Grozin,Marquard,Piclum,Steinhauser'08]
- $q^2 \to 4m^2$, $x \to -1$ ➪ threshold cross section

$x = \frac{2\beta}{1+\beta} - 1$

$\beta = \sqrt{1 - \frac{4m^2}{s}}$

finite! (real rad. has rel. factor β^3)

$$
\begin{aligned}
\sigma(e^+e^- \to Q\bar{Q}) \\
&= \sigma_0 \beta \left[|F_1 + F_2|^2 + \frac{|(1-\beta^2)F_1 + F_2|^2}{2(1-\beta^2)} \right] \\
&= \sigma_0 \frac{3\beta}{2} \left[1 - \frac{\beta^2}{3} + \frac{\alpha_s}{4\pi}\Delta^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \Delta^{(2)} + \left(\frac{\alpha_s}{4\pi}\right)^3 \Delta^{(3)} + \ldots \right]
\end{aligned}
$$

$\Delta^{(1)} = N_c \left[\frac{\pi^2}{\beta} - 8 + \beta \frac{2\pi^2}{3} \right] + \ldots;\ \Delta^{(3)} = \{N_c^3, N_c^2 n_l, N_c n_l^2\}$ including $1/\beta$

agreement with [Pineda,Signer'07],. . . ,[Kiyo,Maier,Maierhöfer,Marquard'09]

Conclusions

- massless 4-loop photon quark vertex
  - large-N_c limit
  - planar and non-planar n_f^2 (also for Higgs-quark FF)
  - $\gamma_{\rm cusp}$ and γ_q
- massive 3-loop photon quark vertex in large-N_c limit
- canonical basis for MIs
- 4-loop results for all planar families (99 MIs) and 2 non-planar families (24 non-planar MIs)
- solution in terms of iterated integrals (HPLs, Goncharov polylogarithms)